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Stability of solutions to the classical SU(3) Yang-Mills theory with external sources

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Stability of solutions to the classical SU(3) Yang-Mills theory with prescribed static sources is studied. It was found before that there existed a number of bifurcating solutions to the SU(3) gauge group equations. The stability of some of these was considered before by showing explicitly their instability. For the rest it is only said that they share stability properties of solutions at the bifurcation point. These solutions are shown here to be absolutely stable under small radial oscillations. The possible consequences of the result are discussed.

There are two outstanding (and interwoven) problems in the study of the classical Yang-Mills (YM) equations in the presence of static external sources.[1,2] One of them addresses the issue of screening of classical color[3] and stems from the belief that color confinement is inherent to QCD, the renormalizable quantum gauge theory of the unbroken local symmetry group SU(3) of color.[4] The other interesting development is aimed at the problem of stability of classical solutions.[1,5]

In this paper we focus on the latter issue and first sketch the most important results of earlier works:[1,6] A spherically symmetric ansatz with the underlying SU(3) gauge symmetry[7]

$$A^0_{ab} = i\,\varepsilon_{abk}\frac{x^k}{r}\frac{f_1(r)}{gr} + \left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right)\frac{f_2(r)}{gr} \tag{1}$$

and

$$A^i_{ab} = i\left(\delta_{bi}\frac{x_a}{r} - \delta_{ai}\frac{x_b}{r}\right)\frac{G(r)-1}{gr} \tag{2}$$

gives rise to a system of nonlinear coupled ordinary differential equations through

$$\mathcal{D}_\mu F^{\mu\nu} = J^\nu \;, \tag{3}$$

where $J^\nu = (\rho,0,0,0)$. Here we are using a compact matrix notation for all objects. Also,

$$\rho_{ab} = i\,\varepsilon_{abc}\frac{x^c}{r}\frac{q_1(r)}{g^2} + \left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right)\frac{q_2(r)}{g^2} \;. \tag{4}$$

The effective Hamiltonian can be written as

$$H_{\text{eff}} = \frac{4\pi}{g^2}\int_0^\infty dr\left[-\tfrac{1}{2}(f_1')^2 - \tfrac{1}{6}(f_2')^2 + (G')^2 + \frac{1}{2r^2}(G^2-1)^2 - \frac{1}{r^2}({f_1}^2+{f_2}^2)G^2 + 2r(f_1 q_1 + \tfrac{1}{3}f_2 q_2)\right] \;. \tag{5}$$

When the well-known constraint (Gauss's law) is satisfied the energy takes the form

$$\mathcal{E} = \frac{4\pi}{g^2}\int_0^\infty dr\left[\tfrac{1}{2}(f_1')^2 + \tfrac{1}{6}(f_1')^2 + (G')^2 + \frac{1}{2r^2}(G^2-1)^2 + \frac{1}{r^2}({f_1}^2+{f_2}^2)G^2\right] \;. \tag{6}$$

By taking a δ-shell source form

$$q_1(r) = Q\,\delta(r-r_1) \;(\text{“monopole”}) \tag{7a}$$

and

$$q_2(r) = R\,\delta(r-r_2) \;(\text{“quadrupole”}) \tag{7b}$$

we find a variety of solutions[6] which could lead to SU(2) solutions[1] by taking the limit $R \to 0$. Above some critical values of Q and R some of the solutions to (3) bifurcate; the energy for such solutions is presented in Fig. 1.

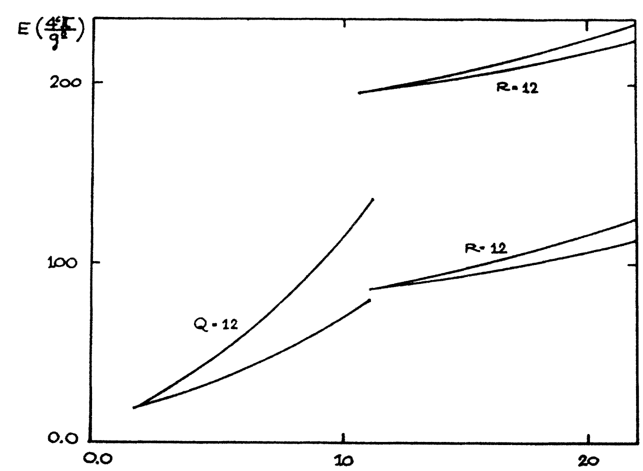

FIG. 1. Energy of bifurcating solutions to the SU(3) Yang-Mills equations.

To understand the stability properties of static solutions for the YM equations we use results of a general analysis done by Jackiw and Rossi.[1] They pointed out that for a general nonlinear theory

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + U'(\phi) = -\rho \,, \tag{8}$$

energy bifurcates

$$\mathcal{E} = \mathcal{E}_c + \varepsilon\int d^3r\,\phi_c\,\delta\rho + \tfrac{2}{3}c\,\varepsilon^{2/3}\int d^3r\,\psi\,\delta\rho \;, \tag{9}$$

where ψ is a normalized zero eigenmode

$$[-\nabla^2+U''(\phi_c)]\psi=0 \tag{10}$$

and ρ_c supports a static solution

$$[-\nabla^2+U'(\phi_c)]\phi_c=0 \ . \tag{11}$$

The expression (9) for the bifurcating energy is obtained when ρ in (8) is replaced by $\rho_c+\varepsilon\delta\rho$, and c is calculated from

$$c=\pm\left|\frac{2\int d^3r\,\psi\delta\rho}{\int d^3r U'''(\phi)\psi^3}\right| \ . \tag{12}$$

For one group of bifurcating solutions[6] for which the energy is plotted in Fig. 1 we found the critical value of Q to be

$$Q_c(R=12)=11.24 \ , \tag{13}$$

and the corresponding energy is

$$\mathcal{E}_c(Q_c,12)=87.64 \ ; \tag{14}$$

and from (9) we conclude that ε is a measure of how far from the bifurcating point the solutions are calculated.

Since c in (12) can have either sign and the zero-eigenvalue solution is proportional to c, $\omega_0{}^2\propto c$, the conclusion is that the energy difference rises as $(Q-Q_c)^{3/2}$ and that the mean energy rises as $Q-Q_c$. Also, it is concluded that the upper-branch solutions are unstable.[1]

In order to determine stability properties of the lower-branch solutions we calculate the oscillation equations

$$\partial^i\delta E^i_{ab}+i[A^i,\delta E^i]_{ab}-ig[\delta A^i,E^i]_{ab}=0 \ , \tag{15a}$$

$$\partial_t\delta E^i_{ab}=(\boldsymbol{\partial}\times\delta\mathbf{B}_{ab})^i-ig\varepsilon^{ijk}[\delta A^k,B^j]_{ab}-ig\varepsilon^{ijk}[A^k,\delta B^j]_{ab} -ig[\delta A^0,E^i]_{ab}-ig[A^0,\delta E^i]_{ab} \ , \tag{15b}$$

$$-\partial_t\delta A^i_{ab}=\delta E^i_{ab}-\partial^i\delta A^0_{ab} +ig[\delta A^i,A^0]_{ab}+ig[A^i,\delta A^0]_{ab} \ , \tag{15c}$$

which follow from $A\to A+\delta A$ and $E\to E+\delta E$. The radial equations to be solved are

$$\begin{aligned}
&-\delta f_1''+\frac{2}{r^2}(2Gf_1\delta G+G^2\delta f_1)=0 \ ,\\
&-\delta f_2''+\frac{6}{r^2}(2Gf_2\delta G+G^2\delta f_2)=0 \ ,\\
&-\delta G''+\frac{1}{r^2}(3G^2-1-f_1{}^2-f_2{}^2)\delta G -\frac{2}{r^2}(f_1\delta f_1+f_2\delta f_2)G=0
\end{aligned} \tag{16}$$

together with

$$\delta f_{1,2}\to 0, \quad \delta G\to 0 \ , \tag{17}$$

as $r\to 0$ and $r\to\infty$.

There is an additional problem in solving the above equations since "coefficients" $f_{1,2}$ and G satisfy static equations and they are not known in their closed analytic form. Fortunately, a numerical code COLSYS[8,9] we were using was able to handle this difficulty.

After careful investigation we found no normalizable solutions to the above equations. If nontrivial normalizable (finite-energy) solutions had been found this would have implied that nonzero-frequency modes must be investigated. Reliability of our calculation is higher at this point since the asymptotic behavior of $f_{1,2}$ and G is much easier to control than the corresponding solutions to the original (unperturbed) system.

As a check of the stability formalism we calculate the bifurcating energy (9) for the SU(3) solutions.[6]

We found that the ratio of the energy difference between the upper and the lower branch was stable and equal to 0.3, as could be seen from Table I. This is in complete agreement with Ref. 1. The same is also true for the genuine SU(3) solutions generated by R in (7b), but for which the ratio is equal to 0.5 (Table II).

In this case

$$R_c(Q=12)=1.81 \tag{18}$$

and

$$\mathcal{E}_c(12,R_c)=18.74 \ .$$

The absence of normalizable solutions to (15) signals stability of the lower-branch solutions. Our conclusion is based on the assumption that radially symmetric oscillations are relevant. Before considering other oscillatory modes one should bear in mind that the situation here is rather complicated by the fact that perturbative methods do not apply here.[10] At the same time it is not quite clear how much physical reality one is able to extract from such investigations even envisaging importance of stability properties of classical solutions within the context of their quantum role.

TABLE I. Stability analysis for the bifurcating solutions: "monopole" (7a).

Q	$(Q-Q_c)^{3/2}$	$\Delta\mathcal{E}$	Ratio
12.0	0.66	0.18	0.3
14.0	4.59	1.33	0.3
16.0	10.39	3.09	0.3
18.0	17.58	5.32	0.3
20.0	25.93	7.97	0.3
22.0	35.30	10.97	0.3
24.0	45.58	14.29	0.3

TABLE II. Stability analysis for the bifurcating solutions: "quadrupole" (7b).

R	$(R-R_c)^{3/2}$	$\Delta\mathcal{E}$	Ratio
2.0	0.08	0.17	0.5
3.2	1.64	3.34	0.5
4.0	3.24	6.58	0.5
5.2	6.24	12.54	0.5
6.0	8.58	17.14	0.5
8.0	15.40	30.39	0.5
10.0	23.44	45.78	0.5
12.0	32.53	63.00	0.5

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