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Variational approach to classical SU_2 gauge theory with spherical symmetry

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We examine solutions to the field equations for an SU_2 gauge theory with spherical symmetry. We find the nonlinear equations governing the behavior of gauge-invariant projections of electric and magnetic field strengths. New solutions are found by linearizing the equations around the simple "vacuum" solution and around the Wu-Yang monopole. The variational approach used is well suited to the study of small fluctuations around more complicated starting points.

I. INTRODUCTION

The study of classical non-Abelian gauge theories has proved to be a rich and fruitful enterprise.[1] Successes associated with the classical approach include the discovery of topologically stable field configurations such as Wu-Yang monopoles[2] and instantons.[3] Other interesting solutions such as merons[4] and non-Abelian plane waves[5] have provided insight into the peculiar nonlinearities of non-Abelian dynamics. However, it has not yet been shown that the knowledge gained from studying classical non-Abelian dynamics has been directly applicable to the problems encountered in the quantum theory.

The basic idea that classical dynamics should be important in understanding quantum dynamics can be easily explained. In the Euclidean path-integral formalism, the expectation value of an observable $O(\Phi)$ is

$$\langle \Omega \mid O(\Phi) \mid \Omega \rangle = \int d[\Phi] O(\Phi) \exp[-S(\Phi)] \;, \qquad (1.1)$$

where $S(\Phi)$ is the action functional. The local minima of S in function space correspond to the classical field configurations obeying the classical equations of motion. In the neighborhood of a classical solution,

$$\Phi = \Phi_{\rm cl} + \delta\Phi \;, \qquad (1.2)$$

then

$$\exp[-S(\Phi)] \simeq \exp(-S_{\rm cl}) \exp[-Q(\delta\Phi)] \times \exp[-R(\delta\Phi)] \;, \qquad (1.3)$$

where $Q(\delta\Phi)$ is a quadratic functional and $R(\delta\Phi)$ is the remainder. Since we understand how to do Gaussian functional integrations we can set up a perturbation expansion based on (1.2) and (1.3) [provided $R(\delta\Phi)$ is well behaved]. In particle physics, the usual perturbation expansion involves an approximation to the free-field, plane-wave solutions of the classical equations. In any theory with intrinsic nonlinearities it is quite possible that alternative classical approximations exist which incorporate essential new dynamics not found in the trivial perturbation expansion. For example, in many systems in condensed-matter physics, it has proved informative to quantize around nontrivial classical solutions representing collective motion. Indeed, the modern picture of spontaneous symmetry breaking relies heavily on the validity of this classical approach.[6]

We will not pursue here the general problem of providing alternative starting points for perturbative quantum field theories. Strictly at the level of the classical field equations, there exists a need for more systematic exploration of non-Abelian dynamics. We will demonstrate this within the context of an SU_2 gauge theory where all observables are constrained to display spherical symmetry. The non-Abelian field equations

$$(D^\mu G_{\mu\nu})^a = j_\nu^a \;, \qquad (1.4)$$

where D^μ is the gauge-covariant derivative, will be constructed in a manner which parallels the formulation of Maxwell's equations as much as possible. It is found to be convenient to deal with gauge-invariant projections of the field strengths. A specific formalism for doing this is used with an eye toward generalization to other types of systems.

We approach the nonlinearities in the system (1.4) by using variational techniques to find approximate solutions. These variational techniques are demonstrated for two simple cases—starting with the "trivial" vacuum solution and starting with the Wu-Yang monopole.

II. SPHERICAL SYMMETRY AND SU_2

Most successful applications of the classical formalism for non-Abelian gauge theories have involved the use of a simplifying *Ansatz*.[7] One of the most useful *Ansätze* for an SU_2 gauge theory involves the assumption that in some frame all observables display a spherical symmetry. This *Ansatz* is one of the simplest available which still allows nonlinear dynamical field equations. Although we will not derive its general properties, we will, for completeness, include some introductory material concerning its formulation.[8] We will be working in Minkowski space with metric diag$(-1,1,1,1)$ and will assume that the SU_2 vector potential can be written in the form

$$gA_0^a(r,t)=A_0(r,t)\hat{r}_a \; ,$$
$$gA_i^a(r,t)=A_1(r,t)\rho_{ia}+\frac{a(r,t)}{r}\sin\omega(r,t)\delta_{ia}^T + \frac{a(r,t)\cos\omega(r,t)-1}{r}\epsilon_{ia}^T \; , \quad (2.1)$$

where we have absorbed the "coupling" g into the vector potential. The structure of Eq. (2.1) can be understood by referring to Fig. 1 which displays pictorially the mapping between a segment of three-dimensional Euclidean space and the SU_2 group manifold which is built into the *Ansatz.* The tensors ρ_{ia}, δ_{ia}^T, and ϵ_{ia}^T which relate spatial vector indices $(i,j,k=1,2,3)$ to SU_2 group indices $(a,b,c=1,2,3)$ in the adjoint representation are defined to be

$$\rho_{ia}=\hat{r}_i\hat{r}_a \; ,$$
$$\delta_{ia}^T=\delta_{ia}-\hat{r}_i\hat{r}_a \; , \quad (2.2)$$
$$\epsilon_{ia}^T=\epsilon_{ial}\hat{r}_l \; .$$

With the identification of SU_2 and SO_3 indicated in Fig. 1 the tensor ρ_{ia} is "radial" in both halves of the direct-product space. Within this *Ansatz,* there is no component which is radial in ordinary space and transverse in group space (or vice versa). The two tensors which are transverse in three-space and group space are designated δ_{ia}^T and ϵ_{ia}^T. These tensors can be written in an alternate form to emphasize their geometrical structure. If we introduce the usual spherical coordinate system and let $\hat{\phi}$ and $\hat{\theta}$ be the transverse unit vectors, then

$$\delta_{ia}^T=(\hat{\theta}_i\hat{\theta}_a+\hat{\phi}_i\hat{\phi}_a) \; ,$$
$$\epsilon_{ia}^T=(\hat{\phi}_i\hat{\theta}_a-\hat{\phi}_a\hat{\theta}_i) \; . \quad (2.3)$$

The reader should keep in mind that $\rho_{ia}\rho_{ia}=1$, while $\delta_{ia}^T\delta_{ia}^T=2$, and $\epsilon_{ia}^T\epsilon_{ia}^T=2$. Since the original form of the *Ansatz* has become conventional, we will not attempt to correct for the difference in normalization between the transverse and radial tensors. We will, however, give an explicit formula with the appropriate normalization when defining "projections."

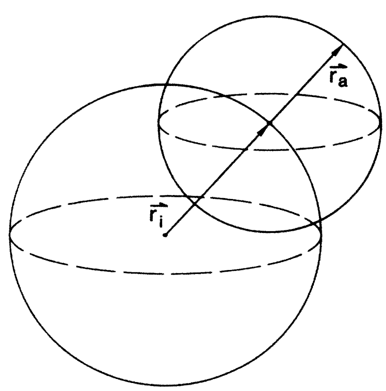

FIG. 1. The sketch indicates the relationship between the group space of SU_2 and ordinary three-space which is inherent in the *Ansatz.* At each point in space, a gauge transformation is used to align the radial direction in group space represented by $\hat{r}_a$ with the radial direction in three-space represented by $\hat{r}_i$. This does not exhaust the gauge freedom of the system since it is possible to make further transformations which preserve $\hat{r}_a$.

One reason why the tensor structures (2.2) simplify the form of the non-Abelian field equations can be seen by using still another representation of the transverse tensors (2.3). Some simple algebra shows that

$$\frac{1}{r}\delta_{ia}^T=\partial_i\hat{r}_a \; ,$$
$$\frac{1}{r}\epsilon_{ia}^T=-i[\hat{r},\partial_i\hat{r}]_a \; , \quad (2.4)$$

where

$$[x_a,x_b]=i\epsilon^{abc}x_c \quad (2.5)$$

is the SU_2 commutator. Equation (2.4) shows how the mapping between three-space and the group manifold built into the *Ansatz* simplifies many of the algebraic manipulations involved in the field equations. Because of the separation of radial and transverse modes, it is simple to consider how a spatial rotation about the $\hat{r}$ direction by an angle δ affects a vector structure

$$B_i'=R_{ij}(\delta)B_j \; , \quad (2.6)$$

where R_{ij} is a rotation matrix which preserves the $\hat{r}$ direction. We see that $R_{ij}(\delta)$ can be written

$$R_{ij}(\delta)=\rho_{ij}+\cos\delta\delta_{ij}^T+\sin\delta\epsilon_{ij}^T \quad (2.7)$$

in terms of the same tensor structures that appear in (2.2) except that both indices now refer to spatial orientations.

Similarly, we can investigate the effect of gauge transformations of the type

$$\Omega(\hat{r},\psi)=\exp\left[i\hat{r}_a\frac{\sigma_a}{2}\psi(r,t)\right] \; , \quad (2.8)$$

which preserve the $\hat{r}$ direction in group space. These gauge transformations on a gauge-covariant object in the adjoint representation (such as $F_{\mu\nu}^a$) yield

$$B_a'=B_bR_{ba}(\psi) \; , \quad (2.9)$$

with

$$R_{ba}(\psi)=(\rho_{ba}+\cos\psi\delta_{ba}^T+\sin\psi\epsilon_{ba}^T) \; , \quad (2.10)$$

where both indices on the tensors now refer to group space.

Comparison of (2.6) and (2.9) points out that we can compensate for the effect of a gauge transformation of the type (2.8) by a spatial rotation or vice versa. Using the algebra

$$\rho_{in}\delta_{na}^T=0, \quad \epsilon_{in}^T\delta_{na}^T=\epsilon_{ia}^T \; ,$$
$$\rho_{in}\epsilon_{na}^T=0, \quad \epsilon_{in}^T\epsilon_{na}^T=-\delta_{ia}^T \; , \quad (2.11)$$
$$\delta_{in}^T\delta_{na}^T=\delta_{ia}^T \; ,$$

we can see that the form of (2.1) is preserved by rotations or gauge transformations which do not change the $\hat{r}$ direction. This invariance suggests that it will be convenient to reformulate the *Ansatz* using a new set of transverse tensors involving the gauge angle $\omega(r,t)$ which appears in the definition of the vector potential (2.1):

$$e_{ia}^S(\omega)=\delta_{ia}^T\cos\omega(r,t)-\epsilon_{ia}^T\sin\omega(r,t)\ ,$$
$$e_{ia}^A(\omega)=\delta_{ia}^T\sin\omega(r,t)+\epsilon_{ia}^T\cos\omega(r,t)\ . \tag{2.12}$$

The superscripts A and S refer to the antisymmetric or symmetric behavior under the interchange of spatial and color indices:

$$e_{ai}^S(\omega)=e_{ia}^S(-\omega)\ ,$$
$$e_{ai}^A(\omega)=-e_{ia}^A(-\omega)\ . \tag{2.13}$$

The two tensors can also be related by a shift in the gauge angle $\omega(r,t)$ since

$$e_{ia}^S(\omega+\pi/2)=-e_{ia}^A(\omega)\ ,$$
$$e_{ia}^A(\omega+\pi/2)=e_{ia}^S(\omega)\ . \tag{2.14}$$

Spherically symmetric objects which transform covariantly under both rotations and gauge transformations can be conveniently written in terms of these gauge-dependent tensors. Examples include the color- electric and color-magnetic components of the SU_2 field strength tensor:

$$gE_i^a=E_L(r,t)\rho_{ia}+E_A(r,t)e_{ia}^A(\omega)+E_S(r,t)e_{ia}^S(\omega)\ ,$$
$$gB_i^a=B_L(r,t)\rho_{ia}+B_A(r,t)e_{ia}^A(\omega)+B_S(r,t)e_{ia}^S(\omega)\ . \tag{2.15}$$

Under a spatial rotation or a gauge transformation which preserves the $\hat{\mathbf{r}}$ direction, the transformations (2.6) and (2.9) induce "rotations" which can be written

$$R_{ij}(\delta)=\rho_{ij}+e_{ij}^S(-\delta)\ ,$$
$$R_{ab}(\psi)=\rho_{ab}+e_{ab}^S(-\psi)\ . \tag{2.16}$$

Using the algebra

$$e_{im}^S(\omega_1)e_{ma}^S(\omega_2)=e_{ia}^S(\omega_1+\omega_2)\ ,$$
$$e_{im}^S(\omega_1)e_{ma}^A(\omega_2)=e_{ia}^A(\omega_1+\omega_2)\ , \tag{2.17}$$
$$e_{im}^A(\omega_1)e_{ma}^A(\omega_2)=-e_{ia}^S(\omega_1+\omega_2)\ ,$$

we see that these transformations merely serve to rotate the basis vectors. The coefficients in (2.15) are invariant under such transformations. The change of basis to e^A and e^S is, in many ways, similar to transforming to body-centered coordinates for a classical extended object. The angle, $\omega(r,t)$, which appears as an argument in the basis tensors is a gauge angle which measures the orientation of the group manifold relative to the three-space manifold. The gauge invariance of the theory ensures that physical observables do not depend on this angle.

The gauge-dependent transverse basis tensors $e^A(\omega)$ and $e^S(\omega)$ appear naturally in the manipulation of gauge-covariant derivatives. Using the definition of the vector potential in (2.1) we can show

$$(D_i^{ab}\hat{r}_b)=\frac{a(r,t)}{r}e_{ia}^S(\omega)\ ,$$
$$-i[\hat{\mathbf{r}},D_i\hat{\mathbf{r}}]^a=\frac{a(r,t)}{r}e_{ia}^A(\omega)\ , \tag{2.18}$$

where

$$D_i^{ab}=\partial_i\delta^{ab}+g\epsilon^{abc}A_i^c\ . \tag{2.19}$$

It is interesting to observe that both of the covariant structures on the left-hand side of (2.18) necessarily vanish when $a(r,t)=0$. From the form of (2.1) we see that when $a(r,t)$ vanishes identically, we have a Wu-Yang monopole[2] or a dyon[9] located at $r=0$. With $a(r,t)\neq 0$, the transverse tensors e^A and e^S obviously provide a convenient basis to exploit the algebraic properties of the theory.

It is useful to rewrite the expression for the vector potential (2.1) in terms of the new transverse basis

$$gA_0^a(r,t)=A_0(r,t)\hat{r}_a\ ,$$
$$gA_i^a(r,t)=A_1(r,t)\rho_{ia}+\frac{a(r,t)}{r}e_{ia}^A(\omega)-\frac{1}{r}e_{ia}^A(0)\ . \tag{2.20}$$

Under a gauge transformation of the form (2.8) which preserves the $\hat{\mathbf{r}}$ direction, the time and radial components of (2.20) transform like a two-dimensional Abelian vector potential

$$A_l(r,t)\xrightarrow{\Omega}\bar{A}_l(r,t)=A_l+\partial_l\psi\quad(l=0,1)\ , \tag{2.21}$$

while the transverse components transform

$$(A_i^a)^T\xrightarrow{\Omega}(\bar{A}_i^a)^T=\frac{a(r,t)}{r}e_{ia}^A(\omega+\psi)-\frac{1}{r}e_{ia}^A(0)\ , \tag{2.22}$$

keeping $a(r,t)$ fixed.

It is not difficult to obtain expressions for the components of the field strength tensor given in (2.15). We have

$$E_L=(\partial A_0/\partial r-\partial A_1/\partial t),\quad B_L=\frac{a^2-1}{r^2}\ ,$$
$$E_A=-\frac{1}{r}\partial a/\partial t,\quad B_A=-\frac{a}{r}(\partial\omega/\partial r-A_1)\ , \tag{2.23}$$
$$E_S=-\frac{a}{r}(\partial\omega/\partial t-A_0),\quad B_S=-\frac{1}{r}\partial a/\partial r\ .$$

Under a radial gauge transformation, the two-dimensional vector potential A_l transforms according to (2.21) while $\omega(r,t)\rightarrow\omega(r,t)+\psi(r,t)$ with $a(r,t)$ fixed. We therefore see that the combinations $(\partial_l\omega-A_l)$ are gauge invariant so that all the structures which appear in (2.23) are not changed by radial gauge transformations as claimed earlier.

We can use the representation of the field strength tensor in (2.23) to explore the connection between this four-dimensional SU_2 theory with spherical symmetry and the Abelian Higgs model in two dimensions.[10] If we write

$$F_{lm}=\partial_l A_m-\partial_m A_l\ ,$$
$$\Phi=ae^{i\omega}\ , \tag{2.24}$$
$$D_l=\partial_l-iA_l\ ,$$

we see that the Lagrangian $\int d^4x\, \mathscr{L}_g(x)$ can be written $4\pi \int dr\, dt(r^2\mathscr{L}_g)$ with

$$r^2\mathscr{L}_g = r^2(F_{lm}F^{lm}) + 2D^l\Phi D_l\Phi^* + \frac{1}{r^2}(\,|\,\Phi\,|^2 - 1)^2 \,. \tag{2.25}$$

This is the Lagrange density for an Abelian Higgs model in a curved two-dimensional space with metric $r^2 g_{lm}$. The transverse components of the field strength tensor correspond to the real and imaginary parts of $D^l\Phi$. This analogy is extremely valuable in working out the properties of the classical theory.

One final feature of the *Ansatz* must be examined before we can proceed to look at the field equations. In non-Abelian gauge theories, one must be aware of the possibility of nontrivial topological connections. In the *Ansatz* (2.1) we can see that there is a "topological current," K_l, $l=0,1$ with divergence

$$\partial^l K_l = g^2 r^2 E_i^a B_i^a \tag{2.26}$$

with components

$$\begin{aligned} K_0 &= (a^2-1)A_1 - a^2\partial\omega/\partial r \,, \\ K_1 &= -(a^2-1)A_0 + a^2\partial\omega/\partial t \,. \end{aligned} \tag{2.27}$$

This means, for example, that the combinations which give the transverse fields

$$\begin{aligned} E_S &= -\frac{a}{r}(\partial\omega/\partial t - A_0) = K_1 - A_0 \,, \\ B_A &= -\frac{a}{r}(\partial\omega/\partial r - A_1) = \frac{K_0 + A_1}{ar} \end{aligned} \tag{2.28}$$

can be written without reference to the gauge angle $\omega(r,t)$. The consequences of this representation are treated more fully in Ref. 11. The arbitrariness of the topological current and the freedom of the gauge angle ω can be seen as two aspects of the same underlying invariance.

Specific dynamical approximations within the framework of classical physics involve the introduction of dynamical sources. We will therefore study the response of the spherically SU_2 system in the presence of a current $j_\mu^a(r,t)$.

The field equations with sources

From our discussion above, it is apparent that we can parametrize the spherically symmetric SU_2-covariant current in terms of the tensors (2.13):

$$\begin{aligned} j_0^a(r,t) &= \frac{1}{r^2}J_0(r,t)\hat{r}_a \,, \\ j_i^a(r,t) &= \frac{1}{r^2}J_1(r,t)\rho_{ia} + p_S(r,t)e_{ia}^S(\omega) + p_A(r,t)e_{ia}^A(\omega) \,. \end{aligned} \tag{2.29}$$

The generalization of Maxwell's equations to the SU_2 system,

$$(D_\mu G_{\mu\nu})^a = j_\nu^a \,, \tag{2.30}$$

with

$$D_\mu^{ab} = \partial_\mu\delta^{ab} + g\epsilon^{abc}A_\mu^c$$

then gives the set of equations

$$\begin{aligned} &-\frac{\partial}{\partial r}(r^2E_L) + 2arE_S = J_0(r,t) \,, \\ &-\frac{\partial}{\partial t}(r^2E_L) + 2arB_A = J_1(r,t) \,, \\ &-\frac{\partial}{\partial t}(arE_S) + \frac{\partial}{\partial r}(arB_A) = arp_S(r,t) \,, \\ &a\Box_{(2)}a - r^2(E_S{}^2 - B_A{}^2) - \frac{a^2(a^2-1)}{r^2} = arp_A(r,t) \,. \end{aligned} \tag{2.31}$$

The generalization of the Bianchi constraints gives

$$\begin{aligned} &\frac{\partial}{\partial r}(arE_A) - \frac{\partial}{\partial t}(arB_S) = 0 \,, \\ &-E_L + \frac{\partial}{\partial r}(arE_S) + \frac{\partial}{\partial t}(arB_A) = \partial_l K^l \\ &\qquad = g^2r^2E_i^aB_i^a \,. \end{aligned} \tag{2.32}$$

The first constraint is trivial in view of the definitions (2.23). The second constraint gives an important relationship which cannot be ignored in our "field strength" formulation of the dynamics.[11] In addition to the equations above, we must for consistency impose the requirement that the SU_2 current (2.29) be covariantly conserved:

$$(D_\mu^{ab}j_\mu^b) = 0 \,. \tag{2.33}$$

Using the definitions above, this gives

$$\partial_l J^l = 2a(r,t)rp^S(r,t) \,. \tag{2.34}$$

Since $a(r,t)$ appears on the right-hand side (RHS), this relationship does not necessarily just give a constraint on the color sources. The most conservative approach, however, is to interpret this as two constraints

$$\begin{aligned} &p^S(r,t) = 0 \,, \\ &\partial_l J^l(r,t) = 0 \,, \end{aligned} \tag{2.35}$$

so that $a(r,t)$ can be given by (2.31). Other possible interpretations are discussed briefly in Ref. 11. The fact that we appear to have already an over-constrained theory may indicate a conceptual flaw in the idea that fields and sources can be clearly separated in non-Abelian theories.

III. SOLUTIONS TO THE FIELD EQUATIONS

Maxwell's equations form the foundation for classical electrodynamics so that the problem of determining classical electric and magnetic field configurations reduces to the study of partial differential equations with specified boundary conditions. An important distinction arises between Maxwell's equations and the classical field equations of a non-Abelian field theory in that the non-Abelian field equations are inherently nonlinear. The nonlinearities arise because the gauge fields themselves carry a non-Abelian charge.

There exists no complete theory of nonlinear particle differential equations and its is therefore not clear that a study of the non-Abelian field equations can lead to any universal insight about non-Abelian dynamics. By and large, the study of nonlinear partial differential equations can be considered a collection of *ad hoc* techniques. One of the most important differences involves the absence of a superposition principle. For a linear operator, if $u_1(r,t)$ and $u_2(r,t)$ are solutions to

$$L(u)=0\ , \tag{3.1}$$

then the principle of linear superposition tells us that

$$U=au_1(r,t)+bu_2(r,t) \tag{3.2}$$

is also a solution. The application of superposition to the study of Maxwell's equations is obvious. We can build up solutions involving complicated current configurations from a set of more elementary solutions. This capability is lost to us in non-Abelian theories.

Occasionally, it is possible to recover some of the useful properties of a superposition principle through the use of nonlinear connecting functions.[12] This, however, does not provide a systematic treatment for constructing complicated solutions. If we scale down our expectations, there are some simple things we can do to expand the range of solutions available to us. We can sometimes find special solutions to nonlinear equations which are somehow "obvious" or have some particular physical interpretation. Using these solutions as a starting point we can utilize a variational approach to deal with small fluctuations around them by means of a linearized system of equations. The linearization leads to useful approximations to new solutions subject to some constraints and restrictions.

We will apply this variational approach to some solutions to the SU_2 field equations with spherical symmetry for the fields and sources given in Sec. II. After some manipulation, the equations (2.31) for the invariant field strength projections can be put in the form

$$\begin{aligned} &E_L=\frac{\epsilon}{r^2},\quad B_L=\frac{a^2-1}{r^2}\ ,\\ &E_S=\frac{1}{2ar}(J_0+\partial\epsilon/\partial r),\quad B_S=\frac{1}{r}\frac{\partial a}{\partial r}\ ,\\ &E_A=-\frac{1}{r}\frac{\partial a}{\partial t},\quad B_A=\frac{1}{2ar}(J_1+\partial\epsilon/\partial t)\ , \end{aligned} \tag{3.3}$$

where $\epsilon=\epsilon(r,t)$ and $a=a(r,t)$ are solutions of the coupled nonlinear wave equations

$$\Box_2\epsilon-\frac{2}{a}(\partial_l\epsilon\partial_l a)-\frac{2a^2\epsilon}{r^2}=-\epsilon^{lm}\partial_l J_m+\frac{2}{a}\epsilon^{lm}J_l\partial_m a\ , \tag{3.4a}$$

$$\Box_2 a-\frac{1}{4a^3}[(\partial\epsilon/\partial t+J_1)^2-(\partial\epsilon/\partial r+J_0)^2]-\frac{a(a^2-1)}{r^2}=rp_A\ . \tag{3.4b}$$

Looking at wave equations suggests that it is instructive to form the generalization of the Poynting vector of classical electrodynamics:

$$P_i=\tfrac{1}{2}(\epsilon_{ijk}E_j^aB_k^a)=\hat{\mathbf{r}}_i(E_SB_A-E_AB_S)\ . \tag{3.5}$$

Inserting the expressions for the transverse fields, (3.3),

$$P_i=\hat{\mathbf{r}}_i\left[\frac{(J_0+\partial\epsilon/\partial r)(J_1+\partial\epsilon/\partial t)}{a^2r^2}+\frac{1}{r^2}\frac{\partial a}{\partial t}\frac{\partial a}{\partial r}\right]. \tag{3.6}$$

This can be formulated in a manner which is more symmetric under electric and magnetic modes by letting $\beta=a^2-1=r^2B_L$:

$$P_i=\frac{\hat{\mathbf{r}}_i}{4(1+\beta^2)r^2}\left[(J_0+\partial\epsilon/\partial r)(J_1+\partial\epsilon/\partial t)+\frac{\partial\beta}{\partial t}\frac{\partial\beta}{\partial r}\right]. \tag{3.7}$$

Our assumptions constrain the direction of this Poynting vector to be radial. If we assume, at large r, that the sources vanish and we have wavelike behavior

$$\beta(r,t)\sim\beta_0e^{i\omega(r-t)}\ ,$$

$$\epsilon(r,t)\sim\epsilon_0e^{i\omega(r-t)}\ ,$$

then

$$P_i=\frac{\hat{\mathbf{r}}_i}{4(1+\beta^2)r^2}\omega^2({\epsilon_0}^2+{\beta_0}^2)\ . \tag{3.8}$$

This is consistent with the idea that the Poynting vector is associated with the energy flow of the fields. One sometimes encounters static (time-independent) solutions to the non-Abelian field equations. The question of whether such solutions are stable can be addressed by looking for time-dependent fluctuations around them.

A. Variational approach to constructing solutions

Given the complicated form of the nonlinear equations (3.4) for $\epsilon(r,t)$ and $a(r,t)$ it is not obvious that the most general solutions can be found for a specific configuration of currents (J_l,p_A). Once a particular solution is found, however, we can use a simple variational approach to study approximate new solutions. Let $\epsilon_0(r,t)$ and $a_0(r,t)$ be solutions to (3.4) for a given J_l^0,p_A^0. Let

$$\begin{aligned} &\epsilon(r,t)=\epsilon_0(r,t)+\lambda\epsilon_1(r,t)\ ,\\ &a(r,t)=a_0(r,t)+\delta a_1(r,t)\ ,\\ &J_l(r,t)=J_l^0(r,t)+\lambda j_l^\lambda(r,t)+\delta j_l^\delta(r,t)\ ,\\ &p_A(r,t)=p_A^0(r,t)+\lambda p_A^\lambda(r,t)+\delta p_A^\delta(r,t)\ , \end{aligned} \tag{3.9}$$

then the equations found by inserting (3.9) into (3.4a) and (3.4b) and neglecting terms of order higher than linear in λ,δ will be linear partial differential equations for $\epsilon_1(r,t)$ and $a_1(r,t)$. The coefficients will depend on $\epsilon_0(r,t)$ and $a_0(r,t)$.

We can illustrate the application of this variational technique by considering two simple starting point with $J_0=J_1=p_A=0$. The first starting point corresponds to the classical vacuum in which all fields vanish. We consider (3.9) with $\epsilon_0(r,t)=0$ and $a_0(r,t)=1$. Variation with λ and δ of Eq. (3.4a) yields, for λ,

$$\Box_2\epsilon_1-\frac{2\epsilon_1}{r^2}=\epsilon^{lm}\partial_l j_m^\lambda\ , \tag{3.10a}$$

for δ

$$\epsilon^{lm}\partial_l j_m^\delta = 0 \, , \qquad (3.10b)$$

while variation with λ and δ for (3.4b) gives, for λ,

$$r p_A^\lambda = 0 \, , \qquad (3.11a)$$

for δ

$$\Box_2 a_1 - \frac{2a_1}{r^2} = r p_A^\delta \, . \qquad (3.11b)$$

Equations (3.10b) and (3.11a) merely restrict the form of the external sources while (3.10a) and (3.11b) are decoupled linear partial differential equations for ϵ_1 and a_1. The linear nature of (3.10a) actually follows from setting $a^0(r,t)=1$ in (3.4a) and does not require any restriction to small λ. This signals an important overall difference in the dynamical role of electric and magnetic degrees of freedom.

The simple wave equations (3.10a) and (3.11b) can be solved by conventional techniques. In regions where $|a_1(r,t)|$ and $|\epsilon_1(r,t)|$ are bounded, the solutions (3.9) with small λ,δ provide a local approximation to a solution of the nonlinear system. Consider, for example, the special case of static solutions and vanishing sources. Equation (3.10a) admits a solution

$$\epsilon_1(r) = \frac{d_1}{r} + d_2 r^2 \qquad (3.12)$$

with d_1 and d_2 arbitrary constants. In regions where $|\lambda\epsilon_1(r)| \ll 1$, we have an approximate solution to the nonlinear system. This constraint excludes regions near $r=0$ or $r=\infty$. We can construct the Green's function

$$\left[\frac{d^2}{dr^2} - \frac{2}{r^2} \right] G(r,r_0) = \delta(r-r_0) \qquad (3.13)$$

subject to the constraint $G(0,r_0)=0$, $G(\infty,r_0)=0$. This gives

$$G(r,r_0) = \frac{-r_0}{6} \left[\frac{r^2}{r_0^2}\theta(r_0-r) + \frac{r_0}{r}\theta(r-r_0) \right] , \qquad (3.14)$$

and we can construct solutions to the linearized inhomogeneous equation. Similar static solutions for $a_1(r)$ also exist. For sources localized around the origin, we can use $\epsilon(r,t)=r^2 E_L(r,t)$ and see from (3.3) that the large r behavior

$$E_L(r) = O(1/r^3) \, ,$$
$$E_S(r) = O(1/r^3)$$

is damped by an extra power of r compared to the static Coulomb solution in electrodynamics.

The general, time-dependent, solution to (3.10a) can also be constructed. In regions of space without sources, a solution to (3.10a) can be written

$$\begin{aligned}\epsilon_1(r,t) = \sum_\omega \{ & C_1(\omega,+)e^{i\omega t}[\omega r \zeta_1^{(1)}(\omega r)] \\ & + C_2(\omega,+)e^{i\omega t}[\omega r \zeta_1^{(2)}(\omega r)] \\ & + C_1(\omega,-)e^{-i\omega t}[\omega r \zeta_1^{(1)}(\omega r)] \\ & + C_2(\omega,-)e^{-i\omega t}[\omega r \zeta_1^{(2)}(\omega r)] \} \end{aligned} \qquad (3.15)$$

in terms of a superposition of spherical Bessel functions. The functions

$$\zeta_n^{(1,2)}(z) = \left[\frac{\pi}{2z} \right]^{1/2} H_{n+1/2}^{(1,2)}(z) \qquad (3.16)$$

are defined as in Ref. 13. The general form of (3.15) is one of the ingoing and outgoing spherical waves. We can use (3.15) to construct Green's functions for various possible harmonic source configurations as in the case of static solutions.

When "linearized" around the simple vacuum solution, $\epsilon_0(r,t)=0$ and $a_0(r,t)=1$, the nonlinear equations (3.4a) and (3.4b) therefore admit solutions which correspond to the propagation of simple waveforms. Solutions with localized sources and fields can radiate energy just as in electrodynamics. This, however, does not give a complete description of the possible solutions. We must also consider fluctuations around alternate starting points.

B. Small fluctuations around a point monopole

The Wu-Yang[2] monopole solution to the Yang-Mills field equations in our *Ansatz* corresponds to $\epsilon(r,t)=0$ and $a(r,t)=0$. It is instructive to look for new classical solutions which constitute small local fluctuations around a monopole field configuration. We will write

$$\begin{aligned} \epsilon(r,t) &= \lambda\epsilon_1(r,t) + \cdots \, , \\ a(r,t) &= \delta a_1(r,t) + \cdots \, . \end{aligned} \qquad (3.17)$$

This gives, for the field strengths,

$$\begin{aligned} & E_L = \frac{\lambda\epsilon_1(r,t)}{r^2}, \quad B_L = -\frac{1}{r^2} + \theta(\delta^2) \, , \\ & E_S = \frac{1}{2\delta a_1 r}(J_0 + \lambda\partial\epsilon_1/\partial r), \quad B_A = \frac{1}{2\delta a_1 r}(J_1 + \lambda\partial\epsilon_1/\partial t) \, , \\ & E_A = -\frac{\delta}{r}\frac{\partial a_1}{\partial t}, \quad B_S = \frac{\delta}{r}\frac{\partial a_1}{\partial r} \, . \end{aligned} \qquad (3.18)$$

In order to consistently interpret the field equations we require

$$E_S = O(\delta,\lambda), \quad B_A = O(\delta,\lambda)$$

or

$$\begin{aligned} J_0 &= -\lambda\partial\epsilon_1/\partial r + O(\lambda\delta,\delta^2) \, , \\ J_1 &= -\lambda\partial\epsilon_1/\partial t + O(\lambda\delta,\delta^2) \, . \end{aligned} \qquad (3.19)$$

This leads to the equations for $a_1(r,t)$:

$$\Box_2 a_1(r,t) + \frac{1}{r^2}a_1(r,t) = \frac{1}{\delta} r p^A(r,t) \, . \qquad (3.20)$$

In order for the RHS of (3.20) to be consistent, we must have $P^A(r,t)=O(\delta)$. It is instructive to look at solutions to the source-free equations of the form

$$a_1(r,t)=a_1(\omega,r)e^{\pm i\omega t} . \tag{3.21}$$

We will choose the causal solutions

$$a_1(r,t)=c_1^+a_1^+(\omega,r)e^{i\omega t}+c_1^-a_1^-(\omega,r)e^{-i\omega t} \tag{3.22}$$

with

$$a_1^{\pm}(\omega,r)=r^{1/2}[J_{+i\nu}(\omega r)\pm J_{-i\nu}(\omega r)] \tag{3.23}$$

with $\nu=\sqrt{3}/2$. The Bessel functions of imaginary order are defined

$$J_{i\sigma}(z)=\frac{\exp[i\sigma\ln(z/2)]}{\Gamma(1+i\sigma)}F_1(1+i\sigma;-z^2/4) . \tag{3.24}$$

We can take the limit $\omega\to 0$ to see the static solutions

$$a_1(r)=r^{1/2}(C_1^+r^{i\nu}+C_1^-r^{-i\nu}) , \tag{3.25}$$

which have been obtained independently from the time-independent form of the linearized equation.[14] Fluctuations around the Wu-Yang monopole can also involve wave propagation. In the absence of additional sources, we have

$$\epsilon_1(r,t)=0$$

and, from (3.18), the nontrivial fields are $E_A(r,t)$ and $B_S(r,t)$. All other modes are "quenched" by the radial magnetic field. In contrast with starting with a simple vacuum solution we have fewer degrees of freedom.

IV. SUMMARY AND DISCUSSION

The study of the spherically symmetric SU_2 gauge theory with nontrivial sources has been approached in many ways before. Specific solutions have been used to illustrate the different dynamic features of the system. For example, in Refs. 15 and 16 certain static solutions corresponding to simple sources were constructed and studied. Any of these time-independent solutions would be a possible starting point for the variational approach to linearization discussed above. It would also be interesting to consider fluctuations around the meron-pair solution to the field equations given by

$$a(r,t)=\frac{1+t^2-r^2}{[(1+t^2-r^2)^2+4r^2]^{1/2}} ,\quad \epsilon(r,t)=0 \tag{4.1}$$

in order to further clarify the issue of whether meron configurations play a special dynamic role in non-Abelian gauge theories.

Our approach to equations for spherically symmetric SU_2 has been to work with the gauge-independent projections of covariant field strengths. This gives a system of two nonlinear coupled wave equations. Because certain explicit solutions to these nonlinear equations are known we can construct new approximate solutions by using linearized forms of the equations. The nonlinearities of the original equations serve to induce nontrivial dependence on the starting point.

For two simple starting points, the fluctuations obey simple wave equations so that familiar solutions involving the radiation of energy are allowed. The non-Abelian system in this approximation is not fundamentally different from complicated configurations of electrodynamic sources.

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