Structure of Yang-Mills field theory in the framework of many-body approximations

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The structure of the many-body problem defined by the regularized Coulomb-gauge Hamiltonian of an SU(n) Yang-Mills field theory is investigated. Results for the glueball spectrum within a recently proposed Bogoliubov approximation are presented. A consistent continuum limit is not reached in this frame because nonlogarithmic divergences do not cancel. In order to guarantee this cancellation structure, an extension of the Bogoliubov scheme, a combination of the exp(S) formalism and cluster expansion (constructed in analogy to the hole-line expansion of standard many-body theory) is introduced. Results for the ground state (= vacuum state) within this new framework are presented and discussed.

I. INTRODUCTION

Recently, one of the authors has proposed to study the structure of SU(n) Yang-Mills (YM) theories with nonperturbative many-body techniques (see Ref. 1, referred to as I in the following). Our aim is to obtain insight into the structure of gauge field theories which are an alternative to lattice calculations. Analogous attempts have been undertaken by Cutkosky[2] using a hyperspherical formalism and by Horn *et al.*[3] using suitable Padé approximations. The basic idea of our approach is to define (in the Coulomb gauge) a regularized field-theoretical Hamiltonian for the SU(n) YM theory,[1]

$$H(g(M),M,\Omega)\ , \tag{1}$$

where M=momentum cutoff, Ω=volume cutoff, g= running coupling constant, and to study the spectrum of H with many-body techniques borrowed (but suitably generalized) from nonrelativisitic many-body theory. Quite analogous to the lattice procedure, $g(M)$ is supposed to be fixed by adjusting the length scale to one observable; other observables should then become independent of M, if M is large enough (i.e., if one is near enough to the continuum limit). From the construction of the many-body techniques, all calculations can be done in the "thermodynamical" limit $\Omega\rightarrow\infty$.

It is the purpose of this paper to report on first numerical results and on some further developments of the many-body techniques which proved necessary in order to be consistent with renormalization. The necessity to consider many-body approaches fulfilling such consistency conditions was recognized by Nojiri.[4] Summarized, the main points of the paper are the following.

Within the Bogoliubov approximation proposed in I, glueball spectra resulting from $H(g(M),M,\Omega)$ have been determined as functions of M (Sec. II). It turns out that there is no consistent scaling of the results for $M\rightarrow\infty$. The reason for this failure is traced back to an inconsistency of the Bogoliubov ansatz for the wave functions (defined in I): Since the three-point interaction V_3 does not contribute within this scheme, there is no cancellation of nonlogarithmic divergences in the gluon self-energies. This is revealed within perturbation theory in Sec. III.

Analogous to the Brueckner approximation of standard many-body theory, it is possible to generalize the Bogoliubov scheme in a canonical way to a suitable exp(S) method which takes into account the three-point interaction (Sec. IV). First results within this "YM-Brueckner theory" for the vacuum structure functions are presented (Sec. V). It turns out that the Bogoliubov structure function induces a suppression of small momenta and a mass gap in the spectrum of physical particles—an expected structure if confinement holds.[2,5] The consistency of the YM-Brueckner theory with renormalization is discussed in Sec. VI: If perturbation theory holds for large momenta—which turns out to be true with all numerical results obtained up to now—and a suitable ansatz for gluon states (including a gluon cloud) is taken, all nonlogarithmic divergences cancel within the expression for the glueball masses. Numerical details for glueball spectra

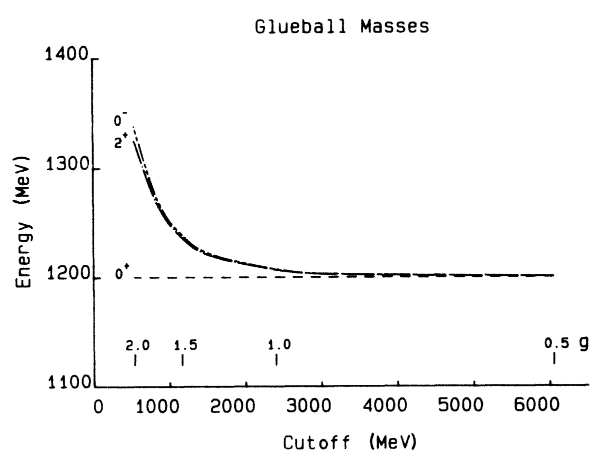

FIG. 1. Glueball mass spectrum within the Bogoliubov scheme as a function of the momentum cutoff M. The running coupling constant $g(M)$ is fixed by setting the energy scale such that the 0^{++} glueball mass is 1200 MeV.

within the YM-Brueckner theory, generalizations to QCD (including quarks), and for temperature $\neq 0$ will be discussed in other publications.[6,7]

II. RESULTS WITHIN THE BOGOLIUBOV SCHEME

The formalism needed in order to apply the Bogoliubov theory in the lowest-order cluster expansion to the Hamiltonian H is described in detail in I. Figure 1 shows the result for the smallest glueball masses $\tilde{\epsilon}_n(g(M),M)$ where n runs through the quantum numbers 0^{++}, 0^{-+}, 2^{++}. The length scale is defined by fixing $\tilde{\epsilon}_{0^{++}}(g(M),M)=1200$ MeV. It is seen that the mass splitting between the glueball states goes to zero for $M\to\infty$. Also, there is no consistent scaling of the function $g(M)$, i.e., we obtain $g(M)\approx 1/M$ rather than $g^2(M)\approx[\ln(M/M_0)]^{-1}$ as expected from perturbation theory and renormalization-group arguments.

The reason for this failure of the Bogoliubov approximation is seen from the divergence structure of the glueball masses when considered as a function of the momentum cutoff M and for $M\to\infty$. According to Eq. (60) of I, the glueball mass $\tilde{\epsilon}$ is split up into single-particle and (irreducible) two-particle terms

$$\tilde{\epsilon}=\tilde{\epsilon}_{\rm sp}+\tilde{\epsilon}_{2p}\ . \tag{2}$$

For $n=0^{++}$ and 2^{++}, the $2p$ term is related to the relative wave function $\gamma(k)$ of the glueball [Eq. (54) of I] by (we take $\Omega\to\infty$ and assume $\int|\gamma|d^3k=1$)

$$\tilde{\epsilon}_{2p}=\frac{\tilde{\alpha}}{4}\int_{|k|<M,|p|<M}d^3p\,d^3k\,\gamma^*(k)\gamma(p)\left[\lambda_k\lambda_p[4-\bar{h}(k,p)]-2\left[\frac{\lambda_k}{\lambda_p}+1\right]\frac{\sigma_{k-p}}{|k-p|^2}\bar{h}(k,p)\right],\quad \tilde{\alpha}=\frac{ng^2}{(2\pi)^3}\ , \tag{3}$$
$$\bar{h}(k,p)=1+(k\cdot p)^2/k^2p^2\ .$$

Here, λ_k is the structure function of the Bogoliubov transformation [Eqs. (8) and (25) of I] which is determined by *minimizing* the vacuum energy expectation value. The result is $\lambda_k{}^{-1}\approx(k^2+m^2)^{1/2}$, $m\approx 600$ MeV in a good approximation. The point we want to make here is that $\tilde{\epsilon}_{2p}$ contains *no* nonlogarithmic divergences, when $M\to\infty$: The only "critical" integration is contained in the integral equations determining the function σ_k [Eqs. (43) and (45) of I],

$$\sigma_k=s_k{}^2[1+Q_k(\sigma)],\quad s_k=[1-Q_k(s)]^{-1}\ , \tag{4}$$

involving the "kernel" Q_k defined by

$$Q_k(f)=\frac{g^2}{4(2\pi)^3}\int_{|q|<M}d^3q\,h(q,k)\frac{f(q-k)}{|k-q|^2}\lambda_q\ ,$$
$$h(q,k)=2-\bar{h}(q,k)\ . \tag{5}$$

If lowest-order perturbation theory holds for large momenta, i.e., if

$$\lambda_k\to 1/k,\quad \sigma_k,s_k\to 1\quad \text{for } k\to\infty\ , \tag{6}$$

which is confirmed by all our numerical computations, it is seen that $Q_k(f)$ is logarithmically divergent for $M\to\infty$ and $f=\sigma$ or s.

However, this divergence structure is not true for the single-particle term $\tilde{\epsilon}_{\rm sp}$ which is given by

$$\tilde{\epsilon}_{\rm sp}=2\int|\gamma(k)|^2\epsilon(k)d^3k\ ,$$
$$\epsilon=\epsilon_0+\epsilon_4+\epsilon_C\ ,$$
$$\epsilon_0(k)=\tfrac{1}{2}(k^2\lambda_k+1/\lambda_k)\ , \tag{7}$$
$$\epsilon_4(k)=\tilde{\alpha}\frac{\lambda_k}{3}\int_{|q|<M}d^3q(\lambda_q-1/q)\ ,$$
$$\epsilon_C(k)=\tilde{\alpha}\tfrac{1}{8}\int_{|q|<M}\bar{h}(k,q)\left[\frac{\lambda_k}{\lambda_q}+\frac{\lambda_q}{\lambda_k}\right]\frac{\sigma_{k-q}}{|k-q|^2}d^3q\ .$$

Here, ϵ_4 is the contribution of the *normal-ordered four-point* interaction V_4. This normal ordering was not taken into account in I and is necessary to be consistent with perturbation theory where tadpole diagrams involving V_4 vanish.[8] ϵ_C is the contribution of the Coulomb interaction [this term was quoted erroneously in Eq. (53) of I].

If $\lambda_k\to 1/k$, $\sigma_k\to 1$ for $k\to\infty$, the first term of ϵ_C is *quadratically* divergent for $M\to\infty$, whereas all other contributions to $\epsilon_{\rm sp}$ are at most logarithmic divergent. This explains our results for the glueball spectrum: For large M, the quadratically divergent term dominates and sets the scale, all the rest, especially the $2p$-interaction term, become negligible relative to this "Coulomb self-energy."

Obviously, this quadratic divergence must be an artifact of our approximation, since perturbation theory yields only logarithmic divergent structures. In the axial gauge, this deficiency of the Bogoliubov approach was noticed by Nojiri.[4]

It must be our next task, therefore, to enlarge the Bogoliubov scheme such that a cancellation of nonlogarithmic divergence occurs. Therefore, we must first understand the structure of this cancellation in perturbation theory.

III. PERTURBATIVE STRUCTURE OF GLUON SELF-ENERGIES

Up to second-order perturbation theory, the contribution to the gluon self-energy (within the Coulomb gauge) is given by the diagrams of Fig. 2. Applying the rules of noncovariant perturbation theory[9,10] for the on-shell gluon mass operator yields, in the notation of Eq. (7),

$$\epsilon^2=\epsilon_0{}^2+\epsilon_4{}^2+\epsilon_C{}^2+\epsilon_3{}^2\ .$$

Here, the first three parts,

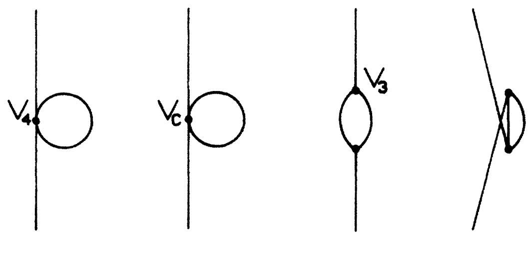

FIG. 2. Diagrams contributing in lowest order to the gluon self-energy: (a) and (b) result from the (normal ordered) four-point interaction V_4 and from the (lowest order) Coulomb interaction V_c, (c) and (d) involve the three-point interaction V_3.

$$\begin{aligned}&\epsilon_0{}^2=k\ ,\\&\epsilon_4{}^2=0\ ,\\&\epsilon_c{}^2=\tilde{\alpha}\tfrac{1}{8}\int \bar{h}(q,k)\left[\frac{|q|}{|k|}+\frac{|k|}{|q|}\right]\frac{1}{|k-q|^2}d^3q\ ,\end{aligned}\tag{8}$$

agree with Eq. (7) for $\lambda_k=1/k$. $\epsilon_3{}^2$ is a new term arising from the three-point interaction V_3. In compact notation, V_3 is given by

$$V_3=\tfrac{1}{6}V_{\alpha\beta\gamma}B_\alpha B_\beta B_\gamma,\ \ B_\alpha=b_\alpha+s_\alpha b_{-\alpha}{}^\dagger\ .\tag{9}$$

Here $\alpha=(k,r,a)$ (=momentum, helicity, color) abbreviates all quantum numbers necessary in order to specify a transverse gluon state (see I), $-\alpha=(-k,r,a)$, $s_\alpha=s_r$ [Eq. (24) of I]. The matrix elements $V_{\alpha\beta\gamma}$ are given by

$$\begin{aligned}&V_{\alpha\beta\gamma}=igU_{\alpha\beta\gamma}(\lambda_\alpha\lambda_\beta\lambda_\gamma)^{1/2}\ ,\\&U_{\alpha\beta\gamma}=[\delta(p_\alpha+p_\beta+p_\gamma)/\sqrt{8\Omega}]\mathrm{sym}f^{a_\alpha a_\beta a_\gamma}[p_\alpha\cdot e(p_\beta,r_\beta)]\\&\qquad\times[e(p_\alpha,r_\alpha)\cdot e(p_\gamma,r_\gamma)]\ ,\end{aligned}\tag{10}$$

$e(p,r)$ $(r=1,2)$ are the transverse polarization vectors, Eq. (21) of I; sym stands for the symmetrization operation with respect to (α,β,γ).

The important "spin sum" occurring in all computations with V_3 is

$$\begin{aligned}\sigma(p_1,p_2)&=\tfrac{1}{2}\sum_{\substack{r_1r_2r_3\\a_1a_2a_3}}|U_{\alpha_1\alpha_2\alpha_3}|^2\\&=\frac{1}{4}\frac{h(p_1,p_2)}{p_3{}^2}[p_1{}^2p_2{}^2h(p_1,p_2)+(p_1{}^4+p_2{}^4+p_3{}^4)],\\&\qquad\qquad -p_3=p_1+p_2\ .\end{aligned}\tag{11}$$

Evaluation of Figs. 2(c) and 2(d) yields then

$$\begin{aligned}\epsilon_3{}^2(k)=\frac{-\tilde{\alpha}}{2}\int\sigma(k,p)\frac{1}{|k|}\frac{1}{|p|}\frac{1}{|k+p|}\\\times\left[\frac{1}{|k|+|p|+|k-p|}\right.\\\left.+\frac{1}{|p|-|k+p|-|k|}\right]d^3p\ .\end{aligned}\tag{12}$$

The important observation now is that all nonlogarithmic divergences, contained in $\Delta\epsilon^2=\epsilon_3{}^2+\epsilon_C{}^2$, exactly cancel and we have $\Delta\epsilon^3(0)=0$. (Thus, also the infrared divergence for $k=0$ cancels, as it should be.)

Applying covariant perturbation theory to the mass operator and dimensional regularization, one gets $\Delta\epsilon(k)\equiv 0$, for all k and in any order of perturbation theory[8]—the mass of a gauge vector meson is not renormalized and remains zero yielding the dispersion relation $\epsilon(k)=k$. Our finding that $\Delta\epsilon^2(k)\neq 0$, therefore, represents an effect of our regularization which breaks the relativistic covariance of the original theory. (The same situation holds for lattice gauge theories.) We have computed $\Delta\epsilon^2(k)$ and found $\Delta\epsilon^2(k)/k\leq 3\%$ (for $g=1$) which shows that the effects of breaking covariance are small. The fact that our numerical results obtained up to now clearly show that lowest-order perturbation theory is valid for large momenta (see Secs. V and VII) makes us confident that these effects of breaking covariance will also stay small within our nonperturbative approach.

Recalling our task to extend the Bogoliubov scheme, we clearly learn from our perturbation exercise that an appropriate generalization will have to to take into account the three-point interaction V_3. We propose, herefore, a suitable "YM-Brueckner" theory which is described in the next section.

IV. THE exp$S$ METHOD AND YM-BRUECKNER THEORY

It is well known from standard many-body theory that there exists a systematic expansion of the ground-state wave function Ψ of an extended many-particle system of the form

$$\Psi=e^S|\Phi_0\rangle\tag{13}$$

yielding linked expressions for observables and, therefore, giving consistent results in the "thermodynamical" limit $\Omega\to\infty$. Hereby, Φ_0 has to be some "quasiparticle vacuum" and S is a function *only* of the *creation* operators.

In nuclear matter theory,[11] Φ_0 is the free ground-state Slater determinant and S has—because of particle-number conservation—an expansion in terms of polynomials $S_2,S_4,S_6,\ldots$, of *even* order in the creation operators

$$\begin{aligned}S=S_2(a_p^\dagger a_h)+S_4(a_p^\dagger a_p^\dagger a_h a_h)\\+S_6(a_p^\dagger a_p^\dagger a_p^\dagger a_h a_h a_h)+\cdots\ .\end{aligned}\tag{14}$$

(Here, p and h refer to particles and holes, respectively, a_h is a creation operator with respect to Φ_0.)

A systematic expansion (hole-line expansion) for the ground-state problem is then defined by truncating the ansatz for S (include only terms up to order $2n$), by approximating the expectation value $E(S)=\langle\Psi|H|\Psi\rangle/\langle\Psi|\Psi\rangle$ via a suitable cluster expansion,[12] and by determining the coefficients yielding the operator S by minimizing $E(S)$.

This yields Hartree-Fock theory for $n=2$, Brueckner theory for $n=4$, and Bethe-Faddeev theory for $n=6$ (Ref. 13). The consistency of this method can be controlled by checking the truncation and the cluster expansion by going to the next order.

For our field-theoretical problem, defined by the Hamiltonian $H(g,M,\Omega)$, the structure (13) of the exact ground state is equally valid (this is a very general result of formal perturbation theory). Due to particle-number nonconservation, we now have to choose $\Phi_0=|0\rangle$ (free vacuum) and

$$S=S_1(a^\dagger)+S_2(a^\dagger a^\dagger)+S_3(a^\dagger a^\dagger a^\dagger)+S_4(a^\dagger a^\dagger a^\dagger a^\dagger)+\cdots \quad (15)$$

In the YM case, $S_1(a^\dagger)=0$ because of color symmetry. Including only S_2 is the Bogoliubov approximation discussed in I [Eq. (23) of I].

A very natural and "canonical" extension of the Bogoliubov scheme is, therefore, to truncate the expansion (15) with S_3. We shall choose the same cluster expansion for the expectation values as in the Bogoliubov case. Our results show that higher orders can be expected to be small (see Sec. V). We shall call this new many-body approach for the YM eigenvalue problem the YM-Brueckner theory.

In order to simplify the expression for the energy expectation value it is convenient to incorporate the operator $S_2(a^\dagger a^\dagger)$ into the definition of the quasiparticle vacuum Φ_0 by replacing $1/|k|$ by an arbitrary function λ_k in the field operators [see Eq. (21) of I]. Thus we write

$$\Psi=e^{S_3}\phi_0,\quad b_\alpha\phi_0=0,\quad b_\alpha=u_\alpha a_\alpha+v_\alpha a^\dagger_{-\alpha}\ ,$$
$$\lambda_k=\lambda_\alpha=(u_k-v_k)^2/|k|\ ,\quad \alpha=(kra)\ , \quad (16)$$
$$u_\alpha=u_k, v_\alpha=s_\alpha v_k\ ,$$
$$S_3=\tfrac{1}{6}S_{\alpha\beta\gamma}b^\dagger_\alpha b^\dagger_\beta b^\dagger_\gamma\ . \quad (17)$$

The ansatz for the coefficients $S_{\alpha\beta\gamma}$ is taken from the structure of the three-point interaction

$$S_{\alpha\beta\gamma}=igU_{\alpha\beta\gamma}\tilde{S}(p_\alpha,p_\beta)\ . \quad (18)$$

If the function $\tilde{S}$ is rotation invariant, S_3 fulfills all symmetry conditions demanded for the vacuum (trivial momentum and angular momentum). The construction of the energy expectation value within the two-cluster approximation

$$E(\lambda,\tilde{S})=\langle\Psi H\Psi\rangle/\langle\Psi\Psi\rangle\,|_{2\text{cl}} \quad (19)$$

is now performed very much in analogy to standard many-body theory.[12–15] Therefore, one first defines "occupation factors" γ_α (wave-function renormalization factors) by

$$\gamma_\alpha=\langle\Psi|b_\alpha b^\dagger_\alpha|\Psi\rangle/\langle\Psi\Psi\rangle\ . \quad (20)$$

The lowest-order cluster expansion of $E(\lambda,\tilde{S})$ is then given by adding to the Bogoliubov energy $E_0(\lambda)$ [Eq. (46) of I] the simplest terms involving V_3, i.e.,

$$E_3(\lambda,\tilde{S})=\langle\phi_0|V_3S_3+S_3V_3|\phi_0\rangle_{\text{linked}} \quad (21)$$

and "inserting" into each "line" occurring both in the evaluation of E_0 and E_3 due to contractions, the occupation factor γ_α. In other words, the prescription for elementary contractions, listed in Eq. (38) of I, are now changed to

$$\begin{aligned}&\underline{b_\alpha b^\dagger_\alpha}=\gamma_\alpha,\quad \underline{b^\dagger_\alpha b_\alpha}=\gamma_\alpha-1\ ,\\ &\underline{A^\mu_i A^\nu_j}=\delta^{\mu\nu}h_{ij}(k)\lambda_k\tilde{\gamma}_k/2\Omega,\quad \tilde{\gamma}_k=2\gamma_k-1\ ,\quad \mu=(k,a)\ ,\\ &\underline{\pi^\mu_i\pi^{-\nu}_j}=\delta^{\mu\nu}h_{ij}(k)\lambda_k{}^{-1}\tilde{\gamma}_k/2\Omega\ ,\\ &\underline{\pi^\mu_j A^{-\nu}_i}=-\underline{A^\nu_i\pi^{-\mu}_j}=-i\delta^{\mu\nu}h_{ij}(k)/2\Omega\ .\end{aligned} \quad (22)$$

Note that for the last contraction the occupation factors disappear because of cancellations. We also have used $\gamma_\alpha=\gamma_k$ [independent of r, a if $\alpha=(kra)$] because of the symmetry of Ψ. This yields for $E(\lambda,\tilde{S})$ the expression

$$E(\lambda,\tilde{S})=(n^2-1)\Omega(e_0+e_1+e_3+e_4+e_c)/(2\pi)^3\ , \quad (23)$$

where

$$e_0=\tfrac{1}{2}\int(k^2\lambda_k+1/\lambda_k)\tilde{\gamma}_k d^3k\ ,$$
$$e_1=\frac{\tilde{\alpha}}{4}\int d^3q\,d^3k\,h(k-q,k)s_ks_q/q^2\ ,$$
$$e_4=\frac{\tilde{\alpha}}{6}\int d^3q\,d^3k\left[\lambda_k\tilde{\gamma}_k-\frac{1}{k}\right]\left[\lambda_q\tilde{\gamma}_q-\frac{1}{q}\right],$$
$$e_c=\frac{\tilde{\alpha}}{8}\int d^3q\,d^3k\left[\frac{\lambda_k}{\lambda_q}\tilde{\gamma}_k\tilde{\gamma}_q-1\right]\bar{h}(q,k)\frac{\sigma_{k-q}}{|k-q|^2}\ ,$$
$$e_3=\sum_{\alpha\beta\gamma}\langle\alpha\beta\gamma|S^\dagger_3V_3+V_3S_3|\alpha\beta\gamma\rangle\gamma_\alpha\gamma_\beta,\gamma_\gamma,\quad \langle\alpha\beta\gamma|=b^\dagger_\alpha b^\dagger_\beta b^\dagger_\gamma|\phi\rangle$$
$$=\frac{\tilde{\alpha}}{3}\int d^3p\,d^3k(\lambda_k\lambda_p\lambda_{p-k})^{1/2}\tilde{S}^*(k,p)\sigma(k,p)\gamma_k\gamma_p\gamma_{k-p}+\text{c.c.}$$

Here, the functions σ_k and s_k are solutions of the integral equation (4), but with the modified kernel

$$Q_k(f)=\frac{\tilde{\alpha}}{2}\int d^3q\,h(q,k)\frac{f(k-q)}{(k-q)^2}\lambda_q\tilde{\gamma}_q\ , \quad (5')$$

e_1 is the contribution of the Faddeev-Popov term, that part of the Hamiltonian which guarantees the gauge invariance of the spectrum (see I).

For consistency, the occupation factor γ_k is determined from (20) within the same cluster expansion yielding

$$\gamma_\alpha = 1 - \sum_{\beta\gamma} \gamma_\alpha \gamma_\beta \gamma_\gamma \langle \alpha\beta\gamma \,|\, S_3^\dagger S_3 \,|\, \alpha\beta\gamma \rangle \,,$$
$$\gamma_k = 1 - \tilde{\alpha} \int d^3p \, \gamma_k \gamma_p \gamma_{k-p} \,|\, \tilde{S}(k,p) \,|^2 \sigma(k,p)$$
$$= 1 - F_k(\gamma, \tilde{S}) \,. \tag{24}$$

This is a nonlinear integral equation relating the function $\tilde{S}(p,q)$ to γ_k.

We mention that exactly the same prescription leads to Brueckner theory in the standard many-body case.[12,15] In that case, it is convenient to do the variation of $E(\lambda,\tilde{S})$ by introducing the relation (24) as a constraint via Lagrange multipliers μ_k, i.e., to determine the functions $\lambda_k, \mu_k, \gamma_k$, $\tilde{S}(k,q)$ by

$$\frac{\delta \tilde{E}}{\delta \lambda} = \frac{\delta \tilde{E}}{\delta \tilde{S}} = \frac{\delta \tilde{E}}{\delta \gamma} = 0, \quad \gamma_k = 1 - F_k(\gamma, \tilde{S}) \,, \tag{25}$$

where

$$\tilde{E}(\lambda, \tilde{S}, \gamma, \mu) = E(\lambda, \tilde{S}) + \int \mu_k (\gamma_k - 1 - F_k(\gamma, \tilde{S})) d^3k$$
$$= \int \mu_k \gamma_k d^3k + E' \,. \tag{26}$$

This allows us to determine $\tilde{S}$ directly,

$$\frac{\delta \tilde{E}}{\delta \tilde{S}} = 0 \rightarrow S_{\alpha\beta\gamma} = \frac{V_{\alpha\beta\gamma}}{\mu_\alpha + \mu_\beta + \mu_\gamma} \,. \tag{27}$$

We do not quote the variation equation for λ and γ, explicitly, we only mention that the Lagrange parameter μ_k is a "single-particle energy" given by

$$\mu_k = \frac{\delta E'}{\delta \gamma_k} = \langle \Psi \,|\, b_\alpha H b_\alpha^\dagger \,|\, \Psi \rangle / \langle \Psi \Psi \rangle \,|_{2\mathrm{Cl}} - E(\lambda, \tilde{S}) \,. \tag{28}$$

It can be shown that the expression (26) for the YM-Brueckner ground-state energy can be constructed by perturbation theory when summing up suitable (infinite) classes of diagrams—quite in analogy to standard Brueckner theory. Here, the perturbation expansion is defined via

$$H = H_0 + H', \quad H_0 = \sum \mu_\alpha b_\alpha^\dagger b_\alpha, \quad \mu_\alpha = \mu_k \,. \tag{29}$$

In contrast to the standard case, however, there is always an essentially nonperturbative part in the theory, namely, the Bogoliubov part, represented by the function λ_k to be determined just variationally.

The relation of the exp(S) method to perturbation theory allows one to construct a generalization of the theory yielding a well-defined "YM-Brueckner" expression for the free energy of the system as a function of the temperature. The basic vehicles, therefore, are contained in an old paper by Bloch and de Dominicis.[16] Details of these perturbative structures and extensions will be given in a separate paper.[6]

V. NUMERICAL RESULTS FOR THE GROUND STATE WITHIN YM-BRUECKNER THEORY

We have determined numerically the solution of the variational equations (25). Results (for $g=1$) are presented in Fig. 3. The most important structures are the following.

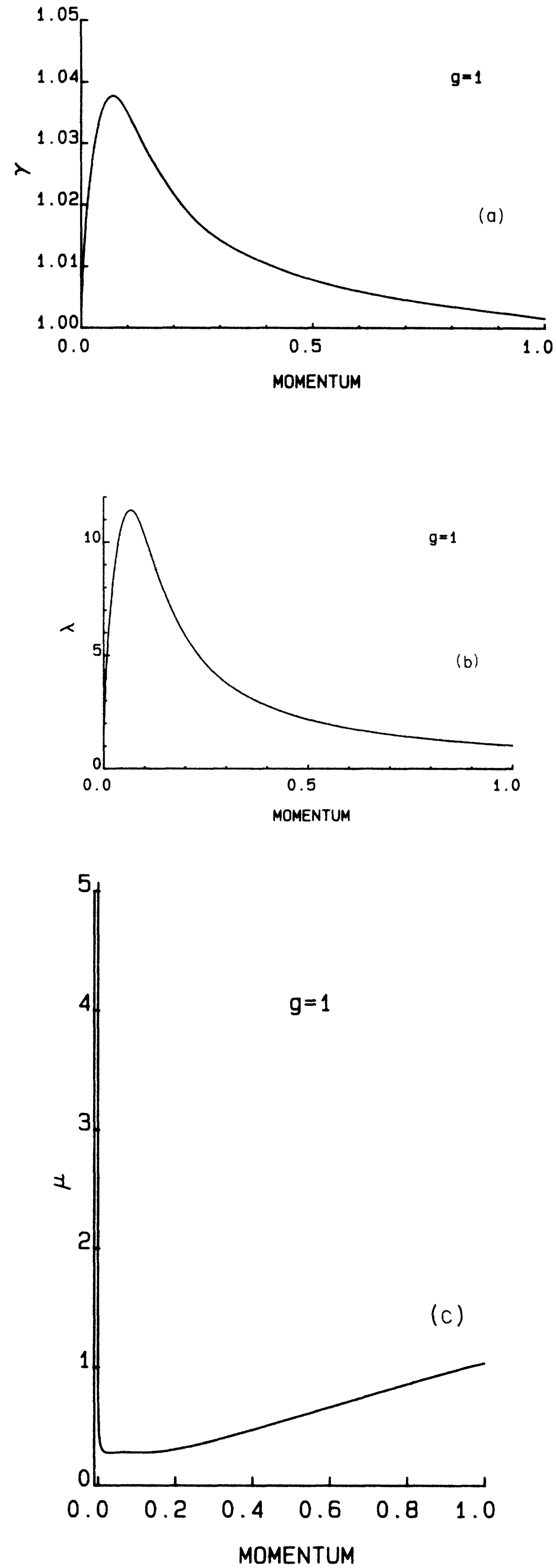

FIG. 3. The vacuum structure functions within the YM-Brueckner theory (units are defined by $M=1$): (a) "occupation factor", Eq. (20); (b) Bogoliubov structure function, Eq. (16); (c) Lagrange multiplier = single-particle energy, Eq. (28).

(1) The occupation factors γ_k always stay very near to 1. This means that only a few (quasi)particles are excited to higher levels due to the operator S_3. (Note $\gamma_k=1$ for $\Psi=\Phi_0$.) Experience of many-body theorists[17] is that the validity of a cluster expansion is governed by the average value of γ_k-1 (the "wound integral"). Since γ_k-1 is always very small in our case, we are confident that our approximation scheme makes sense. Of course, higher orders have to be checked in the future in more detail.

(2) The function λ_k goes to zero for $k\to 0$. Thus defining an "effective mass" $\bar{\mu}(k)$ through

$$\lambda_k=1/[k^2+\bar{\mu}^2(k)]^{1/2}$$

this function is not constant—like in the pure (but inconsistent) Bogoliubov theory—but we have

$$\bar{\mu}(k)\to 0 \quad \text{for } k\to\infty\ ,$$
$$\bar{\mu}(k)\simeq\frac{1}{\sqrt{k}} \quad \text{for } k\to 0\ .$$

This has the consequence that small momenta (large distances) will be suppressed. This structure has been first suggested by Gribov[5] and was also found by Cutkosky[2] who traced it back to the singularity of the Coulomb-gauge Hamiltonian at the Gribov horizon. Our results confirm Cutkosky's analysis. More specifically, our variational principle gives the following structure to this "Gribov singularity" of the function λ_k: Analogous to the standard Hartree-Fock or Hartree-Bogoliubov theory (involving fermions), the condition $\delta E/\delta\lambda=0$ is equivalent to

$$R_k=\sum_{r,a}\langle\Psi\,|\,Hs_\alpha b_\alpha^\dagger b_{-\alpha}^\dagger\,|\,\Psi\rangle=0,\quad \alpha=(k,r,a)\ . \tag{30}$$

For small k, the important contribution to R_k comes from the Coulomb-potential contractions (see I for notation) of the type

$$R_p^c=\overline{\Pi A\overline{R R} R A}\,\Pi b^\dagger b^\dagger=\frac{1}{16}\tilde{\alpha}^2\int d^3k\,d^3q\,\bar{h}(k,k-q)h(k,p)\frac{\lambda_p\tilde{\gamma}_p}{|k-p|^2|k|^2}\left[\frac{\lambda_q}{\lambda_{k-q}}\tilde{\gamma}_q\tilde{\gamma}_{k-q}-1\right](2\sigma_k s_k s_{k-p}+s_k{}^2\sigma_{k-p})\ . \tag{31}$$

This term diverges for $k\to 0$ to $+\infty$, and in order to make $R_k=0$ it has to be compensated by the contribution of the kinetic energy $R_k^{\rm kin}=\frac{1}{2}(k^2\lambda_k-1/\lambda_k)$ by increasing $1/\lambda_k$ correspondingly. In the pure Bogoliubov case, $\gamma_k=1$ and the singularity of R_k^c is smoothened by the factor (λ_p/λ_q-1).

(3) The Lagrange multiplier: μ_k is closely related to $1/\lambda_k$. Therefore, μ_k develops the reciprocal structure of Fig. 3(c). Interpreting μ_k as the gluon single-particle energy, and writing

$$\mu_k=[k^2+m^2(k)]^{1/2} \tag{32}$$

we have—as for $\bar{\mu}(k)$—$m^2\to\infty$ for $k\to 0$ (also the perturbative structure $m^2\to 0$ for $k\to\infty$ clearly holds). Constructing two-particle states out of such gluons, one would, therefore, expect a mass gap in the glueball spectrum. Masses of glueballs should be of the order of $2\mu_0$ where μ_0 yields the minimum of μ_k. Taking $M\simeq 3$ GeV for $g=1$ from lattice calculations, we thus would predict glueballs with masses of 1–2 GeV—a commonly accepted order of magnitude. Details of glueball calculations will be presented in a future paper.[7]

(4) All results are consistent with the restriction of the transverse gluon fields due to the Gribov horizon. As discussed in I, the quantity characterizing how near the ground-state function Ψ is approaching the horizon is given by

$$Q_0(s)=1-1/s_0\ . \tag{33}$$

Our numerical calculations give $Q_0(s)\simeq 0.3$, quite consistent with the condition $Q_0(s)<1$.

VI. COMPENSATION OF NONLOGARITHMIC DIVERGENCES

In this section we will indicate how the necessary cancellations of nonlogarithmic divergences in the expressions for glueball masses emerge, when the YM-Brueckner scheme is applied to these observables. Therefore, we write for the glueball states in analogy to Eq. (54) of I

$$|\,g\,\rangle=\sum f_{\alpha\beta}B_\alpha^\dagger B_\beta^\dagger\,|\,\Psi\rangle\ , \tag{34}$$

where $\Psi=(\exp S_3)\Phi_0$ is the vacuum state of Eq. (16) and $B_\alpha^\dagger\,|\,\Psi\rangle$ describes an (unphysical) gluon state. In order to be consistent with perturbation theory, the operator $B_\alpha^\dagger$ has to contain a "gluon cloud," i.e.,

$$B_\alpha^\dagger=b_\alpha^\dagger+\sum s_{\bar{\alpha}}\bar{S}_{-\alpha\beta\gamma}b_\beta^\dagger b_\gamma^\dagger\ . \tag{35}$$

The functions $f_{\alpha\beta}$ and $\bar{S}_{\alpha\beta\gamma}$ have to be determined by minimizing the expectation value $\langle g\,|\,H\,|\,g\rangle/\langle g\,|\,g\rangle$, approximated within the same "YM-Brueckner" cluster expansion, keeping fixed the vacuum structure functions S_3 and λ_k [Eq. (16)]. We skip the rather lengthy expression for these expectation values here. In second-order perturbation theory, the coefficients $S_{\alpha\beta\gamma}$ [Eq. (17)] and $\bar{S}_{\alpha\beta\gamma}$ become

$$\begin{aligned} S^{(2)}_{\alpha\beta\gamma}&=-V^*_{\alpha\beta\gamma}/(\omega_\alpha+\omega_\beta+\omega_\gamma),\quad \omega_\alpha=|\,k_\alpha\,|\\ \bar{S}^{(2)}_{\alpha\beta\gamma}&=V^*_{\alpha\beta\gamma}/(\omega_\alpha-\omega_\beta-\omega_\gamma)\end{aligned} \tag{36}$$

(of course, $v_k=0$ in this case). $S^{(2)}$ generates the four-particle, $\bar{S}^{(2)}$ the two-particle intermediate states corresponding to Figs. 2(d) and 2(c). Both are *sufficient* contributions to the one-gluon state in order to induce the cancellations of nonlogarithmic divergences within second-order perturbation theory (Sec. III).

The important observation now is that these cancellations are still present under rather weak, i.e., more general, *necessary* conditions. This condition is that the *high-momentum* behavior of the single-particle energies [defined as in Eq. (2)] is that of second-order perturbation theory. This is guaranteed if

$$\begin{aligned} S_{\alpha\beta\gamma} &\rightarrow S^{(2)}_{\alpha\beta\gamma} \\ & \qquad \text{for } k_\alpha \rightarrow \infty \\ \bar{S}_{\alpha\beta\gamma} &\rightarrow \bar{S}^{(2)}_{\alpha\beta\gamma} \end{aligned} \tag{37}$$

and if the single-particle part of the expectation value for the glueball masses tends to the perturbation result for large momenta.

Both conditions turn out to be fulfilled within our YM-Brueckner scheme: The relation (37) is verified within all our numerical computations—in fact the behavior of the vacuum structure functions $\lambda_k, \mu_k, \gamma_k$ for large k display already this perturbative character. On the other hand, the cluster expansion of the expectation value $\langle g \mid H \mid g \rangle / \langle g \mid g \rangle$ also is consistent with perturbation results in the required sense—a well-known structure of this approximation technique (for more details see Ref. 17).

Summarizing, we conclude that it seems possible to define extensions of nonperturbative many-body approximations applicable to gauge field theories which produce results consistent with renormalization. The crucial condition for these many-body techniques is that the correct cancellations of nonlogarithmic divergences should occur. We mention that these cancellations are automatic in lattice calculations since here the entire many-body problem—regularized via the given lattice cutoff—is solved rigorously with much more effort. (Up to now, the computer times needed for the presented results are on the order of standard nuclear physics many-body calculations.) Application of our approximation scheme to the ground-state problem yields a "Gribov singularity" in the structure functions indicating confinement in the sense of a suppression of small momenta and of the occurrence of a mass gap in the spectrum of glueballs.

The important check of the validity of our approach—the consistency of the continuum limit for observables like glueball spectra and a comparison to reliable lattice results—will be investigated in the future.

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