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Removing bag dynamics from chiral bag models: An illustrative example

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We examine a simple Cheshire Cat model in two dimensions, demonstrating that its properties are independent of boundaries by solving for its complete spectrum. This calculation is used to elucidate the more difficult four-dimensional calculations of chiral bag spectra. We also discuss the relationship of the Cheshire Cat boundary conditions to bosonization and the chiral anomaly.

I. INTRODUCTION

There are several quantum field theories in two dimensions which can be described using fermion or boson degrees of freedom. Nadkarni, Nielsen, and Zahed[1] have shown that one can employ different descriptions in different regions of spacetime, if one can also discover the proper boundary conditions to relate the quantum fields in different regions. The boundaries one places between different descriptions are completely arbitrary, and should not affect any observable. This disappearance of the partitions from the observables has led to the name "Cheshire Cat" model (CCM).

Why are CCM's of interest? In the theories one knows how to bosonize, partial bosonization merely complicates calculations, as we shall see. These theories teach one how to perform calculations within a CCM, however. The ultimate utility of partial bosonization may be found in theories which cannot be exactly solved, but which are most easily treated at different distance scales using different degrees of freedom. In trying to describe baryons with quantum chromodynamics (QCD) one encounters the necessity of dealing with a quantum field theory nonperturbatively. No methods exist yet which are adequate to this task, which has led to the development of many models. The bag model[2] allows one to model confinement without spoiling asymptotic freedom; however, it confines everything, making it difficult to include the effects of light mesonic degrees of freedom. The topological chiral-soliton model,[3] motivated by an expansion of QCD in inverse powers of the number of colors, requires an infinite number of mesons and complete knowledge of their interactions to describe the short-distance behavior of baryons. It is natural to seek some description which employs quark and gluon degrees of freedom only at short distance, where they can be treated perturbatively, and which at long distances employs mesonic degrees of freedom, where only the lightest mesons are relevant. The chiral bag model[4] provides a simple version of such a theory, a marriage of the bag and topological chiral soliton models.

It is not clear how various momentum scales can be translated into a separation of the "inner" and "outer" regions of hadrons but if such a separation is possible, the descriptions inside and outside must at least in principle be equivalent. In practice the bag would then form a division of convenience, and one would choose a scale which allows the best approximate treatment of both regions (see the last reference in Ref. 4). This leads us to seek a Cheshire Cat description of hadrons.

Perhaps the most relevant two-dimensional model to examine is QCD_2, but we will be far less ambitious. We will study what may be the simplest model of which one can conceive, free particles confined to a cavity. The cavity is used to avoid the infrared problems associated with massless bosons in two dimensions.[5] We will first demonstrate the equivalence between free massless bosons and free massless fermions by developing a complete map between the two Fock spaces. We then divide the cavity into two regions and study the energy levels of the resultant CCM. We show that these energy levels are independent of the position of the partition, provided a suitable subtraction of the Casimir energy is made.

II. FREE MASSLESS FERMIONS AND FREE MASSLESS BOSONS

Consider a system of massless fermions confined to a box. This confinement is most easily accomplished by letting the mass of the fermions become infinite outside of the box. The appropriate action is

$$S=\int dx \int dt\, \bar{\psi}(x,t)[i\partial\!\!\!/-M(x)]\psi(x,t)\ ,$$

with

$$M(x)=m\theta(|x|-R),\quad m\to\infty\ . \tag{2.1}$$

In the limit of infinite m the equations of motion are

$$\begin{aligned} &i\partial\!\!\!/\psi(x,t)=0,\quad |x|\le R,\\ &(1\pm i\gamma^1)\psi(\pm R)=0\ . \end{aligned} \tag{2.2}$$

With the choices $\gamma_0=\sigma_1$, $\gamma_1=-i\sigma_2$, $\gamma_5=\gamma_0\gamma_1=\sigma_3$, the eigenmodes are

$$\psi_n(u,v)=\left[\frac{1}{4R}\right]^{1/2} e^{i\omega_n R\gamma_5}\begin{bmatrix} e^{-i\omega_n u} \\ -ie^{i\omega_n v}\end{bmatrix},$$

where

$$\omega_n=\frac{(2n+1)\pi}{4R},\quad u=t-x,\ v=t+x\ . \tag{2.3}$$

We have used the light-cone variables, u and v, for later convenience. Note that the free Dirac equation is solved by any spinor whose upper component depends only on u and whose lower component depends only on v. The fermion field operator can then be expanded as

$$\psi(u,v)=\sum_{n=0}^{\infty}[\chi_n(u,v)a_n+\xi_n(u,v)b_n^\dagger]\ ,$$

with

$$\{a_m^\dagger,a_n\}=\{b_m^\dagger,b_n\}=\delta_{mn},\ \ \chi_n=\psi_n,\xi_n=\psi_{-n-1}\ . \tag{2.4}$$

The vacuum $|0\rangle$ satisfies

$$a_n|0\rangle=b_n|0\rangle=0\ . \tag{2.5}$$

Finite observables are easily defined in terms of normal-ordered products of these field operators. The current operators are

$$j_\mu(x)\equiv N_F[\bar\psi(x)\gamma_\mu\psi(x)]\ . \tag{2.6}$$

Here N_F indicates normal ordering with respect to the above one-body fermion operators, as distinguished from boson normal ordering which will also be used later. The unrenormalized energy-momentum tensor, apart from the step-function mass term, can be written in Sugawara form[6] in terms of these currents

$$T^0_{\mu\nu}=\frac{\pi}{2}(2j_\mu j_\nu-g_{\mu\nu}j_\alpha j^\alpha)\ . \tag{2.7}$$

We are now in a position to bosonize the theory by employing the well-known relationships[7]

$$\begin{aligned} j_\mu&=\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi\ ,\\ M\bar\psi\psi&=-ZM\cos(\sqrt{4\pi}\phi)\ . \end{aligned} \tag{2.8}$$

Here $\epsilon_{\mu\nu}$ is the completely antisymmetric tensor with $\epsilon_{01}=-\epsilon_{10}=1$, and ϕ is the boson field operator. Z is a constant which depends on the manner in which normal ordering is implemented. For our purpose Z can be ignored, because M is either zero or infinite. Using (2.7) and (2.8) we obtain, for the unrenormalized energy,

$$E=\int dx\left[\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2+\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2 -ZM(x)\cos(\sqrt{4\pi}\phi)\right]\ . \tag{2.9}$$

From this we can obtain the equations of motion for ϕ:

$$\begin{aligned} &\Box\phi(x,t)=0\ ,\\ &\phi(\pm R,t)=n_\pm\sqrt{\pi}\ \ (n_\pm=0,\pm1,\dots)\ . \end{aligned} \tag{2.10}$$

We can arbitrarily fix $\phi(-R)=0$, and obtain

$$\kappa_n(x,t)=\left[\frac{1}{2\omega_n R}\right]^{1/2}\sin[\omega_n(x+R)]e^{-i\omega_n t}\ ,$$

with

$$\omega_n=\frac{n\pi}{2R}$$

and

$$\sigma_m(x)=\frac{m\sqrt{\pi}}{2}\left[1+\frac{x}{R}\right]\ . \tag{2.11}$$

The first set of modes are the standard boson excitations, while the second are simple topological solitons.

The boson field operator can be expanded as

$$\phi(x,t)=\sum_{n=1}^{\infty}[\kappa_n(x,t)\alpha_n+\kappa_n^*(x,t)\alpha_n^\dagger]+\tilde N\frac{\sqrt{\pi}}{2}\left[1+\frac{x}{R}\right]\ ,$$

with

$$[\alpha_n,\alpha_m^\dagger]=\delta_{nm}\ . \tag{2.12}$$

Here $\tilde N$ is a number operator, which will later be seen to be the fermion number operator. The boson Fock space is built on a set of ground states defined by the relationships

$$\begin{aligned} &\tilde N|n\rangle=n|n\rangle\ \ (n=0,\pm1,\dots)\ ,\\ &\alpha_m|n\rangle=0\ . \end{aligned} \tag{2.13}$$

Once again finite observables are expressed in terms of normal-ordered products of these operators.

Returning to the current bosonization relation (2.8), we note that $\phi(x,t)$ can also be expanded in terms of fermion one-body operators:

$$\phi(x,t)=\sqrt{\pi}\int_{-R}^{x}dy\,N_F[\psi^\dagger(y,t)\psi(y,t)]\ . \tag{2.14}$$

Using this equation it is straightforward to confirm that

$$\begin{aligned} \tilde N&=\sum_{m=0}^{\infty}(a_m^\dagger a_m-b_m^\dagger b_m)\ ,\\ \alpha_n&=\left[\frac{1}{n}\right]^{1/2}\left[\sum_{m=0}^{n-1}b_m a_{n-m-1}+\sum_{m=0}^{\infty}(a_m^\dagger a_{m+n}-b_m^\dagger b_{m+n})\right]\ . \end{aligned} \tag{2.15}$$

We see that $\tilde N$ is indeed the fermion number operator. One can verify that this last expression obeys the proper commutation relations.

Given these identities one can easily map the boson Fock space onto the fermion Fock space. In particular, one will find that $|n\rangle$ corresponds to the ground state with fermion number n. Creating a meson corresponds to exciting fermions and/or antifermions. This means that excited "baryons" can also be thought of as baryon plus meson states. A single boson always consists of states with a single excited fermion, but an m boson excitation consists of states with anywhere from one to m excited fermions and/or antifermions.

Finally, it is possible to find an expression for the fermion field operator in terms of the boson field operator.[8] To do so, divide the boson field operator into two parts, one depending on u only and one on v only. This division is accomplished using

$$\begin{aligned}
\phi_1(u) &= \sum_{n=1}^{\infty} [\xi_n(u)\alpha_n + \xi_n^*(u)\alpha_n^\dagger] , \\
\xi_n(u) &= \frac{1}{2i}\left[\frac{1}{2\omega_n R}\right]^{1/2} e^{-i\omega_n(u-R)} , \\
\phi_2(v) &= \sum_{n=1}^{\infty} [\zeta_n(v)\alpha_n + \zeta_n^*(v)\alpha_n^\dagger] , \\
\zeta_n(v) &= \frac{1}{2i}\left[\frac{1}{2\omega_n R}\right]^{1/2} e^{-i\omega_n(v+R)} , \\
\hat{\phi}_1(u) &= \tilde{N}\frac{\sqrt{\pi}}{4}\left[1-\frac{u}{R}\right] + \frac{1}{\sqrt{\pi}}\tilde{p} , \\
\hat{\phi}_2(v) &= -\tilde{N}\frac{\sqrt{\pi}}{4}\left[1+\frac{v}{R}\right] + \frac{1}{\sqrt{\pi}}\tilde{p} ,
\end{aligned} \tag{2.16}$$

where $\omega_n = n\pi/2R$ and $[\tilde{N},\tilde{p}]=i$.

The variable conjugate to $\tilde{N}$, $\tilde{p}$, has been introduced because the fermion field operator must not commute with the fermion-number operator. The constant in front of $\tilde{p}$ is fixed by insisting that $[\tilde{N},\psi]=-\psi$. With the above definitions, we see that

$$\phi(u,v) = \phi_1(u) - \phi_2(v) + \hat{\phi}_1(u) - \hat{\phi}_2(v) . \tag{2.17}$$

We will not outline the calculation, but the Dirac equation and boundary conditions are then satisfied by

$$\psi(u,v) = \left[\frac{1}{4R}\right]^{1/2} \begin{bmatrix} N_B\{\exp[-i\sqrt{4\pi}\phi_1(u)]\}\exp[-i\sqrt{4\pi}\hat{\phi}_1(u)] \\ -iN_B\{\exp[-i\sqrt{4\pi}\phi_2(v)]\}\exp[-i\sqrt{4\pi}\hat{\phi}_2(v)] \end{bmatrix} . \tag{2.18}$$

Here N_B indicates normal ordering with respect to the boson operators. This expression also satisfies the correct anticommutation relations.

This completes our discussion of the equivalence between the two theories. Using the above operator identities one can readily demonstrate the equivalence of any observable in the two descriptions.

III. THE CHESHIRE CAT MODEL

To develop a CCM one need only introduce a partition into the box, at an arbitrary point y, and use different descriptions on each side of the partition. It would be attractive if one could derive the proper boundary conditions simply by changing variables in part of the box, using (2.18), for example. However, we shall see that this does not work. Instead, we will assume that the chiral bag boundary conditions are correct, and demonstrate that the correct spectrum results.

The chiral bag boundary conditions, for a boundary with normal n_μ are

$$\begin{aligned}
\{\exp[i\sqrt{4\pi}\phi(y,t)\gamma_5]+i\not{n}\}\psi(y,t) &= 0 , \\
n\cdot\partial\phi(y,t) &= \sqrt{\pi}\bar{\psi}(y,t)\not{n}\gamma_5\psi(y,t) .
\end{aligned} \tag{3.1}$$

The fermion current on the right-hand side of the second expression must be regulated in some fashion. It appears as if the second boundary condition will follow directly from bosonization, as it is one component of the current bosonization relations (2.18); and it is indeed obeyed by the bosonized fermion field operator (2.8). However, the bosonized field operator does not obey the first boundary condition. This makes perfect sense because it can know nothing about what direction one has chosen the normal to the partition to point. The boundary conditions depend on which side of the partition one has decided to place the fermions; but, the fermion field operator in (2.18) is completely independent of this choice. In fact, one can verify by substitution that it obeys a boundary condition with $n_\mu=(-i,0)$ (Ref. 9). Thus, the fact that the second boundary condition is obeyed by the fermion field operator, which acts on the whole of space, need not indicate it will be obeyed by the operator, which acts on only one side of the partition.

It is possible to find a bosonized fermion operator which obeys both of the above boundary conditions by changing the constant in front of $\tilde{p}$ in expression (2.18); however, this spoils the fermion anticommutation relations. This will in turn yield an incorrect energy momentum tensor, because the Sugawara form of this tensor (2.7) is derived using the anticommutation relations. We have not been able to discover a Mandelstam-type bosonization operator which satisfies all of the equations of motion and reproduces the correct anticommutation relations.

We will show the equivalence of this CCM to the free fermion/boson model by calculating the complete spectrum. First, let us ask what the states of the theory are? If we have a CCM, specifying the boson part of the state should completely determine the state. That is to say, given $\phi(x,t)$, the fermion part of the state is completely determined. In fact, as the partition is moved from one side of the box to the other, we should be able to watch a boson state evolve into a fermion state, with the correspondence found in the last section. This means that for a fixed y, the individual fermion excitations have no meaning. One cannot excite a fermion mode without changing the boson state and thereby changing the fermion modes.

To make these points explicit, let us first calculate the energy of a ground state with fermion number N. The first problem we face is determining how to handle the boundary conditions (3.1), which are operator boundary conditions. At present we will ignore this difficulty and follow a procedure often used in calculations in four dimensions. It goes as follows.

Treat $\phi(x,t)$ as if it were a c number field, while $\psi(x,t)$

is treated as a quantum field. Erect a partition at y, with fermions to the left and bosons to the right. The equations of motion are given by (3.1) and

$$\begin{aligned} &i\partial\!\!\!/\psi(x,t)=0 \quad (-R\le x\le y)\ ,\\ &(1+in\!\!\!/)\psi(-R)=0\ ,\\ &-\Box\phi(x,t)=0 \quad (y\le x\le R)\ ,\\ &\phi(R)=N\sqrt{\pi}\ . \end{aligned} \tag{3.2}$$

Analytically continue into the left region, assuming with foreknowledge that $\phi(-R)=0$. As we did in the last section, divide $\phi(x,t)$ into two pieces, such that

$$\phi(x,t)=\phi_1(u)-\phi_2(v)\ . \tag{3.3}$$

Then, if ψ_0 satisfies the Dirac equation and boundary conditions when $\phi=0$, the solution to the above equations is

$$\begin{aligned} \psi(x,t)&=\begin{bmatrix}\exp[-i\sqrt{4\pi}\phi_1(u)] & 0\\ 0 & \exp[-i\sqrt{4\pi}\phi_2(v)]\end{bmatrix}\psi_0(x,t)\\ &=\exp\{-i\sqrt{\pi}[\phi(x,t)\gamma_5+\phi_1(u)+\phi_2(v)]\}\psi_0(x,t)\ . \end{aligned} \tag{3.4}$$

Since ψ_0 satisfies the equations when $\phi=0$, it will have an expansion similar to (2.4), but now

$$\psi_{0n}(u,v)=\left[\frac{1}{2(y+R)}\right]^{1/2}\exp\left[i\omega_n\left[\frac{y+R}{2}\right]\gamma_5\right]\times\begin{bmatrix}e^{-i\omega_n u}\\ -ie^{-i\omega_n v}\end{bmatrix},$$

with

$$\omega_n=\frac{(2n+1)\pi}{2(y+R)}\ . \tag{3.5}$$

Next let us study the second boundary condition in (3.1). If we naively substitute the above result into the right-hand side of Eq. (3.1) we get

$$\bar\psi n\!\!\!/\gamma_5\psi=-\psi^\dagger\psi=-\psi_0^\dagger\psi_0\ . \tag{3.6}$$

This result is independent of $\phi(y)$ and cannot equal $\partial^1\phi(y)$. The problem is that the above expression is unregulated, and one must be careful in deciding how it should be regulated. To decide this we return to our original definition of finite currents (2.6), where fermion normal ordering was used. We do not want to normal order with respect to the single-particle operators in ψ_0. Instead, we note that the original fermion normal ordering was equivalent to point splitting in space. (For $\psi^\dagger\gamma_5\psi$ one should use point splitting in time.) Using such point splitting we get

$$\begin{aligned} N_F[\psi^\dagger(y)\psi(y)]&=\tfrac{1}{2}\lim_{\epsilon\to 0^+}[\psi^\dagger(y+\epsilon,t),\psi(y-\epsilon,t)]\\ &=\tfrac{1}{2}\lim_{\epsilon\to 0^+}\left[\psi_0^\dagger(y+\epsilon,t),\begin{bmatrix}\exp[4i\sqrt{\pi}\epsilon\partial^1\phi_1(y)] & 0\\ 0 & \exp[4i\sqrt{\pi}\epsilon\partial^1\phi_2(y)]\end{bmatrix}\psi_0(y-\epsilon,t)\right]\\ &=N_F^0[\psi_0^\dagger(y)\psi_0(y)]+\frac{i}{4\pi}\lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\left(e^{4i\sqrt{\pi}\epsilon\partial^1\phi_1(y)}-e^{4i\sqrt{\pi}\epsilon\partial^1\phi_2(y)}\right)\\ &=N_F^0[\psi_0^\dagger(y)\psi_0(y)]-\frac{1}{\sqrt{\pi}}\partial^1\phi(y)\ . \end{aligned} \tag{3.7}$$

Here N_F^0 is used to refer to normal ordering with respect to the operators in the expansion of ψ_0. We can assume that $N_F^0\psi_0^\dagger\psi_0$ is zero on all physical states (i.e., that there are only induced currents), so that (3.7) is just the second boundary condition. The above calculation works for any $\phi(y)$, but it depends on the assumption that $\phi(-R)=0$, so we have not really lost a boundary condition.

We have satisfied the equations of motion, and next we should calculate the energy spectrum. When $\phi(y)$ is time independent we have a set of eigenstates of the Dirac Hamiltonian. The relevant time-independent boson states are

$$\begin{aligned} &\phi(x)=N\frac{\sqrt{\pi}}{2}\left[1+\frac{x}{R}\right],\\ &\phi_1(x,t)=N\frac{\sqrt{\pi}}{4}\left[1-\frac{t-x}{R}\right],\\ &\phi_2(x,t)=-N\frac{\sqrt{\pi}}{4}\left[1+\frac{t+x}{R}\right]. \end{aligned} \tag{3.8}$$

When these are plugged into (3.4) and the shifted energy levels calculated, we get

$$\Omega_n=\frac{(2n+1)\pi}{2(y+R)}+\frac{\pi N}{2R}\ . \tag{3.9}$$

These energy levels depend on y, and in fact diverge when y goes to $-R$. As we will see below, however, with a suitable subtraction this dependence has no effect on observables.

We can calculate the total energy of the fermion sector by remembering that all of the states which begin at negative energy are filled. Exactly N of these states will move to positive energy as the partition is moved to $y=R$. The total energy is

$$\begin{aligned} E=\sum_{n=1}^{\infty}\Omega_{-n}&=\lim_{s\to 0^+}\frac{\partial}{\partial s}\sum_{n=1}^{\infty}e^{s\Omega-n}\\ &=\lim_{s\to 0^+}\left[-\frac{1}{s^2}\left[\frac{y+R}{\pi}\right]+\frac{\pi N^2}{8R}\left[1+\frac{y}{R}\right]-\frac{\pi}{24(y+R)}\right]. \end{aligned} \tag{3.10}$$

The first and last terms are independent of N, representing the Casimir energy of the fermion vacuum. These terms depend on y despite the fact that the energy should be independent of the position of the partition in a CCM. They represent a false vacuum energy which must be subtracted. This energy is not simply a fraction of the vacuum energy provided by the filled Dirac sea in the whole box, although the divergent part is. *This means that the CCM can be maintained only by employing a subtraction scheme which is dependent on the partitions, and this result will carry through to four dimensions.*

Earlier we saw that normal ordering with respect to the original fermion operators was sufficient to render the currents finite. This procedure could be carried over to the Cheshire Cat model because of its equivalence to a point-splitting procedure which normal orders the energy density, so we know of no method of directly relating the renormalization schemes in the two pictures.

The lesson is that one must be careful when regularizing a CCM to ensure that the regularization scheme does not introduce spurious dependence of the renormalized observables on the positions of the partitions.

The remaining part of the energy (3.10), $(\pi N^2/8R)(1+y/R)$, is exactly what is needed to complement the energy in the boson sector, yielding the correct total energy for the complete set of ground states. To calculate the energy of states containing excited bosons, one needs to modify the above procedure, because eigenstates of the Hamiltonian do not exist. There are several paths one can follow. First, one can derive a Sugawara form for the energy-momentum tensor.[6] We have already seen that

$$j_0 = N_F(\bar{\psi}\gamma_0\psi) = -\frac{1}{\sqrt{\pi}}\partial^1\phi$$

from (3.7), and using time point splitting, one can also show that

$$j_1 = N_F(\bar{\psi}\gamma_1\psi) = \frac{1}{\sqrt{\pi}}\partial^0\phi \ . \qquad (3.11)$$

Using these results, and the Sugawara form of the energy density (2.7), one can easily show that in general

$$E = \int_{-R}^{y} dx[\tfrac{1}{2}(\partial^0\phi)^2 + \tfrac{1}{2}(\partial^1\phi)^2] \ . \qquad (3.12)$$

Once again this correctly complements the energy in the boson sector.

Alternatively, one can use the fact that the Euclidean effective action is related to the ground-state energy of a system,[10] which is all we need. Thus we get

$$Z(\phi) = \exp\left[-\frac{1}{\hbar}\int dt\, E(\phi)\right] = \int D\psi D\psi^\dagger \exp\left[-\frac{1}{\hbar}\int_{-R}^{y} dx \int dt\, \psi^\dagger\partial\psi\right] \ . \qquad (3.13)$$

In this path integral one integrates only over fields satisfying the proper boundary conditions at $x=-R$ and at $x=y$. Making the change of variables found in expression (3.4), we can eliminate the dependence of the path integral on $\phi(y)$. This change of variables produces an anomaly, so that the Jacobian of the transformation must be taken into account properly.[11] The result is

$$Z(\phi) = \exp\left[-\frac{1}{\hbar}\int dt \int_{-R}^{y} dx[\tfrac{1}{2}(\partial_\mu\phi)^2]\right] Z(0) \ . \qquad (3.14)$$

If one calculates $Z(0)$ one will find that it depends on y just as expression (3.10). Dividing by $Z(0)$ to get a renormalized energy is equivalent to the y-dependent vacuum energy subtraction discussed above. The ϕ-dependent piece of $Z(\phi)$ again correctly complements the energy found in the boson sector.

IV. CONCLUSION

We have used a simple model to illustrate bosonization, demonstrating the one-to-one correspondence between states and observables of systems of free massless fermions or bosons in a box. In particular, we have shown how one can easily fermionize the boson field operator (2.15) or bosonize the fermion field operator (2.18). Although the details of such a transformation have long been known, by putting the system in a box the relationship between the states in the two descriptions becomes particularly transparent.

The development of bosonization in a box was a prelude to the main purpose of this work—to illustrate the principles underlying the Cheshire Cat picture in a simple situation, and ultimately to explore what is required to set up a Cheshire Cat model in four dimensions. In the Cheshire Cat picture one describes a single system in two regions using two equivalent field theories. If this is done properly, observables show no dependence on the partitions, as was shown to be the case for the entire spectrum of states in the free theories employed here.

What was required to eliminate any dependence of the energy on the position of the partition? We needed the correct boundary conditions (i.e., chiral bag boundary conditions),[1] the correct renormalization procedure, and two equivalent field theories. We have tried to emphasize that the theories are equivalent quantum theories, so that one must be careful to treat the equations of motion and boundary conditions as operator equations. This means, for example, that one can never treat valence fermions separate from the Dirac sea. In addition we have illustrated the importance of finding the proper renormalization prescriptions. To satisfy one boundary condition, conservation of the axial-vector current at the partition, we had to discover a means of implementing normal ordering without reference to operators (i.e., point splitting). To renormalize the energy we had to subtract a Casimir energy.

There are many issues with which we have not dealt. The only divergences we encountered in our simple calculation were eliminated by point splitting and vacuum energy subtraction. When interactions are added one will see new divergences, which need to match on each side of the bag surface. This issue may be most easily studied using the massive Thirring model, which bosonizes into the sine-Gordon model.[7] Another important issue is that of total momentum. There should be a means of setting up a CCM which does not explicitly break translation invariance. It would also be of interest to study CCM's at finite temperature.[12]

Let us now turn to the more relevant problem of setting up a Cheshire Cat model in four dimensions. Although we are not yet in a position to rigorously generalize our results to four dimensions, we believe that most, if not all, of what one learns in two dimensions will carry over to four dimensions. The same ingredients will be necessary to set up a Cheshire Cat model. One must find the proper boundary conditions, renormalization prescriptions, and two equivalent field theories.

Nadkarni, Nielsen, and Zahed[1] have shown how one can determine the proper boundary conditions to be used at the bag surface. Causality, available symmetries, and renormalizability can be used to eliminate most surface terms. All possible surface terms remaining must be considered. We saw that in two dimensions the proper boundary conditions for the bosonization of fermions with a U(1) symmetry could be determined by considering the chiral anomaly. In a path integral formalism one can seek the correct boundary conditions by first asking what change of variables in the fermion sector has a Jacobian which properly reproduces the free part of a boson action. In general such a change of variables will also induce Wess-Zumino terms,[13] allowing one to infer that except in the U(1) case an equivalent boson theory will include such terms (see the last reference in Ref. 1). We expect that the chiral bag boundary conditions already in use are correct; however, the chiral field in these conditions may correspond to an infinite number of pseudoscalar, isovector fields with the pion being only the lightest particle involved.

What have we learned about renormalization that is relevant to four dimensions? We illustrated that one must include quantum effects when considering current conservation at the bag boundary. In addition we discovered that in calculating the energy one must subtract the fermion energy found at zero chiral angle (i.e., the Casimir energy). These conclusions are valid in four dimensions, but they address only the simplest divergence problems one finds there. New divergences will occur which must be subtracted consistently on each side of the bag. Further study of two-dimensional systems with interactions should clarify this point.

The final ingredient for the successful development of a Cheshire Cat model in four dimensions is two equivalent field theories. Of particular interest is a mesonic theory equivalent to QCD (Ref. 14). This will be the most difficult ingredient to obtain, but it should not be necessary to either solve QCD inside the bag or obtain the full mesonic theory outside the bag. Indeed, perhaps the main utility of the Cheshire Cat picture is that it provides a test of how well one is matching two field theories. It is not necessary to obtain complete independence from the bag. What one desires is a range of bag sizes for which observables are approximately independent of the bag.

In four dimensions one is forced to make several approximations, all of which will introduce some spurious dependence of various observables on the size and shape of the bag. By striving to eliminate such dependence, both one's approximate treatment of QCD inside the bag and one's approximate effective mesonic action outside the bag can be improved. Naturally one must be somewhat optimistic about the possibility of finding a range of bag sizes in which the approximations that cannot be avoided do not spoil the approximate CCM behavior of the system; however, there are already some indications that such hope may be justified.[15]

How should one proceed to determine the proper mesonic theory outside the bag? First, as we have seen, bag boundary conditions are most easily solved if one can treat the bosonic fields entering them as c numbers. This means that outside the bag one should employ an effective action;[10] that is to say, an action which already includes what would normally be regarded as quantum corrections. In fact it is not clear that a renormalizable mesonic theory exists which can give rise to an effective action which is equivalent to QCD. Given that one is working with an effective action, any terms allowed by Lorentz invariance and the other symmetries of the problem should be included, but most terms will be unimportant for sufficiently large bag sizes. After one has determined which terms are relevant in the effective meson action, one should next decide on an approximation to QCD. Finally, one should compute observables such as ground-state energies, and check their dependence on the size and shape of the bag. By varying the mesonic effective action and improving the approximation to QCD one should be able to gradually eliminate any dependence on the bag.

Since the correct meson theory is equivalent to QCD it must reproduce meson scattering data. Therefore to lowest order one expects a nonlinear σ model to arise, with higher derivative self-interactions and interactions with heavier mesonic fields that must be determined. Inside the bag one hopes that to lowest order the quarks can be approximated as noninteracting. Thus, one is led to consider the chiral bag model[4] as a lowest-order approximation to a Cheshire Cat description of QCD. A recent calculation indicates that this may be the case.[15]

Two-dimensional calculations, such as the one we have presented, should continue to serve as important tools for solving many of the formal difficulties which arise in such a program. In two dimensions one is able to separate problems arising from the proper treatment of a field theory, and problems arising from the approximations one is forced to adopt to complete a calculation. For the study of such issues as renormalization in a bag theory this is invaluable.

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