Comment on the finite-temperature behavior of $\lambda(\vec{\Phi}^2)^2{}_4$ theory

William A. Bardeen* and Moshe Moshe
Max-Planck-Institut für Physik und Astrophysik, Werner-Heisenberg-Institut für Physik, Föhringer Ring 6, 8000 München 40, Federal Republic of Germany and Physics Department, Technion–Israel Institute of Technology, Haifa 32000, Israel
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We examine the phase structure and stability of self-interacting scalar field theories at finite temperature. In the large-N limit, the renormalized theory exhibits an intrinsic instability which becomes manifest at high temperature contrary to recent claims in the literature.

In a previous paper,[1] we studied the phase structure and stability of the large-N version of a self-interacting scalar field theory, $\lambda_0\Phi^4$. We found two different situations depending on the sign of the bare coupling constant λ_0. For a positive bare coupling constant the theory is "trivial," which means that the theory can only be defined by introducing an ultraviolet cutoff, and the low-energy field theory describes a free field as the cutoff is removed. For finite cutoff, the theory describes a weakly interacting scalar field with the usual phase structure at zero and finite temperature.

To maintain a finite low-energy effective coupling in the "continuum" limit, the theory must be renormalized. However, renormalization can only be achieved by choosing a negative bare coupling constant which introduces the intrinsic instability of the theory. There is no ground state for the renormalized theory. For a certain range of parameters, we observed the existence of a metastable ground state which was stabilized, in the large-N limit, by a large tunneling barrier. It was subsequently argued that the renormalized theory could be consistently defined using this metastable ground state as the vacuum state for a certain range of couplings.[2]

In a recent paper,[3] the authors claim to be able to extend the range of renormalized couplings allowed for the system at finite temperature. This result contradicts our previous work[1] where we found that even the metastable phase, which can exist at low temperature, will disappear at sufficiently high temperature.[4] In this paper, we will clarify our previous results for the theory at finite temperature and show that the conclusions reached in Ref. 3 result from an incorrect application of thermodynamics.

The theory we will study is the large-N version of the O(N)-symmetric $\lambda_0\Phi^4$ theory which is described by the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\vec{\Phi})(\partial^\mu\vec{\Phi})-\tfrac{1}{2}\mu_0{}^2(\vec{\Phi}^2)-\tfrac{1}{4}\lambda_0(\vec{\Phi}^2)^2 , \qquad (1)$$

where $\vec{\Phi}(x)$ is an N-component real scalar field with μ_0 and λ_0 being the bare mass and coupling constant, respectively. We consider the standard large-N limit where $\mu_0=O(1)$ and $\lambda_0=O(1/N)$ as $N\to\infty$. In this large-N limit, the theory is exactly solvable both at zero and finite temperature. It describes a system of weakly interacting bosons whose mass is determined dynamically as a function of the temperature. In Ref. 1, we used variational methods to study the ground-state structure of the theory for both positive and negative bare coupling. At zero temperature, the vacuum energy is minimized as a function of the dynamical mass and field vacuum expectation $\vec{\Phi}_c$. A system in thermodynamic equilibrium at given temperature is described by minimizing the Helmholtz free energy F, instead of the internal energy E as used by the authors of Ref. 3. The two energies differ at a finite temperature T by the entropy of the system through the relation $F=E-TS$. We will recall our results for the finite-temperature properties of $\lambda\Phi^4$ theory and compare them to the results claimed in Ref. 3.

The Helmholtz free energy F is defined for a system at a finite temperature by

$$\exp(-\beta F)=Z_0=\mathrm{tr}(\exp(-\beta H)) , \qquad (2)$$

where $\beta=1/kT$. For a system of noninteracting massive bosons, the Helmholtz free energy is given by

$$F_0=-(1/\beta)\ln(Z_0)$$
$$=V(2\pi)^{-3}\int d^3k\,[\tfrac{1}{2}\omega_k+(1/\beta)\ln(1-e^{-\beta\omega_k})] , \qquad (3)$$

where $\omega_k=(k^2+m^2)^{1/2}$ for particles of mass m. Using the notation of Ref. 1 [e.g., Eq. (5.4)], we define the "kinetic free energy" density as

$$K_{0\beta}(m)=W_0-\tfrac{1}{2}m^2\langle\vec{\Phi}^2\rangle_{0\beta} , \qquad (4)$$

where W_0 is the free energy density, $W_0=F_0/V$, and

$$\langle\vec{\Phi}^2\rangle_{0\beta}=\mathrm{tr}(\vec{\Phi}^2(0)\exp(-\beta H_0))/Z_0$$
$$=(2\pi)^{-3}\int d^3k\left[\frac{1}{2\omega_k}+(1/\omega_k)[\exp(\beta\omega_k)-1]^{-1}\right]. \qquad (5)$$

We have found the kinetic free energy $K_{0\beta}(m)$ to be a convenient quantity as it depends only on the mass parameter m even when the scalar field develops an expectation value $\langle\vec{\Phi}\rangle_\beta=\vec{\Phi}_c$. The Hartree-Fock free energy density is then given by

$$W_\beta(m,\vec{\Phi}_c)=K_\beta(m)+\langle V(\vec{\Phi})\rangle_\beta . \qquad (6)$$

A system in thermodynamic equilibrium is described by minimizing this free energy as a function of m and $\vec{\Phi}_c$. This Hartree-Fock description is exact in the large-N limit.

In contrast, the work reported in Ref. 3 is based on minimizing the total internal energy of the thermodynamic system. To compare with our results, we can define the "kinetic energy" density,

$$\tilde{K}_{0\beta}(m)=U_{0\beta}-\tfrac{1}{2}m^2\langle\vec{\Phi}^2\rangle_\beta \;, \tag{7}$$

where the internal energy density is given by $U_0=E_0/V$ and

$$E_0=-\partial_\beta \ln(Z_0)=\mathrm{tr}(H_0\exp(-\beta H_0))/Z_0=F_0+TS_0 \;.$$

The confusion between $K_\beta(m)$ and $\tilde{K}_\beta(m)$ is the source of the error in Ref. 3. A direct calculation using Eq. (3) and Eq. (5) [Eqs. (5.1), (5.2), and (5.5) of Ref. 1] gives

$$K_{0\beta}(m)=(2\pi)^{-3}\int d^3k\,\{[\tfrac{1}{2}\omega_k+(1/\beta)\ln(1-e^{-\beta\omega_k})]-\tfrac{1}{2}m^2[(1/2\omega_k)+(1/\omega_k)(e^{\beta\omega_k}-1)^{-1}]\} \tag{8a}$$

and

$$\tilde{K}_{0\beta}(m)=(2\pi)^{-3}\int d^3k\,\{[\tfrac{1}{2}\omega_k+\omega_k(e^{\beta\omega_k}-1)^{-1}]-\tfrac{1}{2}m^2[(1/2\omega_k)+(1/\omega_k)(e^{\beta\omega_k}-1)^{-1}]\} \;. \tag{8b}$$

Clearly,

$$\tilde{K}_{0\beta}(m)-K_{0\beta}(m)=U_0-W_0=TS_0>0 \;.$$

Note that $K_\beta(m)$ satisfies the derivative relation [Eq. (5.5) of Ref. 1], namely,

$$\partial_{m^2}K_{0\beta}(m^2)=-\tfrac{1}{2}m^2\partial_{m^2}\langle\vec{\Phi}^2\rangle_{0\beta} \;, \tag{9}$$

as can be seen from the direct calculation. As noted by Ref. 3, $\tilde{K}_\beta(m)$ does not satisfy this relation.

As a simple example of the variational calculation and a demonstration of the difference between Ref. 1 and Ref. 3, we consider a free bosonic field theory where the potential energy is simply given by the mass term $V(\vec{\Phi})=\frac{1}{2}\mu^2\vec{\Phi}^2$. The free energy density is then given by

$$W_{0\beta}(m)=K_{0\beta}(m)+\tfrac{1}{2}\mu^2\langle\vec{\Phi}^2\rangle_{0\beta}(m) \;. \tag{10}$$

Using the relation of Eq. (9), we can find the condition for the minimum value for the free energy,

$$\begin{aligned}\partial_{m^2}W_\beta(m)&=\partial_{m^2}K_{0\beta}(m)+\tfrac{1}{2}\mu^2\partial_{m^2}\langle\vec{\Phi}^2\rangle_{0\beta}\\ &=-\tfrac{1}{2}(\partial_{m^2}\langle\vec{\Phi}^2\rangle_{0\beta})(m^2-\mu^2)=0 \;,\end{aligned} \tag{11}$$

which gives $m^2=\mu^2=m_{\mathrm{bare}}{}^2$ as expected for a free theory. Inserting $m^2=\mu^2$ into Eq. (10) also gives the correct value of the free energy for a particle of mass μ, as in Eq. (3). Applying this same procedure to the expression for the internal energy as in Ref. 3 does not yield the correct answer for m^2 except at zero temperature where the two approaches coincide. Minimizing the internal energy does not describe a system in thermal equilibrium, even a free system.

The systematic treatment of the interacting case was presented in Sec. V of Ref. 1. We review this study and compare our results to those of Ref. 3. We consider the $\lambda_0\Phi^4$ theory as defined by Eq. (1). The free energy density can be computed using Eqs. (5.4)–(5.6) of Ref. 1 or by using Eqs. (6), (8a), (9), and (5) above. The expectation value of $\vec{\Phi}^2$ can be expressed in the form

$$\begin{aligned}\langle\vec{\Phi}^2\rangle_\beta=&(N/16\pi^2)\Lambda^2-(N/16\pi^2)m^2\ln(e\Lambda^2/m^2)\\ &+(NT^2/24)F(m^2/T^2) \;,\end{aligned} \tag{12}$$

where Λ is an ultraviolet cutoff, and $F(x)$ is the montonically decreasing function of x given in Ref. 1 which has the behavior $F(x)\to 1$ as $x\to 0$ and

$$F(x)\sim\exp(-\sqrt{x})/\sqrt{x} \text{ as } x\to\infty \;.$$

$K_\beta(m)$ is consistently obtained by integrating Eq. (9) [or Eq. (5.5) of Ref. 1] with the result

$$\begin{aligned}K_\beta(m)=&(N/64\pi^2)m^4\ln(e^{1/2}\Lambda^2/m^2)\\ &+(N/48)T^4\int_{m^2/T^2}^\infty dy\,y\,\partial_y F(y) \;.\end{aligned} \tag{13}$$

We note that this expression for $K_\beta(m)$ differs, by an integration constant, from that in Eq. (5.7) of Ref. 1. This change will modify our subsequent expressions for the free energy from those of Ref. 1 but does not modify any of the physical conclusions. At large N, the free energy density becomes

$$\begin{aligned}W_\beta(m,\vec{\Phi}_c)&=K_\beta(m)+\langle V(\vec{\Phi})\rangle_\beta\\ &=K_\beta(m)+\tfrac{1}{2}\mu_0{}^2(\vec{\Phi}_c{}^2+\langle\vec{\Phi}^2\rangle_\beta)+\tfrac{1}{4}\lambda_0(\vec{\Phi}_c{}^2+\langle\vec{\Phi}^2\rangle_\beta)^2\\ &=(N/64\pi^2)m^4\ln(e^{1/2}\Lambda^2/m^2)+(N/48)T^4\int_{m^2/T^2}^\infty dy\,y\,\partial_y F(y)\\ &\quad+\tfrac{1}{4}\lambda_0\{\vec{\Phi}_c{}^2+\mu_0{}^2/\lambda_0+(N/16\pi^2)[\Lambda^2-m^2\ln(e\Lambda^2/m^2)]+(N/24)T^2F(m^2/T^2)\}^2-\tfrac{1}{4}\mu_0{}^4/\lambda_0 \;.\end{aligned} \tag{14}$$

The renormalized theory is defined as in Ref. 1, and in the limit of infinite cutoff, the bare coupling constant vanishes, $\lambda_0\to 0^-$. The free energy density can be expressed in terms of renormalized quantities in this limit [Eq. (5.13) of Ref. 1]:

$$\begin{aligned}V_\beta(m,\vec{\Phi}_c)&=W_\beta(m,\vec{\Phi}_c)+\tfrac{1}{4}\mu_0{}^4/\lambda_0\\ &=\tfrac{1}{2}m^2(\vec{\Phi}_c{}^2\pm\mu^2/\lambda)-\tfrac{1}{4}m^4[1/\lambda+(N/16\pi^2)\ln(e^{3/2}M^2/m^2)]-(N/48)T^4\int_{m^2/T^2}^\infty dy\,F(y) \;.\end{aligned} \tag{15}$$

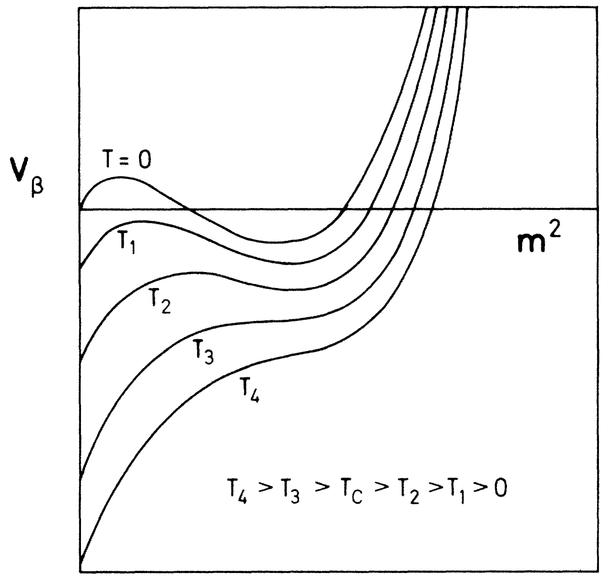

FIG. 1. The disappearance of the false vacuum as the temperature increases beyond T_c.

This form of the free energy density hides a basic instability of the theory which occurs at large $\vec{\Phi}_c$ for large but finite cutoff and $\lambda_0 < 0$, but which is obvious from Eq. (14). However, it is the temperature dependence of the free energy density which is directly related to the criticism of Ref. 3. In Fig. 1, we present a plot of the free energy density as a function of the effective mass m^2 for a range of temperatures. Solutions to the renormalized gap equation represent stationary points on this plot. At low temperatures, the metastable phase exists as the minimum at finite m^2. As the temperature is increased, this secondary minimum disappears, and the metastable phase ceases to exist. Only the minimum at zero mass survives and this phase is unstable to decay to large values of $\vec{\Phi}_c$, see Eq. (14). Hence we conclude that there is a limiting temperature beyond which a false vacuum solution ceases to exist. The result is in direct contradiction to the physics conclusion of Ref. 3 which claimed a new phase structure based on the false vacuum solution at high temperature.

This strange behavior of the renormalized theory can be traced directly to the negative coupling constant. Since the temperature dependence of the effective mass is proportional to the coupling constant, we find that the effective mass decreases as the temperature increases. Hence the system can become more ordered as the temperature increases which is contrary to the usual expectations of thermal systems. Finally, we remark that this strange behavior is not a property of the "trivial" version of the theory where the coupling constant remains positive. Here, the usual thermal properties hold for the system, and no limiting temperature exists within the range of temperatures permitted by the presence of the physical cutoff scale for the theory.

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*On leave from Fermilab, P. O. Box 500, Batavia, Illinois 60510.

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