Compactification of the Chern-Simons term

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Compactifications of Chern-Simons topological Lagrangians are discussed using tori and twisted boundary conditions for the gauge fields.

The Chern-Simons term in three dimensions has been extensively discussed in the literature.[1] In odd dimensions larger than three, Chern-Simons terms were derived in Refs. 2 and 3 and, as in the three-dimensional case, it turns out that they are eligible to be a term in the Lagrangian at both the classical and quantum levels.[2] Here we address ourselves to the question of what is the result of a compactification of such terms from $D=2n-1$ to $D=4$ dimensions.

Given the general form of the Chern-Simons term[2] $(D=2n-1)$,

$$w_{2n-1}=n\int_0^1 dt\,\mathrm{tr}(AF_t{}^{n-1}) ,$$

with

$$A=A_\mu^a\lambda_a\,dx^\mu ,$$
$$F=\tfrac{1}{2}F_{\mu\nu}^a\lambda_a\,dx^\mu\wedge dx^\nu ,$$

and

$$F_t=t[F+(t-1)A^2] ,$$

it is obvious that a trivial compactification to four dimensions in which all "internal" components A_α, $a=5,6,\ldots,2n-1$ are set equal to zero results in no contribution to the four-dimensional Lagrangian. However, if A_α has a nontrivial vacuum expectation value, contributions to the four-dimensional theory can appear. Since CP is violated by Chern-Simons terms,[4] it is natural to expect CP violation in four dimensions; namely, we expect to end up with a four-dimensional θ term with a coefficient that is related to the Chern-Simons coefficient in high dimensions. Because of the quantization condition imposed on the latter, we can also expect a quantized $F\tilde{F}$ coefficient if the vacuum expectation value of the gauge field is also quantized.

Let us start with a $D=9$ gauge theory with a Chern-Simons term

$$\begin{aligned} w_9=\mathrm{tr}[&AF^4-\tfrac{1}{6}(2F^3A^3+AF^2A^2F+AFA^2F^2)\\ &+\tfrac{1}{21}(3F^2A^5+2A^3FA^2F+AFA^4F)\\ &-\tfrac{1}{14}A^7F+\tfrac{1}{126}A^9] . \end{aligned} \tag{1}$$

Since $W(A)=[1/5!(2\pi)^5]\int w_9$ is an integer, we can write the Lagrangian

$$\mathcal{L}_9=M^5\,\mathrm{tr}(F_{\hat\mu\hat\nu}F^{\hat\mu\hat\nu}) + \frac{\mu^5}{M^5}\frac{1}{5!(2\pi)^5}w_9^{\hat\mu_1\cdots\hat\mu_9}\varepsilon_{\hat\mu_1\cdots\hat\mu_9},\ \hat\mu,\hat\nu=1,2,\ldots,9 . \tag{2}$$

The theory is gauge invariant (including large gauge transformations) only if

$$\frac{\mu^5}{M^5}=2\pi N_1 . \tag{3}$$

Compactifying in a five-dimensional torus with all lengths equal to a, topological configurations of the gauge fields appear naturally if one takes into account nontrivial boundary conditions with gauge fields $A_\alpha(x_\beta=0)$ and $A_\alpha(x_\beta=a)$ $(\alpha,\beta=5,\ldots,D)$ equal only up to a gauge transformation. These topological configurations can be given by[5,6]

$$A_\alpha(x_\beta=a)=\Omega_\beta A_\alpha(x_\beta=0) , \tag{4}$$

where $\Omega_\beta=\Omega_\beta(x_5,\ldots,\hat{x}_\beta,\ldots,x_9)$ (meaning that Ω_β does not depend on x_β) represents the gauge transformation. In the case of gauge fields

$$\Omega_\beta A_\alpha=\Omega_\beta(A_\alpha-i\partial_\alpha)\Omega_\beta{}^{-1} .$$

Following 't Hooft,[5] we choose a topological configuration in which only the gauge field in an Abelian subgroup of the gauge group G acquires a (twisted) vacuum expectation value. In this case, terms in w_9 with more than one A_α vanish. The only nonvanishing term is

$$A_\alpha^0F_{\beta\gamma}^0F_{\delta\sigma}^0F_{\mu\nu}^aF_{\lambda\rho}^b\,\mathrm{tr}(\lambda^a\lambda^b)\varepsilon^{\alpha\beta\gamma\delta\sigma\mu\nu\lambda\rho},$$
$$\mu,\nu,\lambda,\rho=1,\ldots,4 ,$$
$$\alpha,\beta,\gamma,\delta,\sigma=5,\ldots,9 ,$$

where A_α^0 is the vacuum expectation value of the (Abelian) gauge field. From the antisymmetric properties of ε we can easily check that all other terms vanish.

A symmetric solution in the extra dimensions with vacuum expectation value $|F_{\alpha\beta}^0|=$const, $\alpha,\beta=5,6,\ldots,9$, can be given by

$$\begin{aligned} A_5^0&=(x_6+x_7)\mathcal{C} ,\\ A_6^0&=(x_7+x_8)\mathcal{C} ,\\ A_7^0&=(x_8+x_9)\mathcal{C} ,\qquad \mathcal{C}=\text{const} ,\\ A_8^0&=(x_9+x_5)\mathcal{C} ,\\ A_9^0&=(x_5+x_6)\mathcal{C} . \end{aligned} \tag{5}$$

Gauge transformations Ω_α that link the fields at $x_\alpha=0$

and $x_\alpha = a$ are given by (4) and (5):

$$\Omega_5 = e^{-ia(x_8+x_9)\mathcal{C}} ,$$
$$\Omega_6 = e^{-ia(x_9+x_5)\mathcal{C}} ,$$
$$\Omega_7 = e^{-ia(x_5+x_6)\mathcal{C}} , \qquad (6)$$
$$\Omega_8 = e^{-ia(x_6+x_7)\mathcal{C}} ,$$
$$\Omega_9 = e^{-ia(x_7+x_8)\mathcal{C}} .$$

Solution (5) is not the only one that gives $|F| =$const for the internal components. But for our purposes the other possibilities with $|F| =$const give the same final result.

A quantization condition for $\mathcal{C}$ emerges if we also consider charged matter fields that may satisfy similar boundary conditions:

$$\Psi(x_\alpha = a) = \Omega_\alpha \Psi(x_\alpha = 0) .$$

Since Ψ must be well defined at the edges [e.g., $(x_5 = a, x_6 = a, x_7, x_8, x_9)$], we have

$$\Psi(x_5 = a, x_6 = a) = \begin{cases} \Omega_5(x_6 = a)\Omega_6(x_5 = 0)\Psi(0,0) , \\ \Omega_6(x_5 = a)\Omega_5(x_6 = 0)\Psi(0,0) . \end{cases}$$

Comparing these expressions and using (6) we get

$$\mathcal{C} = \frac{2\pi N_2}{a^2} , \qquad (7)$$

where N_2 is an integer (number of twists).

The reduction to four dimensions can be done by integrating out the five compactified dimensions, using the twisted configuration described above for the vacuum expectation value of the internal components A_α^0 and assuming $\partial_\alpha A_\mu = 0$ $(F_{\mu\alpha} = 0)$. The kinetic term gives

$$\int dx^5 \cdots dx^9 M^5 \mathrm{tr}(F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}}) = M^5 a^5 \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \Lambda ,$$

where $\mu,\nu = 1,\ldots,4$, $\Lambda \sim M^5 \mathcal{C}^2 a^5$, and $M^5 a^5 = 1/g^2$, with the g coupling constant. The Chern-Simons term is reduced to

$$T_{\mathrm{CS}} \to \frac{\mu^5}{M^5} \int \frac{\mathrm{tr}(AF^4)}{5!(2\pi)^5} = \frac{2\pi N_1 \varepsilon^{\mu\nu\lambda\rho\alpha\beta\gamma\delta\sigma}}{5!(2\pi)^5 2^4} \int dx_5 \cdots dx_9 \mathrm{tr}(F_{\mu\nu}F_{\lambda\rho}) F_{\alpha\beta}^0 F_{\gamma\delta}^0 A_\sigma^0 , \qquad (8)$$

where $\mu,\nu,\lambda,\rho = 1,\ldots,4$ and $\alpha,\beta,\gamma,\delta,\sigma = 5,\ldots,9$. Counting all the possible permutations of indices, we end up with

$$T_{\mathrm{CS}} = \frac{2\pi N_1}{2^2(2\pi)^5} \varepsilon^{\mu\nu\lambda\rho} \mathrm{tr}(F_{\mu\nu}F_{\lambda\rho}) \mathcal{C}^3 a^6$$

and, with $\mathcal{C}$ given by (7),

$$T_{\mathrm{CS}} = 2\pi N \frac{1}{16\pi^2} \varepsilon^{\mu\nu\lambda\rho} \mathrm{tr}(F_{\mu\nu}F_{\lambda\rho}),\ N = N_1 N_2^3 . \qquad (9)$$

The reduced (four-dimensional) Lagrangian is

$$\mathcal{L}_4 = \frac{1}{g^2} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \Lambda + 2\pi N \frac{1}{16\pi^2} \mathrm{tr}(\tilde{F}_{\mu\nu}F^{\mu\nu}) .$$

It is interesting to note that the $\tilde{F}F$ term does not depend on a and has the right coefficient to assure *CP* conservation, despite the fact that this symmetry is not present in $\mathcal{L}_9$.

An extension of this compactification to other (odd) dimensions is straightforward. For $D = 2n - 1$ space-time dimensions and $I = 2n - 5$ internal dimensions, the nonvanishing contribution of the Chern-Simon term is given by

$$T_{\mathrm{CS}} \to \frac{\mu^I}{M^I} \frac{1}{n!(2\pi)^n} \mathrm{tr}(AF^{n-1}) , \qquad (10)$$

with $\mu^I/M^I = 2\pi N_1$. After compactification [using an extension of (5) and (6)] we have roughly

$$AF^{n-1} \to \underset{\text{4D space-time indices}}{F^2} \times \underset{\text{internal indices}}{AF^{n-3}} \to F_{\mu\nu}\tilde{F}^{\mu\nu}\mathcal{C}^{n-2} .$$

All possible permutations of indices must be taken into account. [The factors to be considered are $(2n-1)^{-1}$ for $n-1$ $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$, $2n-5$ because α (in A_α) can be any one of $2n-5$ possibilities, $\frac{1}{2}(n-1)!$ to pick up $n-3$ F's out of $n-1$ F's, and 2^{n-3} for each $F_{\alpha\beta}, F_{\beta\alpha}$.] After integrating over the $2n-5$ internal dimensions, we get

$$T_{\mathrm{CS}} = 2\pi N \frac{(2n-5)}{n} \frac{1}{16\pi^2} \mathrm{tr}(\tilde{F}_{\mu\nu}F^{\mu\nu}) . \qquad (11)$$

To have an integer coefficient for $F\tilde{F}$ we need

$$2 - \frac{5}{n} = \text{integer} ,$$

which has essentially only one solution: $n = 5$ $(D = 9)$. We can also have particular solutions with more twists, that can cancel n or else, with luck, we can have N_1/n = integer, where N_1, coming from quantization of the Chern-Simons coefficient has, *a priori,* nothing to do with n. But excluding these unnatural solutions, $D = 9$ is singled out as the unique possibility for a *CP*-conserving four-dimensional theory coming from the reduction of an odd-dimension theory with Chern-Simons term and twisted gauge fields. Topological and quantum-mechanical requirements provide a quantization mechanism for the θ term and, in the particular case of $D = 9$, they even conspire to give the *CP*-conserving four-dimensional theory. Finally, I point out that, *a priori,* we could expect additional contributions to the four-dimensional θ term coming from Chern-Simons terms other than ω_{2n-1}. In particular, for $D = 9$, we would have Chern-Simons terms $\omega_5 \mathrm{tr}^2$ and $\omega_3 \mathrm{tr}^4$. But if we compactify following the same procedures used with ω_9, these extra terms would only contribute with additional integers to be added to $2 - 5/n$ and all the above conclusions concerning *CP* conservation would remain unchanged.

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