Gravitational coupling at finite temperature

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We discuss a thermodynamic identity which helps explain why $\langle H\rangle \neq \langle T_{00}\rangle$ at finite temperature. In addition we complete the discussion of the gravitational force by including the gravitational variation of the temperature. Gradients in the temperature induce extra forces not accounted for by the usual coupling to the energy-momentum tensor.

It is well known that in general relativity the gravitational field couples to the energy-momentum tensor of matter.[1] That this is the case in a field-theoretic description is evident from one definition of the energy-momentum tensor

$$T^{\mu\nu}\equiv\frac{\partial}{\partial g_{\mu\nu}}(\sqrt{g}\,\mathcal{L}_m)\,, \tag{1}$$

where $g_{\mu\nu}$ is the metric, $g=\det g_{\mu\nu}$, and $\mathcal{L}_m$ is the Lagrangian density of matter. Then, if one works in the weak-field limit about flat space

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\,, \tag{2}$$

with $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$ and $h_{\mu\nu}\ll 1$, the Lagrangian has the expansion

$$L=\int d^4x\,[\mathcal{L}_m^{(0)}+h_{\mu\nu}T^{(0)\mu\nu}+O(h_{\mu\nu}{}^2)]\,. \tag{3}$$

In Eq. (3) the superscript (0) implies that the associated quantity is to be evaluated using $g_{\mu\nu}=\eta_{\mu\nu}$, i.e., in flat space. It is the graviton's coupling to $T^{\mu\nu}$ which permits the equivalence principle to obtain, even after radiative corrections,[2] since in this case the conservation laws for $T^{\mu\nu}$ associated with Lorentz invariance ensure that the tensor has an identical form after renormalization, when written in terms of renormalized quantities.

Recently, the tools of finite-temperature quantum field theory have been used to address the energy and gravitational coupling of a charged particle in QED at $T\neq 0$. The background heat bath of photons introduces a preferred frame (the rest frame of the heat bath), so that the Lorentz and general coordinate invariance of the $T=0$ theory is no longer manifest. At low temperatures ($T\ll m_e$) the electron's energy, as defined by the pole in the propagator, is given by[3,4]

$$E=\left[\mathbf{p}^2+m_0{}^2+\frac{2\alpha\pi T^2}{3}\right]^{1/2}, \tag{4}$$

where m_0 is the renormalized $T=0$ mass. Further work shows that this definition is also equivalent to the inertial mass,

$$m_I{}^2=m_0{}^2+\frac{2\alpha\pi T^2}{3}\,, \tag{5}$$

since the Hamiltonian has a nonrelativistic reduction,

$$H\psi=\left[m_0+\frac{\alpha\pi T^2}{3m_0}+\frac{p^2}{2(m_0+\alpha\pi T^2/3m_0)}+\cdots\right]\psi\,. \tag{6}$$

Another definition of energy is given by the energy-momentum tensor

$$T_{\mu\nu}=\tfrac{1}{2}\bar\psi(\gamma_\mu p_\nu+\gamma_\nu p_\mu)\psi-g_{\mu\nu}\bar\psi(\not p-m)\psi\,. \tag{7}$$

After a detailed calculation[5] it was shown that at the one-loop level the matrix element of $T_{\mu\nu}$ at $T\neq 0$ is given in the rest frame of the heat bath by

$$\langle p\,|\,T_{\mu\nu}\,|\,p\rangle=\frac{p_\mu p_\nu-(2\alpha\pi T^2/3)\delta_{\mu 0}\delta_{\nu 0}}{E}\,, \tag{8}$$

where E is given in Eq. (4). Thus, the energy which follows from T_{00} is not equal to E but rather is given by

$$\langle T_{00}\rangle=E-\frac{2\alpha\pi T^2}{3E}\,. \tag{9}$$

This result was used in Ref. 5 to argue that the gravitational mass [the $p\to 0$ limit of Eq. (9)] and the inertial mass [given by Eq. (5)] differ at finite temperature.

This initially surprising feature has an interesting thermodynamic explanation. The Hamiltonian is identified with the thermodynamic energy appropriate for the given conditions. In the field-theoretic case this is the free energy F, and we would have then

$$F=\left[p^2+m_0{}^2+\frac{2\pi\alpha T^2}{3}\right]^{1/2} \tag{10}$$

for the electron. By contrast, T_{00} measures the internal energy, which is related to F by

$$U=F-T\frac{\partial F}{\partial T}=E-\frac{2\alpha\pi T^2}{3E}\,. \tag{11}$$

Thus the inertial and gravitational masses can be understood as the low-momentum limit of the free energy and internal energy, respectively. The inequivalence of the two quantities arises from the temperature dependence introduced at one loop.

Another feature which is relevant here is the dependence of the local temperature on the value of the gravitational field.[6] Thus if a temperature T_0 is defined by the blackbody photon distribution in a region of space where the gravitational field is absent (say at infinity), then an observer in thermal equilibrium at a location with gravitational field $\phi(\mathbf{r})=-\tfrac{1}{2}h_{00}(\mathbf{r})$ will see a blackbody spectrum at temperature

$$T=\frac{T_0}{1+\phi(\mathbf{r})}\,. \tag{12}$$

That is to say, the temperature has been blue-shifted while

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still preserving the thermal shape of the spectrum. Since quantum corrections to the propagator of a particle in a heat bath will, in general, induce a temperature-dependent component of the mass, gradients in the temperature caused by the presence of a gravitational field will produce corresponding gradients in the effective mass of the particle, which leads to additional forces on the body.

In order to evaluate the gravitational interaction we must include both the explicit coupling $h_{\mu\nu}T^{\mu\nu}$ and also the dependence of the temperature on the gravitational field. The effective Hamiltonian in this case (obtained via a Foldy-Wouthuysen transformation) is[5]

$$i\frac{\partial\psi}{\partial t}=\left[m_0+\frac{\alpha\pi T_0^2}{3m_0}\frac{1}{[1+\phi_g(\mathbf{r})]^2}+\frac{p^2}{2\left(m_0+\frac{\alpha\pi T_0^2}{3m_0}\frac{1}{[1+\phi_g(\mathbf{r})]^2}\right)}+\phi_g(\mathbf{r})\left(m_0-\frac{\alpha\pi T_0^2}{3m_0}\frac{1}{[1+\phi_g(\mathbf{r})]^2}\right)+\cdots\right]\psi$$

$$=\left[m_0+\frac{\alpha\pi T_0^2}{3m_0}+\frac{p^2}{2(m_0+\alpha\pi T_0^2/3m_0)}+\phi_g(\mathbf{r})\left(m_0-\frac{\alpha\pi T_0^2}{m_0}\right)+\cdots\right]\psi=H\psi\,. \quad (13)$$

(Here terms of order ϕ_g^2 have been dropped.) It is the dependence of the temperature on the gravitational field which was missed in our earlier work.[5]

One can now use this effective Hamiltonian to study the validity of the weak equivalence principle. Thus we calculate the gravitational acceleration

$$\mathbf{a}=-[H,[H,\mathbf{r}]]=-\frac{m_0-\alpha\pi T_0^2/m_0}{m_0+\alpha\pi T_0^2/3m_0}\nabla\phi_g(\mathbf{r}), \quad (14)$$

and observe that the ratio of gravitational to inertial mass is not unity but rather

$$\frac{m_g}{m_I}-1=-\frac{4}{3}\frac{\alpha\pi T_0^2}{m_0}, \quad (15)$$

an effect which was noted before.[5] However, the size of the deviation is twice that previously found, due to the proper inclusion of the coordinate dependence of the temperature in the present work.

We have discussed the gravitational coupling of a particle interacting with a heat bath at nonzero temperature. In addition to the radiatively corrected energy-momentum tensor, it is necessary to include the coordinate dependence of the temperature, since the latter enters separately into the effective Hamiltonian.

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