# Electromagnetic mass models in general relativity: Lane-Emden-type models

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(Received 28 October 1985; revised manuscript received 18 February 1986)

Imposing an equation of state $\rho+p=0$, $\rho>0$ ("false vacuum") on a static spherically symmetric charged perfect-fluid distribution, various electromagnetic mass models are derived. It is shown that the modified field equations suggested by Einstein to study the equilibrium structure of an "electron" form a part of the field equations considered by us in this paper. Assuming an implicit relation among the unknown physical parameters, viz., the pressure $p$, charge density $\sigma$, and the electromagnetic potential $\Phi$, it is shown that $\Phi$ satisfies the well-known Lane-Emden equation. Electromagnetic mass models corresponding to the exact solutions of the Lane-Emden equation are obtained. The radii of some of the models are compared with the "classical electron radius."

## I. INTRODUCTION

In a recent investigation,[1] we studied interior charged matter fields described by a static spherically symmetric metric

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2(d\theta^2+\sin^2\theta d\Phi^2)\ , \qquad (1.1)$$

by imposing a condition $g_{00}g_{11}=-1$, on the space-time (1.1). In the case of a charged perfect-fluid distribution described by (1.1), this relation is equivalent to an equation of state[1]

$$\rho=-p,\ \rho>0\ . \qquad (1.2)$$

This equation of state was previously discussed by Gliner[2] in his study of the algebraic properties of the energy-momentum tensor of ordinary matter. According to him, Eq. (1.2) represents a state of matter called the $\rho$ vacuum. The "matter" satisfying this equation of state is also known in the literature as a "degenerate vacuum" or "false vacuum."[3]

We have observed[1] that the imposition of condition (1.2) on charged perfect-fluid distributions leads to a situation wherein all the physical quantities, viz., mass-energy density $\rho$, pressure $p$, and mass, etc., become dependent on charge density $\sigma$ alone and vanish when the charge vanishes. Hence these models have been called electromagnetic mass models.

Various models can be obtained by imposing different conditions—either physical or mathematical—on the set of differential equations for spherically symmetric electromagnetic mass models given in our previous paper.[1] As an example we gave one such solution, the physical behavior of which has been studied by Gautreau[4] and Grøn.[5] Gautreau[4] has also obtained an electromagnetic mass model for a Lorentz extended electron by assuming the structure of the source to be of Weyl type.[6]

In the present paper (Sec. III) we obtain a different class of models which we call "Lane-Emden type" as these models are derived from the Lane-Emden differential equation, well known in classical astrophysics.[7,8] Models of this type, in our opinion, deserve attention. We also show (in Sec. II) that the Einstein-Maxwell field equations for a charged perfect fluid by imposing the condition (1.2) reduce to the system of the modified field equation suggested by Einstein[9] to study the equilibrium structure of an electron.

## II. EINSTEIN'S MODIFIED SYSTEM OF FIELD EQUATIONS

To study the equilibrium structure of an "electron," Einstein, in 1919,[9] suggested a modification of the geometrical terms of the gravitational field equations of general relativity with only the energy-momentum tensor of the electromagnetic field $T_{ij}^{(\mathrm{em})}$ being present in place of the energy-momentum tensor of matter $T_{ij}^{(\mathrm{m})}$. In this section we derive these modified systems of field equations from the Einstein-Maxwell field equations for a charged perfect fluid given by the set of Eqs. (2.2) to (2.4) and (2.5) of Ref. 1. Using condition (1.2), Eq. (2.2) of Ref. 1 takes the form

$$R_{ij}-\tfrac{1}{2}Rg_{ij}=8\pi p g_{ij}-8\pi T_{ij}^{(\mathrm{em})}\ , \qquad (2.1)$$

which on contraction gives $\frac{1}{4}R=-8\pi p$. This, when used in (2.1), yields

$$R_{ij}-\tfrac{1}{4}Rg_{ij}=-8\pi T_{ij}^{(\mathrm{em})}\ . \qquad (2.2)$$

Equation (2.2) and Eqs. (2.3) and (2.4) of Ref. 1 are the modified system of equations suggested by Einstein[9] to study the equilibrium structure of an electron, whereas Eqs. (2.2)-(2.4) of Ref. 1, together with (1.2), are the system of equations considered by us.

In Einstein's theory,[9] as will be pointed out in Sec. IV, the self-stabilizing stresses are of nonelectromagnetic origin, whereas in our case they are of electromagnetic origin. The coincidence of the equations considered by us with those of Einstein's,[9] however, requires further investigation.

## III. VARIOUS LANE-EMDEN-TYPE MODELS

To derive various electromagnetic mass models we shall first write down the necessary field equations in a simplified form as given by Gautreau,[4] viz.,

$$\frac{d}{dr}(re^{-\lambda})+\left[r\frac{d\Phi}{dr}\right]^2=1+8\pi r^2 p \ , \tag{3.1}$$

$$\frac{1}{4\pi r^2}\frac{d}{dr}\left[r^2\frac{d\Phi}{dr}\right]=-\sigma e^{\lambda/2} \ , \tag{3.2}$$

$$\frac{dp}{dr}=\frac{d\Phi}{dr}\frac{1}{4\pi r^2}\frac{d}{dr}\left[r^2\frac{d\Phi}{dr}\right] \ . \tag{3.3}$$

Thus, we have three equations in four unknown quantities: $\lambda$, $p$, $\sigma$, and $\Phi$, $\Phi$ being the electromagnetic potential. It may be noted that Eq. (3.3) can be easily integrated if we assume the term

$$(\tfrac{1}{4}\pi r^2)\frac{d}{dr}\left[r^2\frac{d\Phi}{dr}\right]$$

to be a product of two functions: say, $f(p)$ and $g(\Phi)$. Assuming various functional relations between $p$ and $\Phi$ and using Eq. (3.1) a variety of solutions, some of which may be physically meaningful, can be obtained.

Let us assume an implicit relation among the unknowns as follows:

$$\sigma e^{\lambda/2}=-(n+1)\frac{p}{\Phi} \ , \tag{3.4}$$

where $n$ is a non-negative real number. Let $p_o$, $\Phi_0$, and $\sigma_0$ be the values of $p$, $\Phi$, and $\sigma$, respectively, at $r=0$ [i.e., $\sigma_0=-(n+1)p_0/\Phi_0$]. Using (3.4) in (3.2) and then integrating, we get

$$p/p_0=(\Phi/\Phi_0)^{n+1} \ . \tag{3.5}$$

Equation (3.2), in view of (3.5) and (3.4), can now be written as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\Phi}{dr}\right]+A\Phi^n=0 \ ,$$

$$\text{with } A=-4\pi(n+1)p_0\Phi_0^{-(n+1)} \ . \tag{3.6}$$

Equation (3.6) is in the form of the well-known Lane-Emden equation of classical astrophysics.[7,8] The boundary conditions for this equation are

$$\Phi=\Phi_0,\ \frac{d\Phi}{dr}=0 \text{ at } r=0 \ . \tag{3.7}$$

Once the potential $\Phi$ is known as a solution of Eq. (3.6), the other variables $p$, $\lambda$, and $\sigma$ can be explicitly determined using Eqs. (3.5), (3.1), and (3.4), respectively.

Exact solutions of the Lane-Emden equation (3.6) satisfying the boundary conditions (3.7) are known for the cases $n=0$, 1, and 5.[10] The models corresponding to the cases $n=0$ and $n=1$ have been already obtained by Tiwari, Rao, and Kanakamedala[1] and Gautreau,[4] respectively. The solution of (3.6) for $n=5$ is given by

$$\Phi=\Phi_0(1+\tfrac{1}{3}Kr^2)^{-1/2},\ K=\frac{-24\pi}{\Phi_0^2}p_0 \ .$$

Using this in Eq. (3.5), we get

$$p=p_0(1+\tfrac{1}{3}Kr^2)^{-3} \ . \tag{3.8}$$

In view of $\rho=-p$ (and $\rho_0=-p_0$), $p$ must be negative (as the mass-energy density $\rho$ must be positive). Using the expression for $p$ and $\Phi$ in Eq. (3.1), we get

$$e^{-\lambda}=e^{\nu}=1+\tfrac{1}{2}\Phi^2-\Phi_0^2\left[\frac{\arctan(\frac{1}{3}Kr^2)^{1/2}}{(\frac{1}{3}Kr^2)^{1/2}}\right] \ . \tag{3.9}$$

The charge density $\sigma$ can be obtained from (3.4) and is given by

$$\sigma=\sigma_0 e^{-\lambda/2}(\Phi/\Phi_0)^5 \ .$$

As the pressure must be zero on the boundary of the distribution, (3.8) suggests that the radius of the model is infinitely large. From (3.9), it is evident that the model is asymptotically flat. The total mass and the total charge [Eqs. (3.2) and (3.5) of Ref. 1] for this model are given by

$$m(\infty)=\frac{\pi}{2\rho_0}\left[\frac{\Phi_0}{2}\right]^3 \text{ and } Q(\infty)=\left[\frac{1}{2\pi\rho_0}\right]^{1/2}\frac{\Phi_0}{2} \ ,$$

respectively. These values of mass and charge are finite even though the radius of the model is infinitely large. This model may therefore represent a charged universe with its finite mass arising out of the electromagnetic field alone. Equations (3.4) and (3.5), after a simple algebraic manipulation, give a relation between $p$ and $\sigma$: $p=K(\sigma e^{\lambda/2})^{(n+1)/n}$, where $K$ is an integration constant. This relation, with charge density taking the place of matter density, is analogous to the polytropic equation of state of a perfect gas[7,8] which was extensively used in the study of stellar structure.

## IV. ELECTRON MODELS: A RETROSPECT

One of the classical problems which generated considerable interest at the turn of the century was the structure of an extended "electron" whose mass is believed to be completely of electromagnetic origin.[11] Attempts by Abraham and Lorentz to build a charged spherical model for an extended electron in the realm of classical electrodynamics failed due to divergent self-energies of the models and also due to lack of a self-contained mechanism to avoid the instability caused by Coulomb repulsive forces among the evenly charged parts of a charged sphere. More problems arose after the advent of the special theory of relativity as it became necessary to make the electron theory Lorentz invariant.

Poincaré's ingenious suggestion of nonelectromagnetic cohesive forces (a kind a negative pressure) to keep the repelling parts of electron together not only provided a stable model but also made the electron theory Lorentz invariant.[11-13] However, later it was found independently by Fermi, Wilson, Kwal, and Rohrlich[14] that the relation between the momentum of the Coulomb field of a moving electron and the momentum of the electron itself as considered by Abraham was wrong. By providing necessary corrections, the theory of the electron was made Lorentz invariant without the help of Poincaré stresses, and the role of these stresses was relegated to provide only stability to the electron. But the introduction of nonelectromagnetic stresses to provide stability in a way frustrated the attempts to get an electromagnetic mass model for an electron.

Mie[15] formulated a nonlinear theory of the electron by modifying Maxwell-Lorentz field equations by assuming the self-stresses to be dependent on the electromagnetic potential. This theory could have provided models of an electron dependent on the electromagnetic field alone but the results obtained were physically unsatisfactory.

Einstein, in 1919,[9] discussing the drawbacks of Mie's theory suggested a theory which we have discussed in Sec. II in detail. In his formalism, the gravitational forces provide the necessary stability to the electron and also contribute to its mass. Katz and Horwitz[16] have also studied the problem of the structure of the electron assuming the origin of self-stresses as due to the quantum self-effects of the electromagnetic origin and to interactions within the quantized self-gravitational field. But in the theories of both Einstein[9] and Katz and Horwitz,[16] there is a contribution to the mass of an electron from the self-stabilizing stresses which is of nonelectromagnetic origin.

Thus the original idea of the structure of the electron envisioned by Abraham and Lorentz as a charged spherical particle with its mass arising out of electromagnetic field alone has eluded the above-mentioned theories. However, from Secs. I–III of the present paper it can be seen that it is possible to generate various electromagnetic mass models, some of which may represent extended charged particles such as the electron. This aspect is discussed further in Sec. V.

## V. CERTAIN OBSERVATIONS

(i) The "classical electron radius" $r_0$, in relativistic units, is defined as $r_0=(e^2/m_0)$, where $m_0$ is the rest mass and $e$ is the total charge of an electron. If we identify the "gravitational mass" $m$ and total charge $Q$ of the electromagnetic mass models with $m_0$ and $e$ of an electron, respectively, the radii of some of the models can be compared with the classical electron radius.

For the model corresponding to the case $n=0$, it can be seen that $a=\frac{4}{5}r_0$. This has been also observed by Gautreau,[4] who has further shown that $a=r_0$ for the model corresponding to the case $n=1$. Interestingly, the radius $a=\frac{4}{5}r_0$ is the same that arises in classical electrodynamics from the field momentum considerations of a uniformly moving charged particle when the charge is uniformly distributed throughout the volume of the sphere and on the surface of the spheres, respectively.[11]

It has been pointed out by Gautreau[4] that when $a<r_0$ the effective mass becomes negative in the region outside the source. Grøn[5] suggests that this negative mass is due to the strain of vacuum because of vacuum polarization.

(ii) The homology theorem, as usual, holds for the Lane-Emden equation (3.6) (Ref. 17) here also.

## VI. CONCLUDING REMARKS

It may be remarked that the beauty of all the models obtained here lies in the fact that they are dependent on the electromagnetic field alone. Further, all these models exemplify the phenomenal analogy between gravitational and electromagnetic fields.


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