Finite quantum field theories

Wolfgang Lucha and Helmut Neufeld*
Institut für Theoretische Physik, Universität Wien, Vienna, Austria
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We investigate the relation between finiteness of a four-dimensional quantum field theory and global supersymmetry. To this end we consider the most general quantum field theory and analyze the finiteness conditions resulting from the requirement of the absence of divergent contributions to the renormalizations of the parameters of the theory. In addition to the gauge bosons, both fermions and scalar bosons turn out to be a necessary ingredient in a nontrivial finite gauge theory. In all cases discussed, the supersymmetric theory restricted by two well-known constraints on the dimensionless couplings proves to be the unique solution of the finiteness conditions.

I. INTRODUCTION

The most exciting feature of supersymmetry is its ability to soften the high-energy behavior of quantum field theories by reducing the number of uncorrelated ultraviolet divergences. This property rendered possible the construction of finite supersymmetric quantum field theories in four space-time dimensions:[1] Supersymmetry suffices to ensure finiteness in quantum field theories singled out from the general case by certain relations—called "finiteness conditions"—between the dimensionless couplings in the theory. Simple supersymmetry in cooperation with two finiteness conditions guarantees finiteness up to two loops. Extended supersymmetry imposes, of course, still more restrictions on a theory. As a consequence, $(N=2)$ supersymmetric theories constrained by only a single finiteness condition are finite to all orders of perturbation theory.

In this work we invert the logic and investigate the following problem: Which classes of theories are allowed when imposing the requirement of finiteness upon the most general renormalizable quantum field theory? In particular, we are interested in the question of whether or not finiteness necessarily implies supersymmetry for the particle content and the interactions in the theory. To this end we analyze the finiteness conditions obtained by demanding the absence of divergent contributions to the renormalizations of the parameters of a general gauge theory.

Motivated by the observation that supersymmetric theories are free of quadratic divergences, similar investigations have been performed previously with respect to the absence of quadratic divergences in a renormalizable quantum field theory.[2] In all special cases studied, the requirement of the cancellation of the quadratic divergences—either up to two-loop order or with the additional restriction of renormalization-group invariance of the one-loop conditions—uniquely leads to the (softly broken) supersymmetry of the theory. Supersymmetry then ensures the absence of quadratic divergences to all orders in the loop expansion.

This paper is organized as follows: In order to embed the present investigation in the ongoing developments and to establish our notation we give in Sec. II a brief sketch of supersymmetric finite quantum field theories. Section III is devoted to the renormalization of a general gauge theory. In Sec. IV we formulate the finiteness conditions for an arbitrary gauge theory and deduce some immediate implications. In their most general form these finiteness conditions constitute, however, an extremely complicated nonlinear set of equations for masses and coupling constants. Consequently, in Sec. V, we consider a somewhat restricted class of theories which, nevertheless, still comprehends all supersymmetric theories and hence all finite quantum field theories in four dimensions known so far. Although not fully general, the discussion of these models is quite instructive in order to answer the question of whether or not finiteness implies supersymmetry (SUSY). Our conclusions are summarized in Sec. VI.

II. FINITE SUPERSYMMETRIC QUANTUM FIELD THEORIES

The most general renormalizable, gauge-invariant and $(N=1)$ supersymmetric theory is described by the Lagrangian

$$\mathscr{L}_{(N=1)}=\int d^4\theta\,\Phi^\dagger e^{2gV}\Phi+\left[\int d^2\theta\left[\frac{1}{4T(R)}\operatorname{Tr}W^\alpha W_\alpha+W(\Phi)\right]+\text{H.c.}\right]. \tag{2.1}$$

Here the following notation is adopted: Vector superfields, which represent a massless vector boson V_μ as well as a two-component Weyl spinor λ, both of them transforming according to the adjoint representation G of the gauge group, are denoted by V,

$$V_a=(\lambda_a,V_a^\mu)\sim G,\quad V=V^\dagger\equiv V_aT^a\,. \tag{2.2}$$

Chiral superfields, which represent a two-component Weyl spinor χ as well as a complex scalar boson A, both of them transforming according to some representation R of the gauge group, are denoted by Φ,

$$\Phi_i=(A_i,\chi_i)\sim R,\ \bar{D}_{\dot\alpha}\Phi_i=0\ . \tag{2.3}$$

The chiral field-strength superfield W_α is responsible for the kinetic Lagrangian of gauge bosons and gauge fermions:

$$W_\alpha=-\frac{1}{8g}\bar{D}\,\bar{D}e^{-2gV}D_\alpha e^{2gV}\ .$$

$D_\alpha,\bar{D}_{\dot\alpha}$ label the SUSY-covariant derivatives. $W(\Phi)$ is the so-called superpotential, a gauge-invariant, analytic function of the chiral superfields Φ_i describing all mass terms, Yukawa couplings, and scalar self-interactions in the theory. Renormalizability restricts it to be a polynomial of at most third degree,

$$W(\Phi)=s_i\Phi_i+\tfrac{1}{2}m_{ij}\Phi_i\Phi_j+\tfrac{1}{3}c_{ijk}\Phi_i\Phi_j\Phi_k\ . \tag{2.4}$$

Finally, θ labels the fermionic Grassmann coordinates of the superspace $(x^\mu,\theta_\alpha,\bar\theta_{\dot\alpha})$.

The group invariants for a (possibly reducible) representation R are defined in terms of the generators T^a as usual: The quadratic Casimir operator $C_2(R)$ is defined by

$$\sum_\sigma C_2(R_\sigma)E^\sigma_{ik}\equiv(T^a_R T^a_R)_{ik}\ , \tag{2.5}$$

where E^σ denotes the projector onto the irreducible representation R_σ in the decomposition

$$R=\bigoplus_\sigma R_\sigma\ .$$

The second-order Dynkin index $T(R)$ is defined by

$$T(R)\delta_{ab}\equiv\mathrm{Tr}(T^a_R T^b_R),\ \ T(R)=\sum_\sigma T(R_\sigma)\ . \tag{2.6}$$

These invariants are related to each other by the dimension of the group, $d(G)$, and the dimension of the representation R, $d(R)$, according to

$$T(R)d(G)=\sum_\sigma C_2(R_\sigma)d(R_\sigma)\ . \tag{2.7}$$

Specified to the adjoint representation G, the above definitions read

$$C_2(G)\delta_{ab}\equiv(T^c_G T^c_G)_{ab}=f_{acd}f_{bcd}$$
$$=\mathrm{Tr}(T^a_G T^b_G)\equiv T(G)\delta_{ab}\ , \tag{2.8}$$

i.e.,

$$C_2(G)=T(G)\ , \tag{2.9}$$

where f_{abc} is the structure constant tensor of the gauge group, $[T^a,T^b]=if_{abc}T^c$.

The only possible supersymmetric and gauge-invariant extension of the Lagrangian (2.1) would be a so-called "D term"

$$\mathscr{L}_D=\eta\int d^4\theta\, V \tag{2.10}$$

associated with a U(1) factor of the gauge group. This D term receives, at the one-loop level only, a quadratically divergent contribution proportional to the trace of the U(1) charge.[3] It will thus not be present in any theory based on a semisimple gauge group.

Suppressing all indices, one counts six renormalization constants for the theory characterized by the Lagrangian (2.1), viz., the wave-function renormalizations for vector and chiral superfield,

$$V_0=Z_V{}^{1/2}V,\ \ \Phi_0=Z_\Phi{}^{1/2}\Phi\ , \tag{2.11}$$

the gauge-coupling-constant renormalization,

$$g_0=Z_g g\ , \tag{2.12}$$

as well as the renormalization of the parameters in the superpotential (2.4),

$$s_0=Z_s s,\ \ m_0=Z_m m,\ \ c_0=Z_c c\ . \tag{2.13}$$

However, not all of these renormalization constants are independent. The general line of arguments for this runs as follows.

First of all, in the background-field method—when employed[4]—the product of gauge coupling constant g times vector superfield V is not renormalized at all,[5] $g_0V_0=gV$, which implies the relation

$$Z_g Z_V{}^{1/2}=1 \tag{2.14}$$

for the corresponding renormalization constants.

Furthermore, there is a nonrenormalization theorem, valid for theories invariant under N-extended supersymmetry. In its general form this theorem states that in an N-extended supersymmetric theory any quantum contribution to the effective action must be an integral over the *full* extended superspace $(x^\mu,\theta^L_\alpha,\bar\theta^L_{\dot\alpha})$, $L=1,2,\ldots,N$, and that any contribution arising above the one-loop level must be a gauge invariant function of the matter superfields and the Yang-Mills potentials (connections) A^L_α only.[6]

In the case of $(N=1)$ supersymmetry the above one-loop exception does not exist. The $(N=1)$ nonrenormalization theorem simply states that any quantum contribution to the effective action has to be an integral over the complete superspace $(x^\mu,\theta_\alpha,\bar\theta_{\dot\alpha})$ (Refs. 7 and 8). Consequently, all counterterms in the Lagrangian (2.1) must be of the form

$$\mathscr{L}_{\mathrm{CT}}=\int d^4\theta f(V,\Phi)\ , \tag{2.15}$$

where f is a function of the superfields and their covariant derivatives. This in turn implies that the superpotential $W(\Phi)$—being integrated over only a subspace of the whole superspace—is not renormalized at all, which is expressed by the relations

$$Z_s Z_\Phi{}^{1/2}=1,\ \ Z_m Z_\Phi=1,\ \ Z_c Z_\Phi{}^{3/2}=1 \tag{2.16}$$

for the renormalization constants of the parameters in the superpotential.

As a consequence of Eqs. (2.14) and (2.16), one is left with only two independent renormalization constants, for instance, Z_g and Z_Φ. For a supersymmetric theory the whole renormalization procedure can be carried through with the gauge-coupling-constant renormalization as well as wave-function renormalizations of the chiral superfields. Only the gauge β function $\beta_g\equiv\mu(\partial/\partial\mu)g$ and the anomalous dimensions

$$\gamma_{ij} \equiv (Z_\Phi{}^{-1/2})_{ik}\mu \frac{\partial}{\partial\mu}(Z_\Phi{}^{1/2})_{kj}$$

of the chiral superfields are of interest for the discussion of the high-energy behavior of supersymmetric theories.

The one-loop contributions to the β function and the anomalous dimensions are given by

$$\beta_g^{[1]} = -\frac{g^3}{(4\pi)^2}[3C_2(G) - T(R)] , \tag{2.17}$$

$$\gamma_{ij}^{[1]} = -\frac{2}{(4\pi)^2}[g^2(T^aT^a)_{ij} - c_{ikl}^* c_{jkl}] . \tag{2.18}$$

Hence imposition of the finiteness conditions

$$3C_2(G) = T(R) , \tag{2.19}$$

$$c_{ikl}^* c_{jkl} = g^2(T^aT^a)_{ij} \tag{2.20}$$

guarantees the finiteness of an otherwise arbitrary supersymmetric theory at the one-loop level.[9] Even a certain amount of soft supersymmetry breaking can be tolerated without upsetting one-loop finiteness.[10] The anomaly-free solutions of the finiteness condition (2.19) may be found in Ref. 11.

Moreover, it has been shown that (in supersymmetric theories) one-loop finiteness automatically implies two-loop finiteness, i.e., the finiteness conditions (2.19) and (2.20) suffice to enforce the vanishing of the two-loop contributions to the β function and anomalous dimensions too.[12] Finiteness will, however, in general be destroyed at the three-loop level.[13] Nevertheless, attempts have been undertaken in order to construct a realistic two-loop finite SUSY SU(5) grand unified theory.[14]

Renormalizable $(N=2)$ supersymmetric theories know of two basic building blocks: The $(N=2)$ vector multiplet $V_{N=2}$, transforming according to the adjoint representation G of the gauge group, contains an $(N=1)$ vector superfield V and a chiral superfield Φ in the adjoint representation:

$$V_{N=2} = (V, \Phi) \sim G, \quad V \sim G, \quad \Phi \sim G . \tag{2.21}$$

The hypermultiplet H, transforming according to some representation R_H of the gauge group, contains two chiral superfields Φ_1, $\Phi_2^\dagger$ of opposite chirality—a circumstance which makes every $(N=2)$ supersymmetric theory nonchiral, i.e., vectorlike,

$$H = (\Phi_1, \Phi_2^\dagger) \sim R_H, \quad \Phi_1 \sim R_H, \quad \Phi_2 \sim \bar{R}_H . \tag{2.22}$$

The most general renormalizable, $(N=2)$ supersymmetric Lagrangian, expressed in terms of $(N=1)$ superfields, reads

$$\mathcal{L}_{(N=2)} = \int d^4\theta (\Phi_1^\dagger e^{2gV}\Phi_1 + \Phi_2^\dagger e^{-2gV^T}\Phi_2 + \Phi^\dagger e^{2gV}\Phi) + \left[\int d^2\theta \left[\frac{1}{4T(R)} \mathrm{Tr} W^a W_a + ig\sqrt{2}\Phi_2\Phi\Phi_1 \right] + \mathrm{H.c.} \right] . \tag{2.23}$$

Note that $(N=2)$ supersymmetry restricts the superpotential $W(\Phi)$ to a unique trilinear interaction fixed by gauge invariance,

$$W(\Phi)_{(N=2)} = ig\sqrt{2}\Phi_2\Phi\Phi_1 . \tag{2.24}$$

Hence there is only a single coupling constant, that is the gauge coupling constant g, and only one independent renormalization constant, Z_g. The gauge β function is the only relevant quantity for finiteness considerations.

Now, the application of the nonrenormalization theorem to the $N=2$ case shows, on dimensional grounds, that $(N=2)$ supersymmetric theories are finite above the one-loop level.[15] The only possible contribution to the β function arises from one-loop graphs and can be obtained by setting $T(R) = C_2(G) + 2T(R_H)$ in Eq. (2.17), as demanded by the $(N=1)$ superfield content of the $(N=2)$ supermultiplets:

$$\beta_{(N=2)} = -\frac{2g^3}{(4\pi)^2}[C_2(G) - T(R_H)] . \tag{2.25}$$

Thus, finiteness to all orders of perturbation theory is achieved for

$$C_2(G) = T(R_H) . \tag{2.26}$$

Again, this finiteness is preserved by certain soft supersymmetry breaking operators.[16] The solutions of the finiteness condition (2.26) form a large class of finite $(N=2)$ supersymmetric quantum field theories in four space-time dimensions.[17] However, all efforts to build realistic models, based on $(N=2)$ supersymmetry amended by suitably chosen soft breaking terms, face a number of phenomenological obstacles.[18]

The $(N=4)$ super-Yang-Mills theory proves to be a special case of the $(N=2)$ theories for the hypermultiplet H transforming according to the adjoint representation of the gauge group, i.e., $R_H = G$. Renormalizable $(N=4)$ supersymmetric theories allow for exactly one $(N=4)$ supermultiplet, namely the $(N=4)$ vector multiplet $V_{N=4}$, which consists of an $(N=2)$ vector multiplet $V_{N=2}$ and a hypermultiplet H in the adjoint representation,

$$V_{N=4} = (V_{N=2}, H) = (V, \Phi_1, \Phi_2, \Phi_3) \sim G . \tag{2.27}$$

Finiteness to all orders of perturbation theory, $\beta_{(N=4)} \equiv 0$, is then a trivial consequence of Eq. (2.9) (Ref. 19).

Table I summarizes the evolution of the one-loop contribution to the gauge β function from the well-known expression in a general gauge theory to its automatic vanishing in the $(N=4)$ super-Yang-Mills theory.

III. RENORMALIZATION OF A GENERAL GAUGE THEORY

The most general renormalizable quantum field theory is (equivalent to) a spontaneously broken gauge theory,[20] described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + i\bar{\psi}_L \not{D}\psi_L + \tfrac{1}{2}(D_\mu\phi)^T D^\mu\phi - \tfrac{1}{2}[\bar{\psi}_R^c(m - h^m\phi_m)\psi_L + \mathrm{H.c.}] - V(\phi) + \text{gauge-fixing terms} + \text{ghost terms} + \text{counterterms} . \tag{3.1}$$

The particle content of this theory consists of Hermitian vector gauge fields V_μ^a associated with a compact gauge group, two-component Weyl spinor fields ψ_{iL} transforming according to some representation F of the gauge group, $\psi_{iL} \sim F$, and Hermitian scalar fields ϕ_m transforming according to some representation S of the gauge group, $\phi_m \sim S$. All fermions may be assumed to be, say, left handed because of $\psi_R{}^c \equiv C\overline{\psi}_R{}^T = \psi^c{}_L$. (For simplicity, we consider only the case of a simple gauge group. The generalization to nonsimple gauge groups is straightforward.) The gauge covariant field strength tensor $F^a_{\mu\nu}$ is given by

$$F^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g f_{abc} V^b_\mu V^c_\nu \ . \tag{3.2}$$

The covariant derivatives D_μ acting on the fermion and scalar fields are given by

$$D_\mu \psi_L = (\partial_\mu - ig V^a_\mu T_a)\psi_L \ , \tag{3.3}$$

$$D_\mu \phi = (\partial_\mu - ig V^a_\mu L_a)\phi \ , \tag{3.4}$$

where T_a and L_a are the Hermitian representation matrices for the generators of the gauge group in the fermion representation F and in the scalar boson representation S, respectively. Since, for the sake of generality, the scalars ϕ are assumed to be real, the representation matrices L_a have to be antisymmetric and purely imaginary. The fermion mass matrix m_{ik} as well as the Yukawa coupling matrices h^m_{ik} are symmetric in the fermion indices. The scalar potential $V(\phi)$ is a fourth-order polynomial in the scalar fields ϕ_m,

$$V(\phi) = a_m \phi_m + \tfrac{1}{2} b_{mn} \phi_m \phi_n + \frac{1}{3!} c_{mnp} \phi_m \phi_n \phi_p + \frac{1}{4!} d_{mnpq} \phi_m \phi_n \phi_p \phi_q \ , \tag{3.5}$$

with real and totally symmetric coefficients. Gauge invariance demands

$$m_{ji} T^a_{jk} + m_{jk} T^a_{ji} = 0 \tag{3.6}$$

for the fermion mass,

$$h^m_{ji} T^a_{jk} + h^m_{jk} T^a_{ji} + h^n_{ik} L^a_{nm} = 0 \tag{3.7}$$

for the Yukawa couplings, and

$$a_n L^a_{nm} = 0 \ , \tag{3.8}$$

$$b_{pn} L^a_{pm} + b_{pm} L^a_{pn} = 0 \ , \tag{3.9}$$

$$c_{qnp} L^a_{qm} + \text{cycl. perm. of } mnp = 0 \ , \tag{3.10}$$

$$d_{rnpq} L^a_{rm} + \text{cycl. perm. of } mnpq = 0 \ , \tag{3.11}$$

for the parameters in the scalar potential.

At this point, a closer inspection reveals that the linear term in the scalar potential can be dropped. For gauge nonsinglet scalars, in order not to violate gauge invariance explicitly, a_m has to vanish as a consequence of Eq. (3.8). For gauge singlet scalars the linear term can be made to disappear by an appropriate shift of singlet scalar fields without destroying the manifest gauge invariance of the Lagrangian. Hence $a_m = 0$ in any case.

Furthermore, finiteness is, of course, only relevant for the high-energy limit of the theory, i.e., in the unbroken phase of the gauge symmetry, far above all spontaneous symmetry-breaking thresholds, while at lower energies the decoupling of the comparatively heavy degrees of freedom will result in a nontrivial renormalization-group behavior of the parameters of the theory. According to this spirit, we focus our attention on a theory with an unbroken gauge symmetry. Then the real and symmetric scalar-boson mass-squared matrix $\mu^2{}_{mn}$ is given by $\mu^2{}_{mn} = b_{mn}$.

The most convenient gauge for performing high-energy investigations is the R_ξ gauge, which yields a propagator for massless vector bosons of the form

$$D(k)_{\mu\nu} = -\frac{1}{k^2}\left[g_{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \frac{k_\mu k_\nu}{k^2} \right] , \tag{3.12}$$

where the gauge parameter ξ is left arbitrary. This choice determines the gauge-fixing terms and ghost terms in the Lagrangian (3.1).

The renormalization constants required for the renormalization of a general gauge theory are defined in the

TABLE I. The gauge β function from $N=0$ to $N=4$, $\beta \equiv \mu(\partial/\partial\mu)g \equiv -[g^3/(4\pi)^2]b + O(g^5)$.

Supersymmetry	Supermultiplet	Contribution to b
$N=0$	$V_\mu \sim G$	$\frac{11}{3} C_2(G)$
	$\psi \sim F$	$-\frac{2}{3} T(F)$
	$\phi \sim S$	$-\frac{1}{6} T(S)$
		$\frac{11}{3} C_2(G) - \frac{2}{3} T(F) - \frac{1}{6} T(S)$
$N=1$	$V = (\lambda, V_\mu) \sim G$	$3 C_2(G)$
	$\Phi = (A, \chi) \sim R$	$-T(R)$
		$3 C_2(G) - T(R)$
$N=2$	$V_{N=2} = (V, \Phi) \sim G$	$2 C_2(G)$
	$H = (\Phi_1, \Phi_2^\dagger) \sim R_H$	$-2T(R_H)$
		$2[C_2(G) - T(R_H)]$
$N=4$	$V_{N=4} = (V_{N=2}, H) \sim G$	0

usual fashion:

$$V_0^\mu = Z_V{}^{1/2} V^\mu, \quad \psi_0 = Z_\psi{}^{1/2}\psi, \quad \phi_0 = Z_\phi{}^{1/2}\phi , \tag{3.13}$$

$$m_0 = m - \delta m, \quad \mu^2{}_0 = \mu^2 - \delta\mu^2 , \tag{3.14}$$

$$g_0 = Z_g g, \quad h_0 = Z_h h , \tag{3.15}$$

$$a_0 = \delta a, \quad c_0 = Z_c c, \quad d_0 = Z_d d , \tag{3.16}$$

where all group indices have been suppressed. The one-loop contributions to these renormalization constants, calculated by dimensional regularization in $D = 4 - 2\epsilon$ space-time dimensions[21] in the minimal subtraction scheme, read for the vector wave function

$$Z_V - 1 = \frac{1}{(4\pi)^2\epsilon} g^2 \left[\frac{1}{2}\left[\frac{13}{3} - \frac{1}{\xi}\right] C_2(G) - \tfrac{2}{3}T(F) - \tfrac{1}{6}T(S) \right] , \tag{3.17}$$

for the fermion wave function

$$(Z_\psi - 1)_{ik} = -\frac{1}{(4\pi)^2\epsilon}\left[\frac{g^2}{\xi} T_a T_a + \tfrac{1}{2} h^{n\dagger} h^n \right]_{ik} , \tag{3.18}$$

for the scalar wave function

$$(Z_\phi - 1)_{mn} = \frac{1}{(4\pi)^2\epsilon}\left[\left[3 - \frac{1}{\xi}\right] g^2 (L_a L_a)_{mn} - \mathrm{Re}\,\mathrm{Tr}(h^m h^{n\dagger}) \right] , \tag{3.19}$$

for the fermion mass

$$(\delta m)_{ik} = \frac{1}{(4\pi)^2\epsilon}[3g^2 m T_a T_a - h^n m^\dagger h^n - \tfrac{1}{4}(h^n h^{n\dagger} m + m h^{n\dagger} h^n)]_{ik} , \tag{3.20}$$

for the scalar mass

$$(\delta\mu^2)_{mn} = \frac{1}{(4\pi)^2\epsilon}\{3g^2(\mu^2 L_a L_a)_{mn} + 4\,\mathrm{Re}\,\mathrm{Tr}(m m^\dagger h^m h^{n\dagger}) + 2\,\mathrm{Re}\,\mathrm{Tr}(h^m m^\dagger h^n m^\dagger) - \tfrac{1}{2}[\mu^2{}_{mp}\mathrm{Re}\,\mathrm{Tr}(h^p h^{n\dagger}) + \mu^2{}_{np}\mathrm{Re}\,\mathrm{Tr}(h^p h^{m\dagger}) + d_{mnpq}\mu^2{}_{pq} + c_{mpq}c_{npq}]\} , \tag{3.21}$$

for the gauge coupling constant

$$Z_g - 1 = -\frac{1}{(4\pi)^2\epsilon}\tfrac{1}{2}g^2[\tfrac{11}{3}C_2(G) - \tfrac{2}{3}T(F) - \tfrac{1}{6}T(S)] , \tag{3.22}$$

for the Yukawa coupling constants

$$[(Z_h - 1)h]^m_{ik} = -\frac{1}{(4\pi)^2\epsilon}\tfrac{1}{2}\{3g^2[(h^m T_a T_a)_{ik} + (h^m T_a T_a)_{ki}] - [2h^n h^{m\dagger} h^n + \tfrac{1}{2}h^m h^{n\dagger} h^n + \tfrac{1}{2}h^n h^{n\dagger} h^m + h^n \mathrm{Re}\,\mathrm{Tr}(h^{m\dagger} h^n)]_{ik}\} , \tag{3.23}$$

for the parameter of the linear term in the scalar potential

$$(\delta a)_m = \frac{1}{(4\pi)^2\epsilon}\tfrac{1}{2}[4\,\mathrm{Re}\,\mathrm{Tr}(m^\dagger m m^\dagger h^m) + c_{mnp}\mu^2{}_{np}] , \tag{3.24}$$

for the three-scalar coupling constant

$$[(Z_c - 1)c]_{mnp} = -\frac{1}{(4\pi)^2\epsilon}\tfrac{1}{2}\{[3g^2(L_a L_a)_{mq} - \mathrm{Re}\,\mathrm{Tr}(h^m h^{q\dagger})]c_{npq} - 8\,\mathrm{Re}\,\mathrm{Tr}(m^\dagger h^m h^{n\dagger} h^p) - d_{mnqr}c_{pqr}\} + \text{cycl. perm. of } mnp , \tag{3.25}$$

and for the four-scalar coupling constant

$$[(Z_d - 1)d]_{mnpq} = \frac{1}{(4\pi)^2\epsilon}\tfrac{1}{2}([3g^4\{L_a, L_b\}_{mn}\{L_a, L_b\}_{pq} - 8\,\mathrm{Re}\,\mathrm{Tr}(h^m h^{n\dagger} h^p h^{q\dagger}) + d_{mnrs}d_{pqrs} + \text{cycl. perm. of } npq] - \{[3g^2(L_a L_a)_{mr} - \mathrm{Re}\,\mathrm{Tr}(h^m h^{r\dagger})]d_{npqr} + \text{cycl. perm. of } mnpq\}) . \tag{3.26}$$

Dimensional considerations show that quadratic divergences can only arise in one- and two-point Green's functions of scalar fields as well as in two-point Green's functions of vector fields. More precisely, they can only appear in the renormalization of the parameter of the linear term in the scalar potential and of the scalar and vector-boson masses. The quadratically divergent one-loop contributions to δa and $\delta\mu^2$, evaluated by cutoff regularization,[22] are

$$(\delta a)_m = -\frac{\Lambda^2}{(4\pi)^2}\tfrac{1}{2}[4\,\mathrm{Re}\,\mathrm{Tr}(m^\dagger h^m)+c_{mnn}] + O(\ln\Lambda^2) \tag{3.27}$$

and

$$(\delta\mu^2)_{mn} = \frac{\Lambda^2}{(4\pi)^2}[3g^2(L_a L_a)_{mn} - 2\,\mathrm{Re}\,\mathrm{Tr}(h^m h^{n\dagger}) + \tfrac{1}{2}d_{mnpp}] + O(\ln\Lambda^2)\,, \tag{3.28}$$

where Λ denotes the momentum cutoff. On the other hand, the quadratically divergent contribution to the vector-boson mass—which turns out to be proportional to $C_2(G)-T(F)+\frac{1}{2}T(S)$—violates gauge invariance and is merely an artifact of a bad regularization scheme. In order to restore gauge invariance the renormalized vector-boson mass must be required to vanish (which can be achieved by an appropriate counterterm). The above combination of group invariants vanishes, however, automatically in a supersymmetric gauge theory.

Finally, we will also make use of the two-loop contribution $\beta_g^{[2]}$ to the gauge β function:[23]

$$\beta_g^{[2]} = -\frac{g^5}{(4\pi)^4}\left\{\tfrac{34}{3}[C_2(G)]^2 - [\tfrac{10}{3}C_2(G)+2C_2(F)]T(F) - [\tfrac{1}{3}C_2(G)+2C_2(S)]T(S) + \frac{1}{g^2 d(G)}\mathrm{Tr}(h^{n\dagger}h^n T_a T_a)\right\}, \tag{3.29}$$

where in the terms $C_2(R)T(R)$, $R=F,S$, summation over irreducible representations is implicitly understood.

IV. GENERAL CONSIDERATIONS

The point in the discussion of finite quantum field theories which deserves most care and attention is the definition of what is meant by the term "finiteness." In a more technical sense, the crucial point is the formulation of the finiteness conditions, i.e., the circumstances under which the theory will be regarded as finite. The fundamental problem, which entails a lot of ambiguity, is represented by the gauge dependence of the wave-function renormalizations. The renormalization of fields depends on the chosen gauge, whereas the renormalization of the parameters entering in the Lagrangian, i.e., masses and coupling constants, is gauge independent (in the minimal subtraction scheme).

As far as the renormalization behavior of supersymmetric theories is concerned, the simple state of affairs sketched in Sec. II relies heavily on the maintenance of manifest supersymmetry by adopting supergraph techniques throughout the whole computation of Green's functions. On the other hand, taking into account, in a component formulation of the theory, only the physical degrees of freedom of the $(N=1)$ vector supermultiplet (2.2) corresponds to the employment of the Wess-Zumino gauge which breaks supersymmetry by eliminating all the gauge degrees of freedom that would otherwise show up in the vector supermultiplet (and contribute to the Green's functions). In that case the renormalizations of the component fields belonging to one and the same supermultiplet are no longer identical.[24] In fact, in the nonsupersymmetric Wess-Zumino R_ξ gauge the wave-function renormalization constants read, for the vector supermultiplet (2.2),

$$Z_V - 1 = \frac{1}{(4\pi)^2\epsilon}g^2\left[\frac{1}{2}\left[3-\frac{1}{\xi}\right]C_2(G)-T(R)\right], \tag{4.1}$$

$$Z_\lambda - 1 = -\frac{1}{(4\pi)^2\epsilon}g^2\left[\frac{1}{\xi}C_2(G)+T(R)\right] \tag{4.2}$$

and for the chiral supermultiplet (2.3)

$$(Z_\chi - 1)_{ij} = -\frac{1}{(4\pi)^2\epsilon}\left[g^2\left[1+\frac{1}{\xi}\right](T_a T_a)_{ij} + 2c^*_{ikl}c_{jkl}\right], \tag{4.3}$$

$$(Z_A - 1)_{ij} = \frac{1}{(4\pi)^2\epsilon}\left[g^2\left[1-\frac{1}{\xi}\right](T_a T_a)_{ij} - 2c^*_{ikl}c_{jkl}\right]. \tag{4.4}$$

Even in a finite theory these wave-function renormalizations will not vanish. In contrast to that, the evaluation of the renormalization constants (3.20)—(3.26) for the supersymmetric values of the parameters shows that in an $(N=1)$ SUSY gauge theory all renormalizations of masses and coupling constants vanish at the one-loop level, provided the gauge group, matter representation R, and super-Yukawa couplings c_{ijk} are related by the finiteness conditions (2.19) and (2.20).

In view of the above, an infinity encountered in the computation of Green's functions has to be regarded as an essential divergence with respect to finiteness only if it cannot be absorbed into some wave-function renormalizations. Consequently, we find our finiteness conditions by the requirement of the absence of all divergent contributions to the renormalizations of the physical parameters, i.e., masses and coupling constants, of the quantum field theory. (For coupling constants this requirement is equivalent to demanding finiteness of the S-matrix elements *without* divergent renormalizations of coupling constants.) Accordingly, finiteness means that the bare parameters of the Lagrangian are related to the renormalized ones by a finite amount of renormalization and thus are finite themselves.

In order to construct a nontrivial finite gauge theory, let us start with the requirement of a vanishing one-loop β function [cf. Eq. (3.22)]

$$11C_2(G) - 2T(F) - \tfrac{1}{2}T(S) = 0\,, \tag{4.5}$$

which implies that, in addition to the vector bosons, fermions or scalars must be present in the theory.

To begin with, let us try to find a model without scalars, i.e., a model which contains only gauge bosons and fermions. In this case one has only two one-loop finiteness conditions, viz., from the gauge coupling renormalization (3.22)

$$11C_2(G)=2T(F) \tag{4.6}$$

and from the fermion mass renormalization (3.20)

$$mT_aT_a=0\ , \tag{4.7}$$

which simply states that gauge nonsinglet fermions must be massless. It is not hard to find groups and representations in accordance with Eq. (4.6) which thus constitute a large set of one-loop finite gauge models. However, finiteness is destroyed already at the two-loop level. Inserting (4.6) into the two-loop contribution (3.29) to the β function yields

$$\beta_g{}^{[2]}=\frac{g^5}{(4\pi)^4}\{7[C_2(G)]^2+2C_2(F)T(F)\}\ , \tag{4.8}$$

which shows that β_g will not vanish except in the trivial case $C_2(G)=T(F)=0$. Consequently, we can make the following observations: (i) The occurrence of scalar particles is inevitable in a finite gauge theory. (ii) Contributions from beyond the one-loop level have to be taken into account in order to obtain a definite answer in the search for a finite quantum field theory.

As our next step, we attempt to banish all fermions from the theory. In this case, by inserting the relation which results from (3.28) when one demands a vanishing quadratic mass renormalization of the scalar bosons into $[(Z_d-1)d]_{mmnn}=0$ derived from (3.26), one obtains

$$3g^4(2[\mathrm{Tr}(L_aL_b)]^2+\{L_a,L_b\}_{mn}\{L_a,L_b\}_{mn} +18(L_aL_a)_{mn}(L_bL_b)_{mn})+d_{mnpq}d_{mnpq}=0\ . \tag{4.9}$$

The left-hand side of this condition is a sum of squares of real quantities which implies $g=d_{mnpq}=0$, $\delta\mu^2{}_{nn}$ from Eq. (3.21) then gives $c_{npq}=0$ for the same reason. Thus fermions are elevated to an unavoidable ingredient in a finite gauge theory.

Summarizing, we arrive at the conclusion that in a (nontrivial) gauge invariant quantum field theory vector bosons, fermions, *and* scalars must be present in order to solve the finiteness conditions—the occurrence of one particle species calls for both of the others.

V. SOME SIMPLE MODELS

Unfortunately, the finiteness conditions for a general gauge theory are, already at the one-loop level, extremely complicated. Consequently, we will concentrate our discussion on the investigation of several somewhat simplified models.

The simplest class of gauge theories one can imagine in this respect is characterized by consisting of the particle content of the $(N=1)$ supersymmetric Yang-Mills theory of Sec. II but allowing for arbitrary, i.e., nonsupersymmetric, values of the coupling constants. The Lagrangian defining these models is then, in the notation of Sec. II, given by

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}+i\bar\lambda_L\not{D}\lambda_L+i\bar\chi_L\not{D}\chi_L+(D_\mu A)^\dagger D^\mu A\\&+(2A_i^\dagger\bar\lambda^c_{aR}\Delta^a_{ij}\chi_{jL}+\Gamma_{i,jk}A_i\bar\chi^c_{jR}\chi_{kL}+\text{H.c.})\\&-d_{ij,kl}A_iA_jA_k^\dagger A_l^\dagger+\text{ mass terms}\ .\end{aligned} \tag{5.1}$$

Apart from the obvious symmetry requirements

$$\Gamma_{i,jk}=\Gamma_{i,kj}\ , \tag{5.2}$$

$$d_{ij,kl}=d_{ji,kl}=d_{ij,lk}\ , \tag{5.3}$$

the couplings Δ^a_{ij}, $\Gamma_{i,jk}$, and $d_{ij,kl}$ are completely arbitrary. $(N=1)$ supersymmetry would specify their values to

$$\Delta^a_{ij}=e^{i\delta}\frac{g}{\sqrt2}T^a_{ij}\ , \tag{5.4}$$

$$\Gamma_{i,jk}=-c_{ijk}\ \text{(totally symmetric)}\ , \tag{5.5}$$

$$d_{ij,kl}=c_{ijm}c^*_{klm}+\frac{g^2}{4}(T^a_{ki}T^a_{lj}+T^a_{kj}T^a_{li})\ . \tag{5.6}$$

Since masses and three-scalar couplings are of no relevance for the present discussion, only finiteness conditions relating solely dimensionless coupling constants have to be taken into account. At the one-loop level, as our finiteness conditions for the model (5.1) from the gauge-coupling renormalization (3.22), we obtain

$$3C_2(G)=T(R) \tag{5.7}$$

[which coincides, of course, with Eq. (2.19)], from the Yukawa coupling renormalization (3.23) for Δ^a_{ij}

$$\begin{aligned}&\tfrac{3}{2}g^2C_2(G)\Delta^a_{ik}+\tfrac{3}{2}g^2(\Delta^aT_bT_b)_{ik}-4\Gamma_{j,kl}\Gamma^*_{i,lm}\Delta^a_{jm}-[\Gamma_{j,kl}\Gamma^*_{j,lm}+(\Delta^{b\dagger}\Delta^b)_{mk}]\Delta^a_{im}\\&\qquad-\Delta^a_{jk}[\Gamma_{j,lm}\Gamma^*_{i,lm}+2(\Delta^b\Delta^{b\dagger})_{ij}]-\Delta^b_{ik}\mathrm{Tr}(\Delta^a\Delta^{b\dagger})=0\end{aligned} \tag{5.8}$$

and for $\Gamma_{i,jk}$

$$\tfrac{3}{2}g^2\Gamma_{i,jk}(T_aT_a)_{jl}-4\Gamma_{j,km}\Delta^{a*}_{im}\Delta^a_{jl}-\Gamma_{i,km}[\Gamma^*_{j,mn}\Gamma_{j,nl}+(\Delta^{a\dagger}\Delta^a)_{ml}]-\tfrac{1}{2}\Gamma_{j,kl}[\Gamma^*_{j,mn}\Gamma_{i,mn}+2(\Delta^a\Delta^{a\dagger})_{ji}]+(k\leftrightarrow l)=0\ , \tag{5.9}$$

from the four-scalar coupling renormalization (3.26)

$$\begin{aligned}&\tfrac{3}{4}g^4(T_aT_b)_{pm}\{T_a,T_b\}_{qn}-16\Delta^a_{pi}\Delta^{a*}_{mj}\Gamma_{n,jk}\Gamma^*_{q,ki}-8(\Delta^a\Delta^{b\dagger})_{pm}(\Delta^b\Delta^{a\dagger})_{qn}-4\Gamma_{m,ij}\Gamma^*_{p,jk}\Gamma_{n,kl}\Gamma^*_{q,li}\\&\quad+d_{mn,kl}d_{kl,pq}+4d_{mk,pl}d_{nl,qk}-\tfrac{3}{2}g^2[d_{mn,pj}(T_aT_a)_{qj}+d_{mj,pq}(T_aT_a)_{jn}]\\&\quad+d_{mn,pj}[\Gamma^*_{q,kl}\Gamma_{j,kl}+2(\Delta^a\Delta^{a\dagger})_{qj}]+d_{mj,pq}[\Gamma^*_{j,kl}\Gamma_{n,kl}+2(\Delta^a\Delta^{a\dagger})_{jn}]+(m\leftrightarrow n)+(p\leftrightarrow q)=0\ ,\end{aligned} \tag{5.10}$$

and from the quadratic mass renormalization (3.28) of the scalar bosons

$$\tfrac{3}{4}g^2(T_aT_a)_{ij}-2(\Delta^a\Delta^{a\dagger})_{ij}-\Gamma^*_{i,kl}\Gamma_{j,kl}+d_{jk,ik}=0\ . \tag{5.11}$$

Beside these one-loop finiteness conditions, the vanishing of the two-loop contribution (3.29) to the gauge β function will also be an essential criterion

$$3[C_2(G)]^2-2C_2(G)T(R)-3C_2(R)T(R)+\frac{2}{g^2d(G)}\{C_2(G)\mathrm{Tr}(\Delta^a\Delta^{a\dagger})+(T_aT_a)_{ij}[\Gamma^*_{l,jk}\Gamma_{l,ki}+(\Delta^{b\dagger}\Delta^b)_{ji}]\}=0\ . \tag{5.12}$$

One solution of the above finiteness conditions is certainly provided by the two-loop finite ($N=1$) supersymmetric gauge theory of Sec. II. It is easy to check that the supersymmetric values (5.4)—(5.6) of the dimensionless coupling constants, when restricted by the constraints (2.19) and (2.20), satisfy Eqs. (5.8)—(5.12). Soft supersymmetry breaking of the form mentioned in Sec. II is, of course, always possible. However, as can be explicitly seen already from Eqs. (3.22), (3.23), (3.26), (3.28), and (3.29), the corresponding mass terms and three-scalar couplings do not affect the finiteness conditions for dimensionless couplings.

Now the following question arises: Is the supersymmetric theory defined by (5.4)—(5.6) and (2.19), (2.20) the unique solution of the finiteness conditions (5.8)—(5.12) or do other, nonsupersymmetric solutions exist? To investigate this problem we consider several special cases which we obtain from the class of models (5.1) by imposing some group-theoretically motivated constraints in order to simplify the analysis.

A. Model I

This model is defined by the Lagrangian (5.1) and the following additional constraints.

(i) R is an irreducible representation r of the gauge group, $R=r$.

(ii) The adjoint representation occurs only once in the Kronecker product $r\times\bar{r}$.

(iii) The singlet occurs at most once in the cubic Kronecker product $r\times r\times r$.

The smallest possible (anomaly-free) group and representation which solve $3C_2(G)=T(r)$, i.e., the requirement (5.7) of a vanishing one-loop gauge β function, and satisfy constraints (i)—(iii) is SO(9) with $r=\mathbf{84}$ (Ref. 25). Here the adjoint representation is $G=\mathbf{36}$ and $\mathbf{84}\times\mathbf{84}=\mathbf{1}_S+\mathbf{36}_A+\mathbf{44}_S+\mathbf{84}_A+$ higher representations.

Constraint (ii) implies that Δ^a_{ij} is proportional to the generator t^a_{ij} in the irreducible representation r,

$$\Delta^a_{ij}=\tau t^a_{ij}\ , \tag{5.13}$$

where the parameter τ can be made real by an appropriate phase transformation on the fermions λ_a. Constraint (iii) leaves the two possibilities open that either there exists no invariant tensor of the form γ_{ijk} at all, $1\not\subset r\times r\times r$, or γ_{ijk} is uniquely determined up to an arbitrary factor, $1\subset r\times r\times r$. In the latter case we normalize γ_{ijk} according to $\gamma^*_{ikl}\gamma_{jkl}=\delta_{ij}$. Then the Yukawa coupling $\Gamma_{i,jk}$ will have the form

$$\Gamma_{i,jk}=\kappa\gamma_{ijk}\equiv -c_{ijk}\ , \tag{5.14}$$

where γ_{ijk} must be completely symmetric in order to meet the symmetry requirement (5.2). Inserting (5.7), (5.13), and (5.14) into (5.8), (5.9), and (5.12) yields

$$\begin{gathered}\tau\left[\tau^2-\frac{g^2}{2}\right]=0\ ,\\ \kappa[\,|\kappa|^2-g^2C_2(r)]=0\ ,\\ \frac{2}{g^2}\{\,|\kappa|^2+\tau^2[C_2(G)+C_2(r)]\}-C_2(G)-3C_2(r)=0\ ,\end{gathered} \tag{5.15}$$

which has the unique solution

$$\begin{gathered}\tau^2=\frac{g^2}{2}\ ,\\ |\kappa|^2=g^2C_2(r)\ ,\end{gathered} \tag{5.16}$$

corresponding just to the supersymmetric form (5.4) of the Yukawa coupling Δ^a_{ij} and to the finiteness condition (2.20).

Our last task is to satisfy Eqs. (5.10) and (5.11). The form of the four-scalar coupling $d_{ij,kl}$, i.e., the number of different types of vertex structures as well as their explicit expressions, will, in general, depend on the gauge group and on the representation R according to which the scalar fields A_i transform. Here we assume that $d_{ij,kl}$ contains only those interaction terms which one encounters also in a supersymmetric theory. Hence we make the ansatz

$$d_{ij,kl}=\alpha c_{ijm}c^*_{klm}+\beta g^2(T^a_{ki}T^a_{lj}+T^a_{kj}T^a_{li})\ , \tag{5.17}$$

where α, β are two parameters to be determined. Equation (5.11) then implies $\alpha+\beta=\frac{5}{4}$, while inserting this ansatz into (5.10) gives $\alpha=1$, $\beta=\frac{1}{4}$, i.e., the supersymmetric form (5.6) of $d_{ij,kl}$, as the only possible solution.

B. Model II

In addition to the Lagrangian (5.1) the group-theoretic constraints defining this model are (i) R is the direct sum of two identical irreducible representations r, $R=r+r$, where again (ii) the adjoint representation occurs only once in the Kronecker product $r\times\bar{r}$, and (iii) the singlet occurs at most once in the cubic Kronecker product $r\times r\times r$.

Here the smallest possible (anomaly-free) group and representation satisfying $3C_2(G)=2T(r)$ is SO(10) with $r=\mathbf{54}$ (Ref. 25). The adjoint representation is $G=\mathbf{45}$ and $\mathbf{54}\times\mathbf{54}=\mathbf{1}_S+\mathbf{45}_A+\mathbf{54}_S+$ higher representations.

For our further discussion we replace the index i by (I,α), where $I=1,2$ is a multiplicity index corresponding to $R=r+r$ while α is related to the irreducible representation r with generators $t^a_{\alpha\beta}$. With this convention

$$T^a_{ij}=\delta_{IJ}t^a_{\alpha\beta}\ , \tag{5.18}$$

$$\Delta^a_{ij}=\tau_{IJ}t^a_{\alpha\beta}\ , \tag{5.19}$$

$$\Gamma_{i,jk}=\kappa_{I,JK}\gamma_{\alpha\beta\gamma}\ , \tag{5.20}$$

where $i=(I,\alpha)$, $j=(J,\beta)$, $k=(K,\gamma)$. By an appropriate rotation of the fields $A_{I,\alpha}$ and $\chi_{I,\alpha}$, τ_{IJ} can be transformed into a real, positive-semidefinite diagonal matrix:

$$\hat{\tau}_{IJ}=\rho_I\delta_{IJ},\ \ \rho_I\geq 0\ \ (\text{no sum over } I)\ . \tag{5.21}$$

The normalization of $\gamma_{\alpha\beta\gamma}$ again reads $\gamma^*_{\alpha\gamma\delta}\gamma_{\beta\gamma\delta}=\delta_{\alpha\beta}$.

Remembering the relation $3C_2(G)=2T(r)$ which guarantees the one-loop finiteness of the gauge coupling constant, the finiteness conditions (5.8), (5.9), and (5.12) now take the form

$$3\left[g^2C_2(G)+g^2C_2(r)-2C_2(r){\rho_I}^2-\tfrac{2}{3}T(r)\sum_L {\rho_L}^2\right]\rho_I\delta_{IK} +2\sum_{L,M}(2\kappa_{L,KM}\kappa^*_{I,LM}\rho_L-\kappa_{L,MK}\kappa^*_{L,MI}\rho_I-\kappa_{K,LM}\kappa^*_{I,LM}\rho_K)=0\ \ (\text{no sum over } I,K)\ , \tag{5.22}$$

$$(3g^2-2{\rho_I}^2-{\rho_K}^2-{\rho_L}^2)C_2(r)\kappa_{I,KL}+2C_2(r)(\kappa_{L,IK}\rho_I\rho_L+\kappa_{K,IL}\rho_I\rho_K) -\sum_{M,N,P}(\kappa_{I,KN}\kappa^*_{M,NP}\kappa_{M,LP}+\kappa_{M,KN}\kappa^*_{M,NP}\kappa_{I,LP}+\kappa_{M,KL}\kappa^*_{M,NP}\kappa_{I,NP})=0\ \ (\text{no sum over } I,K,L)\ , \tag{5.23}$$

$$3[C_2(G)]^2+9C_2(G)C_2(r)-\frac{2T(r)}{g^2}\left[[C_2(G)+C_2(r)]\sum_I{\rho_I}^2+\sum_{I,K,L}|\kappa_{I,KL}|^2\right]=0\ . \tag{5.24}$$

The invariant tensor $\gamma_{\alpha\beta\gamma}$ is either totally symmetric or totally antisymmetric. Let us look at these two cases in more detail.

If $\gamma_{\alpha\beta\gamma}$ is totally antisymmetric, $\kappa_{I,JK}$ must be antisymmetric in J,K in order to preserve the symmetry of (5.2). In this case a supersymmetric solution of our finiteness conditions cannot exist because in two dimensions a totally antisymmetric object κ_{IJK} vanishes identically so that the finiteness condition (2.20) cannot be satisfied. Requiring, however, only antisymmetry of $\kappa_{I,JK}$ in J,K, our independent variables are ρ_1, ρ_2, $\kappa_{1,12}$, and $\kappa_{2,12}$. A straightforward calculation then shows that Eqs. (5.22)–(5.24) have no solution at all.

If $\gamma_{\alpha\beta\gamma}$ is totally symmetric, $\kappa_{I,JK}$ must be symmetric in the last two indices and our independent variables are ρ_1, ρ_2, $\kappa_{1,11}$, $\kappa_{1,12}$, $\kappa_{1,22}$, $\kappa_{2,11}$, $\kappa_{2,12}$, and $\kappa_{2,22}$. In this case a tedious calculation gives as the only solution of Eqs. (5.22)–(5.24)

$$\begin{gathered}{\rho_1}^2={\rho_2}^2=\frac{g^2}{2}\ ,\\ \kappa_{1,12}=\kappa_{2,11},\ \ \kappa_{1,22}=\kappa_{2,12}\ ,\end{gathered} \tag{5.25}$$

corresponding to the supersymmetric form (5.4), (5.5) of the Yukawa couplings, and

$$\sum_{K,L}\kappa^*_{I,KL}\kappa_{J,KL}=g^2C_2(r)\delta_{IJ}\ , \tag{5.26}$$

which is just the finiteness condition (2.20). Again, ansatz (5.17), when inserted into Eqs. (5.10) and (5.11), leads to the supersymmetric form (5.6) of the quartic scalar self-coupling.

C. Model III

This model is characterized by the following two basic assumptions regarding the Yukawa couplings in the Lagrangian (5.1).[26]

(i) The coupling Δ^a_{ij} is proportional to the group generator T^a_{ij} in the (arbitrary) representation R,

$$\Delta^a_{ij}=\tau T^a_{ij}\ . \tag{5.27}$$

(ii) The coupling $\Gamma_{i,jk}$ is totally symmetric,

$$\Gamma_{i,jk}=-c_{ijk}\ \ (\text{totally symmetric})\ . \tag{5.28}$$

In this case, introducing for convenience the abbreviation

$$K_{ij}\equiv c^*_{ikl}c_{jkl}\ , \tag{5.29}$$

the finiteness conditions (5.8)–(5.12) read

$$\tau\{[\tfrac{3}{2}g^2C_2(G)-|\tau|^2T(R)]T^a_{ik}+3(\tfrac{1}{2}g^2-|\tau|^2)(T^aT^bT^b)_{ik}\}=0\ , \tag{5.30}$$

$$\{c_{ikm}[(\tfrac{3}{2}g^2-|\tau|^2)(T^aT^a)_{ml}-K_{ml}]+(k\leftrightarrow l)\}+c_{jkl}[2|\tau|^2(T^aT^a)_{ji}-K_{ji}]=0\ , \tag{5.31}$$

$$\tfrac{3}{4}g^4(T^aT^b)_{pm}\{T^a,T^b\}_{qn}-8|\tau|^4(T^aT^b)_{pm}(T^bT^a)_{qn}-4|\tau|^2(T^aT^a)_{ji}c^*_{pqi}c_{mnj} -4c_{mij}c^*_{pjk}c_{nkl}c^*_{qli}+d_{mn,kl}d_{kl,pq}+4d_{mk,pl}d_{nl,qk}+d_{mn,pj}[(2|\tau|^2-\tfrac{3}{2}g^2)(T^aT^a)_{qj}+K_{qj}] +d_{mj,pq}[(2|\tau|^2-\tfrac{3}{2}g^2)(T^aT^a)_{jn}+K_{jn}]+(m\leftrightarrow n)+(p\leftrightarrow q)=0\ , \tag{5.32}$$

$$(\tfrac{3}{4}g^2-2|\tau|^2)(T^aT^a)_{ij}-K_{ij}+d_{jk,ik}=0\ , \tag{5.33}$$

$$3[C_2(G)]^2+2\left[\frac{|\tau|^2}{g^2}-1\right]C_2(G)T(R)+\left[2\frac{|\tau|^2}{g^2}-3\right]\sum_\sigma C_2(R_\sigma)T(R_\sigma)+\frac{2}{g^2d(G)}\mathrm{Tr}(T^aT^aK)=0\ . \tag{5.34}$$

Now, first of all, for representations satisfying the finiteness condition for the gauge coupling constant, $3C_2(G)=T(R)$, Eq. (5.30) allows for the two solutions $\tau=0$ and $|\tau|^2=\frac{1}{2}g^2$. When, however, inserted into Eqs. (5.31) and (5.34), $\tau=0$ results in a trivial theory, leaving thus the supersymmetric value

$$|\tau|^2=\tfrac{1}{2}g^2 \tag{5.35}$$

as the only possible solution of Eq. (5.30). For this value of τ the remaining finiteness conditions are given by

$$c_{ikm}[g^2(T^aT^a)_{ml}-K_{ml}]+\text{cycl. perm. of } ikl=0\ , \tag{5.36}$$

$$\tfrac{3}{4}g^4(T^aT^b)_{pm}\{T^a,T^b\}_{qn}-2g^4(T^aT^b)_{pm}(T^bT^a)_{qn}-2g^2(T^aT^a)_{ji}c^*_{pqi}c_{mnj}-4c_{mij}c^*_{pjk}c_{nkl}c^*_{qli}+d_{mn,kl}d_{kl,pq}$$
$$+4d_{mk,pl}d_{nl,qk}+d_{mn,pj}[K_{qj}-\tfrac{1}{2}g^2(T^aT^a)_{qj}]+d_{mj,pq}[K_{jn}-\tfrac{1}{2}g^2(T^aT^a)_{jn}]+(m\leftrightarrow n)+(p\leftrightarrow q)=0\ , \tag{5.37}$$

$$d_{jk,ik}-\tfrac{1}{4}g^2(T^aT^a)_{ij}-K_{ij}=0\ , \tag{5.38}$$

$$\mathrm{Tr}[T^aT^a(K-g^2T^bT^b)]=0\ . \tag{5.39}$$

Once again we adopt the ansatz

$$d_{ij,kl}=\alpha c_{ijm}c^*_{klm}+\beta(T^a_{ki}T^a_{lj}+T^a_{kj}T^a_{li}) \tag{5.40}$$

for the quartic scalar coupling $d_{ij,kl}$ entering in Eqs. (5.37) and (5.38), the latter of which then reads

$$(\alpha-1)K_{ij}+(\beta-\tfrac{1}{4}g^2)(T^aT^a)_{ij}=0\ . \tag{5.41}$$

At this point one has to distinguish the following two possibilities.

Case A. The parameters α and β take the supersymmetric values

$$\alpha=1,\ \ \beta=\tfrac{1}{4}g^2 \tag{5.42}$$

which solve Eq. (5.41). Defining for brevity the Hermitian matrix

$$\Omega_{ij}\equiv K_{ij}-g^2(T^aT^a)_{ij},\ \ \Omega^\dagger=\Omega\ , \tag{5.43}$$

one is left with the finiteness conditions

$$c_{ikm}\Omega_{ml}+\text{cycl. perm. of } ikl=0\ , \tag{5.44}$$

$$c^*_{pjk}c_{mnk}\Omega_{jq}+c^*_{pqk}c_{mjk}\Omega_{jn}+c^*_{pqk}c_{mnj}\Omega_{jk}+(m\leftrightarrow n)+(p\leftrightarrow q)=0\ , \tag{5.45}$$

$$\mathrm{Tr}(T^aT^a\Omega)=0\ . \tag{5.46}$$

By making use of the gauge invariance of the Yukawa coupling c_{ijk}, Eq. (5.45) can be reduced to Eq. (5.44). Equations (5.44) and (5.46) can be combined to $\mathrm{Tr}(\Omega^2)=0$, which implies $\Omega=0$ due to the Hermiticity of Ω, i.e., the finiteness condition (2.20),

$$K_{ij}=g^2(T^aT^a)_{ij}\ . \tag{5.47}$$

Case B. If $\alpha\neq 1$, $\beta\neq\frac{1}{4}g^2$, Eq. (5.41) shows that K_{ij} has to be proportional to $(T^aT^a)_{ij}$; the factor of proportionality may be read off from Eq. (5.39):

$$K_{ij}=g^2(T^aT^a)_{ij} \tag{5.48}$$

satisfies Eqs. (5.36) and (5.39). By a careful analysis one can then convince oneself that in case relation (5.48) holds the remaining finiteness conditions, (5.37) and (5.38), have no solution except the supersymmetric one,

$$\alpha=1,\ \ \beta=\tfrac{1}{4}g^2\ . \tag{5.49}$$

Thus, in any case one ends up with the supersymmetric four-scalar interaction (5.6) and the finiteness condition (2.20).

VI. SUMMARY

In the present work we have discussed the conditions for the finiteness of the most general renormalizable quantum field theory. This finiteness criterion yields, order by order in the loop expansion, a set of relations between the parameters of the theory. A large class of solutions of these finiteness conditions, provided by the supersymmetric theories described in Sec. II, has already been known for some time. We were thus particularly interested in the question whether or not nonsupersymmetric finite quantum field theories exist.

General considerations showed that within a nontrivial gauge theory, in addition to the gauge bosons, fermions as well as scalar bosons have to be present in order to be able to solve the finiteness conditions at all. The most general case being, already at the one-loop level, rather involved, we focused our attention to a somewhat simplified class of theories. In all models studied we were unambiguously led to supersymmetric relations between the dimensionless coupling constants. Supersymmetry together with the finiteness conditions (2.19) and (2.20) proved to be the unique solution of the finiteness requirement for these models. Consequently, all our finite models belong to the class of constrained supersymmetric theories of Sec. II. Thus the analysis of Ref. 12 applies. Although we did not need the full two-loop divergence structure for our investigation, the models discussed will nevertheless be finite up to the two-loop level.

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*Present address: Fakultät für Physik, Universität Bielefeld, Bielefeld, Federal Republic of Germany.

[1]P. C. West, in *Proceedings of the 1983 Shelter Island II Conference,* edited by R. Jackiw, N. N. Khuri, S. Weinberg, and E. Witten (MIT, Cambridge, Mass., 1985), p. 127; A. Sagnotti, University of California Report No. UCB-PTH-84/26, LBL-18406, 1984 (unpublished).

[2]T. Inami, H. Nishino, and S. Watamura, Phys. Lett. **117B**, 197 (1982); N. G. Deshpande, R. J. Johnson, and E. Ma, *ibid.* **130B**, 61 (1983); Phys. Rev. D **29**, 2851 (1984).

[3]W. Fischler, H. P. Nilles, J. Polchinski, S. Raby, and L. Susskind, Phys. Rev. Lett. **47**, 757 (1981).

[4]Of course, the dependence of the wave-function renormalizations on the gauge, manifesting itself in this particular case as the freedom of choice to adopt the background-field gauge or not, expresses nothing else but the fact that the corresponding divergences are not essential ones.

[5]L. F. Abbott, Nucl. Phys. **B185**, 189 (1981).

[6]M. T. Grisaru and W. Siegel, Nucl. Phys. **B201**, 292 (1982).

[7]J. Wess and B. Zumino, Phys. Lett. **49B**, 52 (1974); J. Iliopoulos and B. Zumino, Nucl. Phys. **B76**, 310 (1974); S. Ferrara, J. Iliopoulos, and B. Zumino, *ibid.* **B77**, 413 (1974); S. Ferrara and O. Piguet, *ibid.* **B93**, 261 (1975).

[8]M. T. Grisaru, W. Siegel, and M. Roček, Nucl. Phys. **B159**, 429 (1979).

[9]Note that $\beta_g^{[1]}=0$ demands $C_2(G)\neq 0$, i.e., a non-Abelian gauge group, while $\gamma_{ij}^{[1]}=0$ requires nonvanishing super-Yukawa couplings c_{ijk}. The latter is the reason why the search for a two-loop finite $(N=1)$ supersymmetric pure Yang-Mills theory without superpotential was doomed to fail. See D. R. T. Jones, Nucl. Phys. **B87**, 127 (1975).

[10]D. R. T. Jones, L. Mezincescu, and Y.-P. Yao, Phys. Lett. **148B**, 317 (1984); J. León and J. Pérez-Mercader, *ibid.* **164B**, 95 (1985).

[11]S. Hamidi, J. Patera, and J. H. Schwarz, Phys. Lett. **141B**, 349 (1984); S. Rajpoot and J. G. Taylor, *ibid.* **147B**, 91 (1984).

[12]D. R. T. Jones and L. Mezincescu, Phys. Lett. **136B**, 242 (1984); P. West, *ibid.* **137B**, 371 (1984); A. Parkes and P. West, *ibid.* **138B**, 99 (1984); D. R. T. Jones and L. Mezincescu, *ibid.* **138B**, 293 (1984).

[13]A. J. Parkes and P. C. West, Nucl. Phys. **B256**, 340 (1985); A. J. Parkes, Phys. Lett. **156B**, 73 (1985); D. R. T. Jones and A. J. Parkes, *ibid.* **160B**, 267 (1985).

[14]D. R. T. Jones and S. Raby, Phys. Lett. **143B**, 137 (1984); S. Hamidi and J. H. Schwarz, *ibid.* **147B**, 301 (1984); J. León, J. Pérez-Mercader, M. Quirós, and J. Ramírez-Mittelbrunn, *ibid.* **156B**, 66 (1985); J. E. Björkman, D. R. T. Jones, and S. Raby, Nucl. Phys. **B259**, 503 (1985).

[15]P. S. Howe, K. S. Stelle, and P. C. West, Phys. Lett. **124B**, 55 (1983).

[16]A. Parkes and P. West, Phys. Lett. **127B**, 353 (1983); J.-M. Frère, L. Mezincescu, and Y.-P. Yao, Phys. Rev. D **29**, 1196 (1984); **30**, 2238 (1984).

[17]I. G. Koh and S. Rajpoot, Phys. Lett. **135B**, 397 (1984); F.-x. Dong, T.-s. Tu, P.-y. Xue, and X.-j. Zhou, *ibid.* **140B**, 333 (1984); J.-P. Derendinger, S. Ferrara, and A. Masiero, *ibid.* **143B**, 133 (1984); X.-d. Jiang and X.-j. Zhou, *ibid.* **144B**, 370 (1984); S. Kalara, D. Chang, R. N. Mohapatra, and A. Gangopadhyaya, *ibid.* **145B**, 323 (1984).

[18]F. del Aguila, M. Dugan, B. Grinstein, L. Hall, G. G. Ross, and P. West, Nucl. Phys. **B250**, 225 (1985); K. Enqvist and J. Maalampi, Phys. Lett. **157B**, 43 (1985).

[19]P. S. Howe, K. S. Stelle, and P. K. Townsend, Nucl. Phys. **B236**, 125 (1984).

[20]C. H. Llewellyn Smith, Phys. Lett. **46B**, 233 (1973); J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. D **10**, 1145 (1974).

[21]G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).

[22]Quadratic divergences do not contribute in dimensional regularization since there $\lim_{m^2\to 0}\int d^Dk/(k^2-m^2)=0$.

[23]M. E. Machacek and M. T. Vaughn, Nucl. Phys. **B222**, 83 (1983).

[24]R. Barbieri, S. Ferrara, L. Maiani, F. Palumbo, and C. A. Savoy, Phys. Lett. **115B**, 212 (1982).

[25]R. Slansky, Phys. Rep. **79**, 1 (1981).

[26]Note that, in contrast with the previously discussed models, here we do not impose any (explicit) constraint on the representation R.