Radiative isotropic cosmologies with extra dimensions

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We solve Einstein's equations in an n-dimensional vacuum with the simplest *Ansatz* leading to a Friedmann-Robertson-Walker (FRW) four-dimensional space-time. We show that the FRW model must be of radiation. For the open models the extra dimensions contract as a result of cosmological evolution. For flat and closed models they contract only when there is one extra dimension.

Kaluza-Klein theories have been studied recently as an attractive way to unify all gauge interactions with gravity. In these theories gauge symmetries are seen as coordinate transformations on an extra-dimensional space. One of the current problems facing these theories is to explain why the extra-dimensional space is so small, since otherwise it would be observed. One usually assumes that the extra-dimensional space is a compact space with the topology of $S_{(n-4)}$. Such compactification may arise naturally if the space-time has $n=11$ dimensions and there is a supergravity field of seven indices.[1,2] But another mechanism is to explain the compactification as a result of cosmological evolution in a universe satisfying n-dimensional Einstein equations.[3] Although the ultimate reason for the compactification and the size of the extra space, of the scale of the Planck length, may have a quantum origin,[4-7] it is of interest to derive classical solutions undergoing compactification.

The first simple cosmological model in which the extra dimensions contract as a result of cosmological evolution was proposed by Chodos and Detweiler.[3] Their model was a generalization of the anisotropic Kasner solution in an n-dimensional vacuum. In this solution one can have three of the space dimensions expanding (the observable universe), while the extra dimensions contract. This solution considerably improves the usual four-dimensional Kasner solution, which has contraction in one dimension and does not describe the real Universe, without need of introducing matter fields. The expansion of the Universe can then be seen as a consequence of matter sources. Towards the initial cosmological singularity the Universe however, looks very different from today; it has essentially the dimensions of the extra space.

There is now extensive literature dealing with different aspects of higher-dimensional cosmologies. Some authors explain entropy production[8,9] and inflation[10,11] as a result of the contraction of the extra space. A great number of exact cosmological solutions of Einstein equations with different equations of state and different symmetries, including or not a cosmological constant, has been found with five[12-14] dimensions and, also, with an arbitrary number of dimensions.[15-18] Most solutions present a time-varying extra-dimensional radius, although some solutions have been found with a constant extra-dimensional radius and expanding three-dimensional space,[19,20] which is consistent with the lack of variability of gauge coupling with time.

Exact cosmological solutions have also been found for the bosonic part of supergravity in $n=11$ dimensions[21-27] using mainly the Freund and Rubin ansatz.[1] Some solutions have also a constant extra-dimensional radius.

Here we consider a set of classical homogeneous cosmological models which give expanding three-dimensional isotropic spaces, that is, Friedmann-Robertson-Walker (FRW) models, which are solutions of an n-dimensional vacuum. We will see that all vacuum solutions reproduce a FRW model with a radiative perfect fluid. For other equations of state other *Ansätze* are needed. The extra dimensions contract as a result of cosmological evolution in certain cases only, depending on the number of extra dimensions and the type of FRW model.

We look for solutions to Einstein's equations of the form

$$ds^2=(ds^2)_{\rm FRW}+\sum_{i=1}^{d} a_i(t)(dx^i)^2 \ , \qquad (1)$$

where $(ds^2)_{\rm FRW}$ is the line element of the FRW metrics in four dimensions, and d is the number of extra dimensions ($d=n-4$). That is, we assume a homogeneous space-time with a diagonal extra metric whose coefficients depend on t only. This is the simplest *ansatz* which reduces to the FRW models in four dimensions.

The vacuum Einstein's equations in n dimensions reduce to the following set of equations:

$$3\left[\frac{\dot R}{R}\right]^2+\frac{3Kc^2}{R^2}=\frac{\ddot D}{D}+\frac{1}{8}\left[\sum_{i=1}^{d}\frac{\dot a_i}{a_i}\right]^2-\frac{1}{8}\sum_{i=1}^{d}\left[\frac{\dot a_i}{a_i}\right]^2 \ , \qquad (2a)$$

$$2\frac{\ddot R}{R}+\left[\frac{\dot R}{R}\right]^2+\frac{Kc^2}{R^2}=\frac{\dot R}{R}\frac{\dot D}{D}-\frac{1}{8}\left[\sum_{i=1}^{d}\frac{\dot a_i}{a_i}\right]^2+\frac{1}{8}\sum_{i=1}^{d}\left[\frac{\dot a_i}{a_i}\right]^2, \qquad (2b)$$

$$R\ddot D=-3\dot R\dot D \ , \qquad (2c)$$

$$\ddot a_i+\frac{3\dot R}{R}\dot a_i+\frac{\dot D}{D}\dot a_i-\frac{\dot a_i^{\,2}}{a_i}=0 \ , \qquad (2d)$$

where $R(t)$ is the radius of the FRW universe, c is the speed of light, K is a constant characterizing the different models ($K=0$ flat, $K=-1$ open, $K=1$ closed), and $D(t)$ is a function defined as

$$D^2(t)=\prod_{i=1}^{d} a_i(t) .$$

From (2c) and (2d) it is easy to see that

$$a_i=D^{2p_i},\ \sum_{i=1}^{d} p_i=1 , \tag{3}$$

where p_i are constants. Equations (2a) and (2b) may be recognized as Einstein's equations in four dimensions where the right-hand side of these equations are the components of a source tensor T_{ab} ($a,b=0,1,2,3$) with a trace

$$T=\left[\frac{\dot{D}}{D}\right]^2\left[1-\sum_{i=1}^{d} p_i{}^2\right] . \tag{4}$$

We may identify such a tensor with the energy-momentum tensor of a perfect fluid T'_{ab} with an equation of state $p=u\rho$ ($u\in[0,1]$), whose trace is

$$T'=(3u-1)\rho . \tag{5}$$

Equating (4) and (5) and taking into account (2a) and (2b) one can see that the equations are only compatible when $u=\frac{1}{3}$, which corresponds to a radiative perfect fluid. Thus, from (4),

$$\sum_{i=1}^{d} p_i{}^2=1 . \tag{6}$$

Therefore, the *Ansatz* (1) and the n-dimensional vacuum field equations are compatible with a four-dimensional FRW model for a radiative perfect fluid only. This was also noted in five dimensions by Davidson, Sonnenschein, and Vozmediano.[13]

Now we can determine the function $D(t)$. The final solution is

$$ds^2=(ds^2)_{\rm FRW}+\sum_{i=1}^{d}\left[\frac{c(A-2Kt)}{R(t)}\right]^{2p_i}(dx^i)^2 ,\quad \sum_{i=1}^{d} p_i=\sum_{i=1}^{d} p_i{}^2=1 , \tag{7}$$

where A is a constant which appears in the radius of the radiative universe:

$$R(t)=c(At-Kt^2)^{1/2},\ A>0 . \tag{8}$$

The constant A can be related to the Hubble constant H_0 and age of the Universe t_0 (Weinberg[28]):

$$A=2Kt_0(1-H_0t_0)(1-2H_0t_0)^{-1} .$$

The relations (7) for the parameters p_i imply that for $d>3$ at least one of the p_i is negative, unless one takes the trivial solution $p_1=1$, $p_2=\cdots=p_d=0$ (this is the only solution for $d=1$ and $d=2$) which implies a $(d-1)$-dimensional Euclidean space.

We can now analyze the behavior of the extra dimensions for the three different FRW models.

(i) Flat ($K=0$). By redefining the extra coordinates we obtain

$$a_i=t^{-p_i} , \tag{9}$$

so that the extra space behaves as the Kasner solution. This means that, except for $d=1$ or the trivial Euclidean solution, at least one of the extra dimensions expands with time for $d\geq 3$, since one of the p_i is negative. Therefore, in this model there is no compactification as a result of time evolution when $d\geq 3$. The five-dimensional case ($d=1$), which admits extra dimensional contraction, corresponds to the Chodos and Detweiler[3] solution for an isotropic three-dimensional space.

(ii) Open ($K=-1$). In this case from (7) and (8) we have

$$a_i=[(A+2t)^2/(At+t^2)]^{p_i} . \tag{10}$$

Thus, when $t\rightarrow\infty$, all a_i go to some constants, and when $t\rightarrow 0$, $a_i\rightarrow(A/t)^{p_i}$. When $p_i>0$, $a_i\rightarrow\infty$, and when $p_i<0$, $a_i\rightarrow 0$. Therefore, for open models the extra dimensions evolve with time towards a constant value, so that eventually they will get smaller than the four-dimensional FRW model. Near the cosmological singularity some of the extra dimensions become larger than the four-dimensional space-time, and this should have cosmological implications.

(iii) Closed ($K=1$). Now we have from (7) and (8)

$$a_i=[(A-2t)^2/(At-t^2)]^{p_i} . \tag{11}$$

According to the sign of p_i we have the following.

For $p_i>0$ near the cosmological singularity ($t\rightarrow 0$) the corresponding extra dimension goes to infinity. Afterwards, a_i decreases with time, and when the radius of the Universe gets its maximum ($t=A/2$), $a_i=0$. For $t>A/2$, a_i increases with time approaching infinity when $t\rightarrow A$.

For $p_i<0$, the corresponding a_i starts from zero at the cosmological singularity and increases towards infinity when $R(t)$ reaches its maximum. After this time, $t=A/2$, it decreases to zero when $t\rightarrow A$.

Thus, as a result of dynamical evolution, only the open FRW models contract all the extra dimensions independently of their number. The sizes of the extra dimensions become nonzero constants and one can adjust the parameters to fit cosmological observations. Near the cosmological singularity some of the extra dimensions become dominant.

For the flat models there is no compactification unless one takes $d=1$ or the trivial case $p_1=1$ and $p_2=\cdots=p_d=0$.

For the closed model there is no compactification either, unless $d=1$ or the trivial case as before. However, one may envisage a situation in which all extra dimensions are much smaller than the radius $R(t)$, some of them growing and the others decreasing. The dimensions that contract go to zero radius. Before they reach zero radius, quantum effects should be considered.

Some points suggested by the above analysis are the following. We see that a radiative four-dimensional FRW model can be obtained as a consequence of an n-dimensional vacuum. FRW models with other equations of state do not allow the simplest *Ansatz* (1) and they may be obtained introducing an n-dimensional energy-momentum tensor or perhaps vacuum with a more complicated *Ansatz*. Another source of matter may come from quantum effects due to the contracted extra dimensions as follows. Our calculation has ignored any dependence on the extra coordinates; it corresponds to the zero mode of the Fourier expansion in terms of the extra coordinates. The higher-order modes can be interpreted in the four-dimensional picture as an infinite tower of massive particles whose masses are quantized in terms of the inverse size of the extra space. When this size is small those massive particles may produce an important effect.

Another quantum effect may come from the Casimir potential associated with the size of the extra space,[4,5] which will make the extra dimensions contract to a size on the order of the Planck length.

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