PHYSICAL REVIEW D VOLUME 34, NUMBER 2 15 JULY 1986

Generalized monopole in $(4+K)$-dimensional Abelian theories

Li Xin-zhou and Yu Feng
Department of Physics, Fudan University, Shanghai, People's Republic of China

Zhang Jian-zu
*Department of Physics, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada
and Ames Laboratory and Department of Physics, Iowa State University, Ames, Iowa 50011*

(Received 9 October 1985)

In this paper we show the generalized monopole in $(4+K)$-dimensional Abelian theories which is dependent on the y parameter and the average field strength is defined on the point of four-dimensional spacetime. Like the Gross-Perry-Sorkin (GPS) solution the generalized y-dependent monopole solutions of GPS type are nonsingular. However, the Schwarzschild type has naked singularities.

I. INTRODUCTION

Kaluza-Klein theories,[1,2] which unify Einstein's theory of gravity with other gauge interactions, have been the subject of revived interest in recent years. The classical vacuum solution of the Kaluza-Klein theory is assumed to be $M^4\times B$, the direct product of four-dimensional Minkowski space and the compact manifold B. Any classical field configuration which approaches the vacuum solution at special infinity thus defines a B "bundle" over the sphere at spatial infinite S^2. If the B bundle over S^2 cannot be continuously deformed to the global direct product $S^2\times B$, then the field configuration cannot be continuously deformed to the vacuum solution. We conclude that it is a topological soliton. We can associate with every B bundle over S^2 a loop in the isometry group H. The B is a circle and H is U(1) in the five-dimensional theory, and the monopoles are Z monopoles as would be expected from topological arguments, $\pi_1(\mathrm{U}(1))=Z$. In the $(4+K)$-dimensional pure Kaluza-Klein theory, the ground-state configuration may be defined on the partially compactified manifold, $M^4\times S^1\times\cdots\times S^1$, where $S^1\times\cdots\times S^1$ is interpreted as the $\mathrm{U}(1)\times\cdots\times\mathrm{U}(1)$ gauge group.

In this paper we consider pure Kaluza-Klein theory in $4+K$ dimensions, where the signature of the metric $g_{\hat\mu\hat\nu}$ is $-++\cdots+$, and the caret denotes a coordinate $x^{\hat\mu}=(x^\mu,y^i)$, $\mu=0,1,2,3$, and $i=1,\ldots,K$. In the last couple of years, many authors have suggested the interest of multidimensional monopole solutions which are independent on the y^i parameters. However, we show the general monopole solution which is dependent on the y parameter and the average field strength is defined on the point of four-dimensional spacetime.

In the next section we review the $(4+K)$-dimensional Abelian theories and three different metrics in Kaluza-Klein theories. In Sec. III we discuss the radial equations in $4+K$ dimensions. In Sec. IV the generalized y-dependent monopole solutions for $K=1,2$ Schwarzschild type are carried out. However, these solutions have naked singularities. Other generalized y-dependent monopole solutions are Gross-Perry-Sorkin (GPS) type for $K=1,2$. Like the GPS solution these monopoles are nonsingular. Section V is devoted to the discussion of the average field strength. A summary is given in Sec. VI. We take the formalism in parallel with Refs. 3 and 4.

II. REVIEW OF $(4+K)$-DIMENSIONAL ABELIAN THEORIES

The $(4+K)$-dimensional Abelian theories are neither realistic nor typical of Kaluza-Klein theories of interest, but they provide a useful theoretical "laboratory" for illustrating certain ideas. First, let us consider pure gravity in five dimensions described by the action

$$S=-\frac{1}{2\pi G_5}\int d^5x(-g_5)^{1/2}R_5\ , \tag{2.1}$$

where R_5 is the five-dimensional curvature scalar. To avoid ghosts and tachyons, extra dimensions must be spacelike, i.e., $-++++$ signature, the caret denotes a coordinate $x^{\hat\mu}=(x^\mu,y)$, $\mu=0,1,2,3$. There are three different metrics which enter the discussion in Kaluza-Klein theories and they should not be (but frequently are) confused.

(1) The full metric:

$$g_{\hat\mu\hat\nu}(x,y)=\begin{bmatrix} g_{\mu\nu}+\phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{bmatrix}, \tag{2.2}$$

where $g_{\mu\nu}$, A_μ, and ϕ depend on x^μ and y. The fields $g_{\mu\nu}$, A_μ, and ϕ may be Fourier expanded in the form

$$\begin{aligned} g_{\mu\nu}(x,y)&=\sum_{n=-\infty}^{\infty} g_{\mu\nu}^{(n)}(x)\exp(iny/R)\ ,\\ A_\mu(x,y)&=\sum_{n=-\infty}^{\infty} A_\mu^{(n)}(x)\exp(iny/R)\ ,\\ \phi(x,y)&=\sum_{n=-\infty}^{\infty} \phi^{(n)}(x)\exp(iny/R)\ , \end{aligned} \tag{2.3}$$

with $g_{\hat{\mu}\hat{\nu}}^{(n)*}=g_{\hat{\mu}\hat{\nu}}^{(-n)}$, etc. The full metric describes the ground state plus massless ($n=0$) and massive ($n\neq 0$) modes.

(2) The Kaluza-Klein ansatz. As above, but discarding the massive modes, this describes the "low-energy theory" of the graviton $g_{\mu\nu}^{(0)}$, photon $A_\mu^{(0)}$, and scalar $\phi^{(0)}$. In the case of S^i, this ansatz is the same as dimensional reduction,[5] i.e., taking fields to be independent of the extra coordinates, discarding the $n\neq 0$ modes in (2.3) so that $g_{\mu\nu}$, A_μ, and ϕ depend only on x^μ.

(3) The ground-state metric

$$\langle g_{\mu\nu}\rangle = \begin{bmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{bmatrix} . \tag{2.4}$$

This describes the vacuum expectation value (VEV) of $g_{\hat{\mu}\hat{\nu}}(x,y)$ and contains neither massless nor massive fluctuations. It is this "ground-state metric" which determines the unbroken symmetries of the vacuum not the Kaluza-Klein ansatz nor the full metric.

The action (2.1) is invariant under five-dimensional general coordinate transformations with parameters $\xi^{\hat{\mu}}$:

$$\delta g_{\hat{\mu}\hat{\nu}}=\partial_{\hat{\mu}}\xi^{\hat{\rho}}g_{\hat{\rho}\hat{\nu}}+\partial_{\hat{\nu}}\xi^{\hat{\rho}}g_{\hat{\mu}\hat{\rho}}+\xi^{\hat{\rho}}g_{\hat{\mu}\hat{\nu}} . \tag{2.5}$$

Dolan and Duff[6] have analyzed the four-dimensional symmetries of the theory which result from retaining the $n\neq 0$ modes of (2.3) and which describe, in addition to the above massless states an infinite tower of charged, massive, purely spin-2 particles.[7] Following Ref. 6 we also expand the general coordinate parameter $\xi^{\hat{\mu}}(x,y)$ of (2.5) in the form

$$\xi_{\hat{\mu}}(x,y)=\sum_{n=-\infty}^{\infty}\xi_{\hat{\mu}}^{(n)}(x)\exp(iny/R) , \tag{2.6}$$

with $\xi_{\hat{\mu}}^{(n)*}=\xi_{\hat{\mu}}^{(-n)}$. An important observation is that the assumed topology of the ground state, namely, $M^4\times S^1$, restricts us to general coordinate transformations periodic in y. Since the parameters $\xi_\mu^{(n)}(x)$ and $\xi_5^{(n)}(x)$ each correspond to spontaneously broken generators except for $n=0$, it follows that for $n\neq 0$, the fields $A_\mu^{(n)}(x)$ and $\phi^{(n)}(x)$ are the corresponding Goldstone-boson fields. The corresponding gauge fields $g_{\mu\nu}^{(n)}$, with two degrees of freedom, will then each acquire masses by absorbing the two degrees of freedom of each vector Goldstone boson $A_\mu^{(n)}$ and the one degree of freedom of each scalar Goldstone boson $\phi^{(n)}$ to yield a pure spin-2 massive particle with five degrees of freedom. It is easy to see that we generalize these results to $(4+K)$-dimensional Abelian theories.

In the $(4+K)$-dimensional Kaluza-Klein theories, $R_{\hat{\mu}\hat{\nu}}=0$ is consistent with the ground state $M^4\times S^1\times\cdots\times S^1$ and is metrically flat. The gauge group would be $[\mathrm{U}(1)]^K$. The flatness of the extra dimensions implies no potential for the scalar fields. This in turn means that the cosmological constant is zero and there is no Higgs mechanism.

III. RADIAL EQUATIONS IN 4+K DIMENSIONS

A. Action

Let us consider pure gravity in $4+K$ dimensions described by the action

$$S=-\frac{1}{16\pi G}\int d^{4+K}x\sqrt{-g}R_{4+K} , \tag{3.1}$$

where R_{4+K} is the $(4+K)$-dimensional curvature scalar. It is assumed on the partially compactified manifold, $M^4\times S^1_{(1)}\times\cdots\times S^1_{(K)}$, where M^4 denotes four-dimensional Minkowski space and $S^1_{(i)}$ is a circle of radius R_i. The metric $g_{\hat{\mu}\hat{\nu}}$ becomes a periodic function of y^i with a periodicity of $2\pi R_i$. Next consider the change of variables

$$\begin{aligned} g_{\hat{\mu}\hat{\nu}}(x,y)&=g_{\mu\nu}(x,y)dx^\mu\otimes dx^\nu \\ &\quad+\Phi_{ij}(x,y)[dy^i+A^i_\mu(x,y)dx^\mu] \\ &\quad\otimes[dy^j+A^j_\mu(x,y)dx^\mu] . \end{aligned} \tag{3.2}$$

The fields $g_{\mu\nu}$, A^i_μ, and Φ_{ij} may be Fourier expanded in the form

$$g_{\mu\nu}(x,y)=\sum_{n_1\cdots n_K} g_{\mu\nu}^{(n_1\cdots n_K)}(x)\exp[i(n_1y^1/R_1+\cdots+n_Ky^K/R_K)] , \tag{3.3}$$

$$A^i_\mu(x,y)=\sum_{n_1\cdots n_K} A_\mu^{i(n_1\cdots n_K)}(x)\exp[i(n_1y^1/R_1+\cdots+n_Ky^K/R_K)] , \tag{3.4}$$

$$\Phi_{ij}(x,y)=\sum_{n_1\cdots n_K} \Phi_{ij}^{(n_1\cdots n_K)}(x)\exp[i(n_1y^1/R_1+\cdots+n_Ky^K/R_K)] , \tag{3.5}$$

with

$$g_{\hat{\mu}\hat{\nu}}^{(n_1\cdots n_K)*}(x)=g_{\hat{\mu}\hat{\nu}}^{(n_1\cdots n_K)}(x) \quad\text{and}\quad -\pi R_i\le y^i\le \pi R_i .$$

B. Schwarzschild type

We make the static and spherically symmetric ansatz

$$\begin{aligned} &g_{\mu\nu}dx^\mu dx^\nu=-e^{2a}dt^2+e^{2b}dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2 , \\ &A^i_r=tp^i(r)h(y^i),\quad A^i_t=0 , \\ &A^i_\phi=-g^i\cos\theta h(y^i),\quad A^i_\theta=0 , \\ &\Phi_{ij}=h^{-1}(y^i)h^{-1}(y^j)\phi_{ij}(r) , \end{aligned} \tag{3.6}$$

where $h(y^i)$ is only dependent on the ith parameter y^i and $h(\pi R_i)=h(-\pi R_i)$ because of $h(y^i)$ should be a monotropic function.

The Riemann tensor is

$$R^{\hat{\mu}}_{\hat{\nu}\hat{\rho}\hat{\sigma}}=\partial_{\hat{\rho}}\Gamma^{\hat{\mu}}_{\hat{\nu}\hat{\sigma}}-\partial_{\hat{\sigma}}\Gamma^{\hat{\mu}}_{\hat{\nu}\hat{\rho}}+\Gamma^{\hat{\mu}}_{\hat{\delta}\hat{\rho}}\Gamma^{\hat{\delta}}_{\hat{\nu}\hat{\sigma}}-\Gamma^{\hat{\mu}}_{\hat{\delta}\hat{\sigma}}\Gamma^{\hat{\delta}}_{\hat{\nu}\hat{\rho}} , \tag{3.7}$$

$$R_{\hat{\nu}\hat{\rho}}=R^{\hat{\mu}}_{\hat{\nu}\hat{\mu}\hat{\rho}} , \tag{3.8}$$

$$R_{4+K}=R^{\hat{\nu}}{}_{\hat{\nu}} . \tag{3.9}$$

With this ansatz, we find the following nonvanishing Christoffel connection:

$$\begin{aligned}
&\Gamma^0_{10}=a', \quad \Gamma^0_{11}=e^{-2a}\phi_{ij}\dot{A}^i_r A^j_r, \quad \Gamma^0_{31}=e^{-2a}\phi_{ij}A^i_\phi \dot{A}^j_r/2 ,\\
&\Gamma^0_{i1}=e^{-2a}\phi_{ij}\dot{A}^j_r/2, \quad \Gamma^1_{00}=a'e^{2a-2b} ,\\
&\Gamma^1_{10}=e^{-2b}\phi'_{ij}\dot{A}^i_r A^j_r/2, \quad \Gamma^1_{30}=e^{-2b}\phi_{ij}A^i_\phi \dot{A}^j_r/2 ,\\
&\Gamma^1_{i0}=e^{-2b}\phi_{ij}\dot{A}^j_r/2 , \quad \Gamma^1_{11}=b'-e^{-2b}\phi'_{ij}A^i_r A^j_r/2 ,\\
&\Gamma^1_{31}=-e^{-2b}\phi'_{ij}A^i_r A^j_\phi/2, \quad \Gamma^1_{i1}=-e^{-2b}\phi'_{ij}A^j_r/2 ,\\
&\Gamma^1_{22}=-re^{-2b}, \quad \Gamma^1_{33}=-e^{-2b}(r\sin^2\theta+\phi'_{ij}A^i_\phi A^j_\phi/r) ,\\
&\Gamma^1_{i3}=-e^{-2b}\phi'_{ij}A^j_\phi/2, \quad \Gamma^1_{ij}=-e^{-2b}\phi_{ij}/2 ,\\
&\Gamma^3_{31}=\Gamma^2_{21}=1/r, \quad \Gamma^2_{31}=-\phi_{ij}A^i_r(\partial_\theta A^j_\phi)/2r^2 ,\\
&\Gamma^2_{33}=-\sin\theta\cos\theta-(\phi_{ij}A^i_\phi\partial_\theta A^j_\phi)/r^2 ,\\
&\Gamma^i_{j1}=(M^i_j+e^{-2b}\phi'_{Kj}A^i_r A^K_r)/2 ,\\
&\Gamma^2_{i3}=-\phi_{ij}(\partial_\theta A^j_\phi)/2r^2, \quad \Gamma^3_{21}=D\phi_{ij}A^i_r\partial_\theta A^j_\phi ,\\
&\Gamma^3_{32}=\cot\theta+D\phi_{ij}A^i_\phi\partial_\theta A^j_\phi , \quad \Gamma^3_{i2}=D\phi_{ij}\partial_\theta A^j_\phi ,\\
&\Gamma^i_{00}=-a'A^i_r e^{2a-2b}, \quad \Gamma^i_{10}=(1-e^{-2b}\phi_{Kj}A'^K_r A^j_r)\dot{A}^i_r/2 ,\\
&\Gamma^i_{30}=-e^{2b}\phi_{jk}\dot{A}^j_r A^K_\phi A'^i_r/2, \quad \Gamma^i_{j0}=-e^{-2b}A^i_r\phi_{jK}\dot{A}^K_r/2 ,\\
&\Gamma^i_{11}=-b'A^i_r+M^i_K A^K_r+A'^i_r+e^{-2b}\phi'_{Kj}A^K_r A^j_r A^i_r ,\\
&\Gamma^i_{21}=-D\phi_{jK}A^i_\phi A^j_r\partial_\theta A^K_\phi ,\\
&\Gamma^i_{31}=-A^i_\phi/r+(M^i_K+e^{-2b}\phi_{Kj}A^i_r A^j_r)A^K_\phi/2 ,\\
&\Gamma^i_{22}=e^{-2b}rA^i_r ,\\
&\Gamma^i_{32}=-\tan\theta A^i_\phi+\partial_\theta A^i_\phi/2-D\phi_{jK}A^i_\phi A^j_\phi\partial_\theta A^K_\phi/2 ,\\
&\Gamma^i_{33}=(r\sin^2\theta+\phi'_{jK}A^j_\phi A^K_\phi/2)e^{-2b}A^i_r ,\\
&\Gamma^i_{j2}=-D\phi_{Kj}A^i_\phi\partial_\theta A^K_\phi ,\\
&\Gamma^i_{j3}=e^{-2b}\phi'_{Kj}A^i_r A^K_\phi/2 , \quad \Gamma^i_{jK}=e^{-2b}\phi'_{jK}A^i_r+\delta_{ij}\delta_{jK}\chi ,
\end{aligned} \tag{3.10}$$

where

$$D=(2r^2\sin^2\theta)^{-1}, \quad M^i_j=\phi^{iK}\phi'_{Kj} ,$$

$$\phi^{iK}\phi_{Kj}=\delta^i_j, \quad \chi=h(y^i)\partial h^{-1}(y^i)/\partial y^i ,$$

and

$$a'=da/dr, \quad \dot{A}_r=\partial A_r/\partial t, \text{ etc.}$$

In the absence of other fields, the equations of motion are, of course, $R_{\hat{\mu}\hat{\nu}}=0$. The field equations can be put in the following form:

$$\frac{dp^i}{dr}+M^i_j p^j=\left[a'+b'-\frac{2}{r}-\tfrac{1}{2}M^K_K\right]p^i , \tag{3.11}$$

$$M'^i_j+M^i_j\left[a'-b'+\frac{2}{r}+\tfrac{1}{2}M^K_K\right] =\left[\frac{1}{r^4}e^{2b}g^l g^i-e^{-2a}p^l p^i\right]\phi_{jl} , \tag{3.12}$$

$$2a''+2a'^2-2(ab)'-4b'/r =-M'^K_K+e^{2a+4b}\phi_{ij}p^i p^j+b'M^K_K-M^i_j M^j_i/2 , \tag{3.13}$$

$$2a''+2a'^2-2(ab)'+4a'/r=e^{2a+4b}\phi_{ij}p^i p^j-M^K_K a' , \tag{3.14}$$

$$e^{2b}=e^{2b}\phi_{ij}g^i g^j/2r^2+r(a'-b'+M^K_K/2)+1 . \tag{3.15}$$

C. GPS type

When we used isotropic coordinates, the static and spherically symmetric ansatz of Gross-Perry-Sorkin type is

$$\begin{aligned}
&g_{\mu\nu}dx^\mu dx^\nu=-e^{2a}dt^2+e^{2b}(dr^2+r^2\alpha\theta^2+r^2\sin^2\theta\, d\phi^2) ,\\
&A^i_\phi=-g^i\cos\theta h(y^i), \quad A^i_t=p^i h(y^i) ,\\
&\Phi_{ij}=h^{-1}(y^i)h^{-1}(y^j)\phi_{ij}(r), \quad A^i_r=A^i_\theta=0 ,
\end{aligned} \tag{3.16}$$

where $h(y^i)$ is only dependent on the ith parameter y^i and $h(\pi R_i)=h(-\pi R_i)$ because of $h(y^i)$ should be a monotropic function, and p^i are constants. The equations of motion can be put in the following form:

$$a''+a'\left[\frac{2}{r}+a'+b'+\tfrac{1}{2}M^K_K\right]=0 , \tag{3.17}$$

$$a''+a'^2+2b''+\frac{2}{r}b'+\tfrac{1}{2}M'^K_K+\tfrac{1}{4}M^K_i M^i_K -a'b'-\tfrac{1}{2}M^K_K b'=0 , \tag{3.18}$$

$$b''+b'\left[\frac{2}{r}+a'+b'+\tfrac{1}{2}M^K_K\right]+\frac{1}{r}(a'+b'+\tfrac{1}{2}M^K_K) +\frac{1}{2r^4}\phi_{ij}g^i g^j e^{-2b}=0 , \tag{3.19}$$

$$M'^i_j+\left[\frac{2}{r}+a'+b'+\tfrac{1}{2}M^K_K\right]+\frac{1}{r^4}\phi_{Kj}g^i g^k e^{-2b}=0 , \tag{3.20}$$

where $M^i_j=\phi^{iK}\phi'_{Kj}$.

IV. SOLUTIONS FOR $K=1,2$

A. Schwarzschild type

In the $K=1$ case, straightforward calculation gives complete integration of field equations (2.11)—(2.15) for generalized monopole solutions:

$$p'(r)=0 , \tag{4.1}$$

$$\lambda^2(g^1)^2\exp(-2a)\alpha^{-2}\sinh^2\alpha(a+\beta) \times[(\lambda/\delta)\sinh(\delta a/\lambda)]^{-4}=r^4 , \tag{4.2}$$

$$\exp\phi_{11} = -(\delta/r)\exp(-a)[\sinh(\delta a/\lambda)]^{-1} , \tag{4.3}$$

$$\exp b = -ra'[(\lambda/\delta)\sinh(\delta a/\lambda)]^{-1} , \tag{4.4}$$

where λ, α, β are integration constants and the following necessary condition is satisfied:

$$4(\delta/\lambda)^2 - \alpha^2 = 3 . \tag{4.5}$$

In the $K=2$ case, we shall define

$$\begin{aligned} \phi_{11} &= t\exp(\tfrac{1}{2}\mathrm{Tr}\ln\phi) , \\ \phi_{12} = \phi_{21} &= s\exp(\tfrac{1}{2}\mathrm{Tr}\ln\phi) , \\ \phi_{22} &= q\exp(\tfrac{1}{2}\mathrm{Tr}\ln\phi) , \end{aligned} \tag{4.6}$$

where $tq - s^2 = 1$ and we also write

$$g_1 = g^1 t + g^2 s, \quad g_2 = g^1 s + g^2 q . \tag{4.7}$$

Change the variable to z such that

$$z = \int dr (1/r^2)\exp(b - a - \tfrac{1}{2}\mathrm{Tr}\ln\phi) . \tag{4.8}$$

Straightforward calculation gives complete integration of field equations for generalized monopole solutions:

$$r^2 = [\alpha/\sin(\alpha z)^2]\beta^{2/3}\exp(-2a/3)\Phi_-(z) , \tag{4.9}$$

$$4(\alpha^2 - 2\dot{a}^2/3) = -(\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1) , \tag{4.10}$$

$$\dot{a} = -3\gamma/4g^1 g^2 , \tag{4.11}$$

$$(g^1 g_1 - g^2 g_2)^2 = (2g^1 g^2)^2 (1/\Phi_-)(\dot{\Phi}_+)^2 , \tag{4.12}$$

$$\begin{aligned} 4g^1 g^2 s &= (g^1 g_1 + g^2 g_2) \\ &\quad - (2g^1 g^2)^2 \Phi_+ \ddot{\Phi}_+ / (g^1 g_1 + g^2 g_2)\Phi_- , \end{aligned} \tag{4.13}$$

$$(g^1 g_1 + g^2 g_2)\exp[\tfrac{3}{2}(\mathrm{Tr}\ln\phi) + 2a] = -\Phi_+/\Phi_-{}^2 , \tag{4.14}$$

$$(g^1 g_1 + g^2 g_2)^2 \Phi_- = \beta^2 \Phi_+{}^2 , \tag{4.15}$$

$$r\sinh(\alpha z) \doteq \alpha\exp(b - a\tfrac{1}{2}\mathrm{Tr}\ln\phi) , \tag{4.16}$$

$$\begin{aligned} \Phi_\pm = {} & \exp[\pm\omega_1(z - z_1)]/(\omega_1 - \omega_2)(\omega_3 - \omega_1) \\ & + \exp[\pm\omega_2(z - z_2)]/(\omega_1 - \omega_2)(\omega_2 - \omega_3) \\ & + \exp[\pm\omega_3(z - z_3)]/(\omega_2 - \omega_3)(\omega_3 - \omega_1) , \end{aligned} \tag{4.17}$$

where α, β, γ, $\omega_1 > \omega_2 > \omega_3$, and z_i are integration constants and the following necessary conditions are satisfied:

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad \omega_1 z_1 + \omega_2 z_2 + \omega_3 z_3 = 0 . \tag{4.18}$$

However, these solutions have naked singularities.

B. GPS type

In the $K=1$ case, we shall take $a=0$, $\phi_{11} = e^{-2b}$ and the boundary condition $b(\infty) = 0$. Equations (3.17) and (3.18) would be of the form

$$\begin{aligned} & b'' + \frac{2}{r}b' + b'^2 = 0 , \\ & b' = -\frac{|g^1|}{2r^2}e^{-2b} . \end{aligned} \tag{4.19}$$

The required solution takes the form

$$b = \tfrac{1}{2}\ln\left[1 + \frac{|g^1|}{r}\right] . \tag{4.20}$$

The y-dependent monopole solution may be written as

$$ds^2 = -dt^2 = \left[1 + \frac{|g^1|}{r}\right](dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2) + \left[\left[1 + \frac{|g^1|}{r}\right]f^2(y^1)\right]^{-1}[dy + g^1\cos\theta f(y^1)d\phi]^2 , \tag{4.21}$$

which is the direct generalization of the Gross-Perry-Sorkin solution. Like that solution these monopoles are nonsingular.

In the $K=2$ case, straightforward calculation gives a class of solutions which are nonsingular. These solutions are

$$\begin{aligned} & a = 0, \quad b = \tfrac{1}{2}\ln\left[1 + \frac{1}{r}[\phi_{ij}(\infty)g^i g^j]^{1/2}\right] , \\ & \phi_{ij} = \left[1 + \frac{1}{r}[\phi_{ij}(\infty)g^i g^j]^{1/2}\right]^{-1/2}(F^T e^{-\sigma_3 b}F) \\ & \qquad\qquad (i,j = 1,2) , \end{aligned} \tag{4.22}$$

where σ_3 is the Pauli matrix and $F^T \cdot F = \phi(\infty)$.

V. AVERAGE FIELD STRENGTH

We have shown the general monopole solution which is dependent on y^i in the Abelian Kaluza-Klein theory. Next, the average field strength is defined on the point of four-dimensional spacetime. Let us consider the field multiplet $(i,j,K) \equiv (A_\mu^{(i)}, g_{\mu\nu}^{(j)}, \phi^{(K)})$ in five-dimensional Kaluza-Klein theory. As yet the Kaluza-Klein monopole solution of the field multiplet (0,0,0) was investigated by many authors.[2–4,8–12] Now, for a general solution, we have

$$F(\theta,\phi) = g^1 h(y^1)\sin\theta , \tag{5.1}$$

where $-\pi R_1 \le y^1 \le \pi R_1$ and $f(\pi R_1) = f(-\pi R_1)$. On the point of four-dimensional spacetime, the average field strength is defined as

$$\bar{F}_{\theta\phi} = \int_{-\pi R_1}^{\pi R_1} g^1 h(y^1)\sin\theta\, dy^1/2\pi R_1 . \tag{5.2}$$

If the $h(y^1)$ is continuous, then there exists at least one point y_0 on the circle such that (the mean value theorem)

$$\bar{F}_{\theta\phi} = gh(y_0)\sin\theta . \tag{5.3}$$

We choose $h(y^1) = \exp(i\xi y^1/R_1)$ where ξ is any real number; then

$$\bar{F}_{\theta\phi} = g^1 \sin\pi\xi \sin\theta / 2\pi\xi . \tag{5.4}$$

If we take $\xi=n$ (n is a nonzero integer), then the solution of $(n,0,-2n)$ is described by (3.6) and (4.1)–(4.4) but their average field strength vanishes.

It is easy to see that we generalize the average field strength to $(4+K)$-dimensional theory.

VI. SUMMARY

The classical vacuum solution of the Kaluza-Klein theory is assumed to be $M^4\times B$. Any field configuration which approaches the vacuum solution at spatial infinity thus defines a B bundle over the sphere at spatial infinity S^2. If the B bundle over S^2 cannot be continuously deformed to the global direct products $S^2\times B$, then the field configuration cannot be continuously deformed to the vacuum solution. We conclude that it is a topological soliton. In the effective low-energy five-dimensional theory, the simplest case is the Gross-Perry-Sorkin monopole. It is well known that the effective low-energy theory can be deduced by considering the metric $g_{\hat{\mu}\hat{\nu}}$ to be independent of y.

In this paper, we consider the full metric $g_{\hat{\mu}\hat{\nu}}$ depending on x and y. This describes the ground state plus both massless ($n=0$) and ($n\neq 0$) modes. Following Ref. 6, an important observation is that the assumed topology of the ground state, namely, $M^4\times S^1\times\cdots\times S^1$ restricts us to general coordinate transformations periodic in y^i. The y-dependent generalized monopole solutions are the field configurations of the vector Goldstone boson $A_\mu^{(n)}$, scalar Goldstone boson $\phi^{(n)}$, and massless fields ($n=0$). In fact, $g_{\hat{\mu}\hat{\nu}}^{(n)}$ with two degrees of freedom will then each acquire masses by absorbing the two degrees of freedom of each $A_\mu^{(n)}$ and the one degree of freedom of each $\phi^{(n)}$ to yield a pure spin-2 massive particle with five degrees of freedom. For this configuration we have shown the simplest case, i.e., y-dependent Gross-Perry-Sorkin solution which is nonsingular.

Recently, there has been much interest in the problem of bound states of a fermion in the field of a fixed Dirac monopole[13,14] or in a monopole of the unification of the fundamental interactions.[15–17]

The above generalized y-dependent monopole may be considered as a background field which is topologically different from the vacuum background field. It is interesting to examine the dynamical property of a fermion in this monopole background field which will be reported elsewhere.

[1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. **Kl**, 966 (1921); O. Klein, Z. Phys. **37**, 895 (1926).
[2] T. Appelquist and A Chodos, Phys. Rev. D **28**, 772 (1983).
[3] H. M. Lee and S. C. Lee, Phys. Lett. **149B**, 95 (1984).
[4] S. C. Lee, Phys. Lett. **149B**, 100 (1984).
[5] E. Cremmer, in *Supergravity '81*, proceedings of the Trieste Supergravity School, 1981, edited by S. Ferrara and J. G. Taylor (Cambridge University Press, Cambridge, England, 1982).
[6] L. Dolan and M. J. Duff, Phys. Rev. Lett. **52**, 14 (1984).
[7] A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1984).
[8] D. J. Gross and M. J. Perry, Nucl. Phys. **B226**, 29 (1983).
[9] R. D. Sorkin, Phys. Rev. Lett. **51**, 87 (1983).
[10] S. C. Lee, Phys. Lett. **149B**, 98 (1984).
[11] Z. F. Ezawa and I. G. Koh, Phys. Lett. **140B**, 205 (1984).
[12] I. G. Angus, California Institute of Technology Report No. CALT-68-1276, 1985 (unpublished).
[13] Li Xin-zhou, Wang Ke-lin, and Zhang Jian-zu, Phys. Lett. **148B**, 89 (1984).
[14] Li Xin-zhou and Zhang Jian-zu, Phys. Rev. D **33**, 562 (1986).
[15] Li Xin-zhou, Wang Ke-lin, and Zhang Jian-zu, Phys. Lett. **140B**, 209 (1984); Nuovo Cimento A **75**, 87 (1983); **80**, 311 (1983); **82**, 377 (1984); Kexue Tongbao **29**, 1307 (1984).
[16] Li Xin-zhou, Nuovo Cimento A **83**, 166 (1984).
[17] Z. F. Ezawa and A. Iwazaki, Phys. Lett. **138B**, 81 (1984).