Field quantization for accelerated frames in flat and curved space-times

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We formulate quantum field theory for a wide class of accelerated space-times in four dimensions and describe its thermal properties in terms of analytic mappings. We demonstrate that four-dimensional Rindler space uniquely satisfies the condition of global thermal equilibrium, while spaces which are asymptotically Rindler have thermal equilibrium asymptotically. We discuss the renormalization of the quantum energy-momentum tensor with application to situations in two, four, and ν dimensions and specifically refer to the general result for conformally flat two-dimensional space-times. Covariant and noncovariant regularization schemes are presented and compared, and we make an improvement to proposals for covariantly regularizing the stress tensor by point separation.

I. INTRODUCTION

In previous publications,[1,2] analytic mappings have been used to define a wide class of accelerated space-times preserving the light-cone structure of flat Minkowski space-time. Two-dimensional quantum field theory (QFT) was then formulated in these accelerated spaces and Bogoliubov coefficients, relating a positive-frequency description for accelerated and inertial bases, were given explicitly in terms of the mappings. In this paper we extend to four dimensions a series of results stemming from this approach: four-dimensional QFT and its thermal properties are analyzed in terms of analytic mappings relating some manifold to its global analytic extension. Following a description of the relevant formalism (in Sec. II), we give a specific demonstration (in Sec. III) that the four-dimensional Rindler space (described by the exponential mapping) uniquely satisfies the condition of global thermal equilibrium. Spaces which are asymptotically Rindler have thermal equilibrium asymptotically. In Sec. IV we give a general discussion of renormalization in a curved space-time where there may be no unique state singled out as a ground state, and where (even for a free scalar field) coupling to the curvature may force us to regard gravity as interacting. We examine several properties of some specific regularizing schemes which indicate how the renormalization might be effected, noting of course, that final results must be independent of any scheme we use to obtain it. A refinement to proposals for covariantly regularizing the stress tensor by point separation is given in Sec. V.

Generalizations of this work including rotating accelerated frames and even nonanalytic mappings are given elsewhere.[3]

A. Contextual background

Before presenting our work in the next sections, we pause here to survey the context in which it has been developed.

While a full quantum theory of gravity is still nonexistent, continuous effort over the last quarter of a century has demonstrated the many difficulties encountered in repeated attempts to construct such a theory and have also indicated some of the particular properties which an eventually complete theory will have to possess. Complementary to these approaches, there are investigations for problems in flat space-time which can throw light on both classical and quantum results in curved space-time: Quantum field theory developed for curvilinear (accelerated) coordinates in flat space in a way which can be directly generalized to curved space-time may be useful for a physical and mathematical discussion of the full theory.

The genuine coordinate independence which is so familiar in the classical theory of general relativity is not a particular property of gravity but a fundamental principle prevalent in all descriptions of physical laws. On the other hand, the apparent difference which results from the treatment of a quantum field theory in a variety of coordinate systems (in either curved or flat space-time) is not a coordinate effect at all, but is a consequence of the fact that physically different quantum states are correctly described by the quantum theory as being physically distinct. "Canonical" states for different coordinate systems are physically different (each timelike vector field leads to a separate indication of what constitutes a definition of positive frequency).

It seems difficult to give sensible meaning to the following question: how should we describe quantum field theory for an (accelerated) *observer following a particular world line*? Even for a uniformly accelerated observer, some subtle assumptions go into the handling of this question. First, although one might use a coordinate system giving the world lines for an infinite family of uniformly accelerated observers, this system actually refers only to *accelerated frames*, since there would need to be collusion (spacelike correlations) between observers in order to have and maintain them as uniformly accelerating. In addition, to use standard quantum-field-theory techniques, one would impose boundary conditions such that, asymptotically, modes appear as free waves in these coordinates. But for an *arbitrary accelerating observer,* both the coordinate system of his connected region and the asymptotic boundary conditions we might assign for him can actually be chosen in an infinite variety of ways. However, a perfectly well-posed problem is as follows: what description is there for a quantum field theory in accelerated *frames*? We associate a physical meaning to this question by describing the boundary conditions of the quantum theory in terms of the asymptotic behavior of those frames. This automatically specifies the quantum state to be examined. In addition it allows us to use appropriate coordinates in examining physical consequences of the quantum theory for the chosen state. Of course, these consequences are completely independent of the coordinates used to evaluate them.

Further considerations arise for curved space, in which acceleration and gravitation are locally indistinguishable but globally inequivalent. On the one hand, we have just imposed a connection between the boundary conditions of the space-time and a quantum state in which to discuss physics. We have done it by using the asymptotic behavior of the accelerated frames. On the other hand, in curved space, we shall have to decide whether there may be some link between the global properties of the gravitational field and some specific state which may be regarded (gravitationally) as a global ground state in our space-time. The alternative to such a normalization is to regard gravity as essentially interacting with (otherwise free) fields propagating in a curved space-time. These questions become unavoidable when one tries to handle the infinities, inevitably arising in theories of field quantization.

The usual purpose for examining QFT in flat space is to obtain an understanding about the particles which we use the fields to represent. In curved space-time, this same reason exists, with the added fact that we can look at (linearized) curvature effects on the theory. But a more interesting reason in the context of general relativity is to be able to determine some nonlinear effects of the coupling between matter fields and geometry through the stress energy of the quantum fields. Of course, one should include the quantum effects of gravitation itself but technically this is rather difficult and it has often been felt that dealing simply with a scalar-field source first would give a guide to the treatment of some of the difficulties. Even to consider the back reaction of the scalar field, progress is not altogether straightforward, the problem being that the stress tensor for a quantum field is a formally divergent operator, just as it is in flat space. Whereas one has a clear idea of what the vacuum (i.e., zero-energy) state is in flat space, and therefore can give a well-defined procedure for rendering quantum operators finite, in a curved space-time this is not the case, and there has arisen some discussion over whether one should use a normalization procedure, equivalent to defining some specific state as having zero energy or whether a renormalization of the theory is necessary, equivalent to regarding the curvature as introducing an interaction between the scalar field and the geometry. With regard to normalization, it is not at all clear how to determine which state might naturally be regarded as having zero energy, or even whether there is any such state. On the other hand, renormalization of a theory is usually considered in the context of a perturbative expansion. However, in the case of gravitation, apart from the infinite changes which will be introduced into the quantities originally appearing in the Lagrangian, new counterterms will be required at each level of a loop expansion about some classically valued geometry (say Minkowski space). Because of this the full quantum theory of gravity is often described as being "unrenormalizable." It has also been argued that if only the matter fields are quantized, even an expansion to one loop may not make sense since the quantum fluctuations of the geometry should also be considered at exactly that level where back-reaction effects for the matter fields become significant. Nevertheless there is ample reason to believe that knowledge gained from a treatment of the "semiclassical" Einstein equations will be useful in any perturbative discussions of the "full" theory. In this context, some form of the "absolute" renormalization would seem to be required, and we shall refer to methods which have been used for carrying this out. However, it simply is not clear that geometry will respond to quantum matter in the same way as does an observer in a laboratory which is accelerating. Thus, at least in any discussion of quantum fields on a fixed space-time background, some form of normalization may be more in order. We will make further reference to the choices which arise here in Secs. IV and V.

II. QUANTUM STATES AND VACUUM SPECTRA IN ACCELERATED FRAMES

The extension of earlier results to four dimensions is embodied in the following coordinate transformations:

$$x-t=f(x'-t'),\quad x+t=g(x'+t'),\quad y=y',\quad z=z', \tag{1}$$

so that the metric takes the form

$$ds^2=f'(x'-t')g'(x'+t')dx'^2-dt'^2+dy^2+dz^2,$$

where $f(g)$ is a strictly monotonic function, unprimed coordinates refer to all of inertial Minkowski space and primed coordinates to an accelerated space-time. (Below, we shall use $u=x-t, v=x+t, u'=x'-t', v'=x'-t'$, for convenience.) Singularities of the inverse mapping $u'=F(u)$ $[v'=G(v)]$ at u_+ and u_- (v_+ and v_-) give the (x',t') boundaries of the accelerated space, thus

$$u_\pm=f(\pm\infty),\quad v_\pm=g(\pm\infty). \tag{2}$$

Here $u_\pm$ ($v_\pm$) can take finite or infinite values. Future and past boundaries at $u=u_-$ and $v=v_-$ are defined by different types of singularities of f and g, respectively, and they can have (as we will see) different associated temperatures. The definitions proceed analogously for future and past boundaries at $v=v_+$ and $u=u_+$. Boundaries can be horizons or infinites. For finite $u_\pm, v_\pm$ the accelerated coordinates cover a bounded region (a parallelogram)

$$u_-<|x-t|<u_+,\quad v_-<|x+t|<v_+,$$

of Minkowski space-time; $u=u_\pm$ and $v=v_\pm$ represent two event horizons. (If $f=g$, the bounded region is a rhombus.) There can be horizons on u but no horizons on v and vice versa, in which case the coordinates cover an infinite strip at 45° angle with the x axis. If $u_\pm=\pm\infty$ and $v_\pm=\pm\infty$, there are no horizons. Conditions (2) guarantee that the accelerated coordinates (x',t') range over all values from $-\infty$ to $+\infty$ (light rays take an infinite time t' to reach the boundaries of the accelerated space). For $t'\to\pm\infty$ the world lines of the accelerated observers defined by $x'=$const tend asymptotically to the characteristic lines $u=u_\pm$ $(v=v_\pm)$ where its velocity given by

$$V=\frac{g'(v')-f'(u')}{g'(v')+f'(u')}$$

reaches the values $\pm c$. These conditions ensure that one can formulate QFT in these accelerated spaces in a consistent way. In accelerated spaces for which the mappings $f(g)$ do not satisfy conditions (2), self-adjointness of propagation equations, completeness and orthogonality of their solutions cease to hold, unless additional assumptions on the wave functions are imposed. This can be clearly illustrated by comparing the mappings

$$f_1=\frac{\alpha u'+\beta}{\gamma u'+\delta}\ \ (\alpha\delta-\beta\gamma\neq 0)\ \text{ and } f_2=\beta e^{\alpha u'}$$

$(\alpha,\beta,\gamma,\delta)$ being real constants. Both mappings describe uniform accelerations (f_2 gives the Rindler frame). However, f_1 does not satisfy conditions (2) and then solely quantum effects of Casimir type can be described in terms of f_1.

In four dimensions as in the two-dimensional massive case, knowledge of $f'(\pm\infty)$ $[g'(\pm\infty)]$ is also required to proceed with a discussion of QFT. Different choices are possible. For definiteness in our discussion here we will take $f=g, u_-=0$ (then $u'=-\infty$ is a critical point) and $u_+=+\infty$, so that the accelerated space covers the right-hand wedge of Minkowski space-time. Then

$$f'(-\infty)=0 \tag{3a}$$

and we choose

$$f'(+\infty)=+\infty \tag{3b}$$

since the Rindler space is included by this as are also nonuniformly accelerated space-times which are asymptotically Rindler. The cases with two or zero event horizons can be easily solved from the discussion given below.

In the accelerated space-time, the minimally coupled scalar-field equation

$$\Box\hat{\Psi}=m^2\hat{\Psi} \tag{4}$$

becomes

$$\left[\frac{1}{\Lambda}(-\partial_{t'}{}^2+\partial_{x'}{}^2)+\partial_{y'}{}^2+\partial_{z'}{}^2-m^2\right]\hat{\Psi}=0 \tag{5}$$

where $\Lambda=f'(x'-t')f'(x'+t')$.

The substitution

$$\hat{\Psi}=\frac{1}{2\pi}e^{i(\lambda_2 y+\lambda_3 z)}\phi(x',t')\quad -\infty\leq\lambda_2,\lambda_3\leq+\infty \tag{6}$$

yields

$$(-\partial_{t'}{}^2+\partial_{x'}{}^2-\Lambda M^2)\phi(x',t')=0$$

with

$$M^2=\lambda_2{}^2+\lambda_3{}^2+m^2\ . \tag{7}$$

Conditions (3) mean that the effective mass ΛM^2 is zero on the horizon and infinite at infinity preventing particle escape there. Thus we can choose as a complete set of in-basis solutions, the functions $\phi^{\rm in}$ satisfying

$$\lim_{v'\to-\infty}\phi^{\rm in}_{\lambda_1}=\frac{1}{2\sqrt{\pi\lambda_1}}e^{i\lambda_1 u'}\ , \tag{8}$$

$$\lim_{u'\to+\infty}\phi^{\rm in}_{\lambda_1}=0,\ \ \lambda_1>0 \tag{9}$$

and given completely for any mapping by[2]

$$\phi^{\rm in}_{\lambda_1}=\frac{i}{2}\left[\frac{\lambda_1}{\pi}\right]^{1/2}\int_{+\infty}^{u'}d\xi\, e^{i\lambda_1\xi}J_0(M\{v[f(\xi)-u]\}^{1/2})\ , \tag{10}$$

where J_0 stands for the Bessel function.

This can be also written as

$$\phi^{\rm in}_{\lambda_1}=\frac{1}{2\sqrt{\pi\lambda_1}}\left[e^{i\lambda_1 F(u)}+M\sqrt{v}\int_{+\infty}^{u}d\eta\, e^{i\lambda_1 F(\eta)}\times\frac{J_1\{M[v(\eta-u)]^{1/2}\}}{\sqrt{\eta-u}}\right],$$

in terms of the inverse mapping $F=f^{-1}$. Near the horizon, for $uv\to 0$, $\phi^{\rm in}_{\lambda_1}$ behaves as

$$\phi^{\rm in}_{\lambda_1}\underset{uv\to 0}{=}-\frac{1}{2\sqrt{\pi\lambda_1}}(e^{i\lambda_1 u'}-e^{2i\delta(\lambda)}e^{-i\lambda v'})\ , \tag{11}$$

where $\delta(\lambda)$ is real. (Because of the infinite potential barrier existing at $uv\to+\infty$, the waves coming from the past horizon leave out through the future horizon.)

The functions $\Psi^{\rm in}_\lambda$ are orthonormal with respect to the scalar product

$$\langle\Psi_\lambda,\Psi_{\lambda'}\rangle=i\int\Psi^*_\lambda\overleftrightarrow{j}_\mu\Psi_{\lambda'}d\Sigma^\mu\ ,$$

$$(\overleftrightarrow{j}_\mu=\sqrt{-g}\,\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu\sqrt{-g}\,)\ .$$

With two event horizons, conditions (3) are modified to

$$f'(\pm\infty)=0$$

and besides the solutions $\overrightarrow{\phi}^{\rm in}_\lambda\equiv\phi^{\rm in}_\lambda$, we would also need the solutions $\overleftarrow{\phi}^{\rm in}_\lambda$ satisfying

$$\lim_{u'\to+\infty}\overleftarrow{\phi}^{\rm in}_{\lambda_1}=\frac{1}{2\sqrt{\pi\lambda_1}}e^{-i\lambda_1 v'}\ ,\qquad \lim_{v'\to-\infty}\overleftarrow{\phi}^{\rm in}_1=0,\ \ \lambda_1>0 \tag{12}$$

to form a complete basis.

Alternatively, with no event horizons, one could choose

$$f'(\pm\infty)=\text{finite constant } C_{\pm}\neq 0 \ ,$$

i.e., asymptotically inertial frames, and then $\{\vec{\phi}_{\lambda}^{\rm in},\overleftarrow{\phi}_{\lambda}^{\rm in}\}$ would describe asymptotically massive plane waves (with effective mass not necessarily equal in the left and right asymptotic regions).

The solutions ϕ_{λ_1} are described for frequencies λ_1 which are positive with respect to the accelerated time t'. As is well known a complete set of solutions describing positive frequencies ω with respect to the inertial time t is given by

$$\varphi_k=\frac{1}{4\pi\sqrt{\pi E_k}}e^{i(k_1x+k_2y+k_3z-E_kt)}$$

with $-\infty<k_1,k_2,k_3<+\infty$,

$$E_k=+(k_1{}^2+k_2{}^2+k_3{}^2+m^2)^{1/2}>0 \ . \tag{13}$$

We recall that in the formulation of QFT in accelerated spaces, the dynamical operators are defined in terms of the accelerated creation and annihilation operators $C_\lambda, C_\lambda^\dagger$ associated with the accelerated modes ϕ_λ. The vacuum state of the theory ($|0\rangle$) is defined by the inertial operators a_k associated with the inertial modes φ_k, i.e.,

$$a_k\,|0\rangle=0,\quad \forall k \ .$$

The state $|{}^{\rm in}0'\rangle$ such that $C_\lambda^{\rm in}\,|{}^{\rm in}0'\rangle=0$, $\forall\lambda$, is an excited state with respect to the true vacuum $|0\rangle$. A Bogoliubov transformation relates $C_\lambda^{\rm in}$ to a_k and $a_k^\dagger$,

$$C_\lambda^{\rm in}=\int_{-\infty}^{\infty}d^3k\,[A_\lambda(k)a_k+B_\lambda(k)a_k^\dagger] \ , \tag{14}$$

where

$$A_\lambda(k)=\langle\phi_\lambda^{\rm in},\varphi_k\rangle,\quad B_\lambda(k)=\langle\phi_\lambda^{\rm in},\varphi_k^*\rangle \ . \tag{15}$$

Alternatively to the solutions $\phi_\lambda^{\rm in}$, one can define solutions $\phi_\lambda^{\rm out}$ by fixing the positive-frequency boundary condition at the future rather than at the past, thus

$$\lim_{u'\to-\infty}\phi_\lambda^{\rm out}=\frac{1}{2\sqrt{\pi\lambda}}e^{-i\lambda v'} \ ,\qquad \lim_{v'\to+\infty}\phi_\lambda=0 \ . \tag{16}$$

They satisfy $\phi_\lambda^{\rm out}(u',v')=\phi_\lambda^{\rm in*}(v',u')$. Analogously, we could define $C_\lambda^{\rm out}$ and $|0'^{\rm out}\rangle$ such that $C_\lambda^{\rm out}\,|0'^{\rm out}\rangle=0$. Note that $|0'^{\rm out}\rangle\neq|0'^{\rm in}\rangle$. (Only in the Rindler frame $|0'^{\rm out}\rangle=|0'^{\rm in}\rangle$ up to a phase factor.) As we always deal with the in formulation we omit in what follows the superscript "in."

Before proceeding further it will be useful, as in the two-dimensional case, to introduce the functions

$$N(\lambda,\lambda')\equiv\langle 0\,|\,C_\lambda C_{\lambda'}^\dagger\,|\,0\rangle=\int_{-\infty}^{\infty}d^3k\,B_\lambda(k)B_{\lambda'}^*(k) \ , \tag{17}$$

$$R(\lambda,\lambda')\equiv\langle 0\,|\,C_\lambda C_{\lambda'}\,|\,0\rangle=\int_{-\infty}^{\infty}d^3k\,A_\lambda(k)B_{\lambda'}(k) \ . \tag{18}$$

$N(\lambda,\lambda')$ and $R(\lambda,\lambda')$ describe interferences between the created modes with different frequencies λ, λ'. $N(\lambda,\lambda')$ is the production function. For $\lambda=\lambda'$ it gives the number $N(\lambda)$ of λ quanta in the vacuum $|0\rangle$ on the total volume. The number $N_v(\lambda)$ of λ quanta per unit volume is obtained by introducing wave packets; i.e.,

$$N_v(\lambda)=\lim_{\xi\to\infty}\int\int_0^\infty d\lambda'd\lambda''W_\xi(\lambda,\lambda')W_\xi^*(\lambda',\lambda'')\times N(\lambda',\lambda'')$$

W_ξ is such that

$$\int_0^\infty d\lambda\,|\,W_\xi(\lambda,\lambda')\,|^2=1 \ .$$

For instance

$$W_\xi(\lambda,\lambda')=\sqrt{2\xi/\pi}\exp[-\xi(\lambda-\lambda')^2] \ .$$

From Eqs. (10), (13), and (15), we find

$$B_\lambda(k)=B_{\lambda_1}(k_1,M)\delta(k_2+\lambda_2)\delta(k_3+\lambda_3),\quad M^2=\lambda_2{}^2+\lambda_3{}^2+m^2 \ , \tag{19a}$$

where

$$B_{\lambda_1}(k_1,M)=-\frac{(k_1+E_k)}{4\pi\sqrt{\lambda_1E_k}}\int_{u_-}^{u_+}du\,\exp\left[-i\lambda_1F(u)-\frac{i}{2}(k_1+E_k)u\right] \ . \tag{19b}$$

Then, we get

$$R(\lambda_1,\lambda_1')=\int_{-\infty}^{\infty}dk_1A_{\lambda_1}(k_1)B_{\lambda_1'}(k_1)$$
$$=+\frac{1}{4\pi^2}\frac{1}{\sqrt{\lambda_1\lambda_1'}}\int_0^\infty du\,du_1\frac{\exp[i\lambda_1F(u+i\epsilon)-i\lambda_1'F(u_1-i\epsilon)]}{(u-u_1+i\epsilon)^2},\quad \epsilon>0 \ . \tag{20b}$$

$$R(\lambda_1,\lambda_1')=\int_{-\infty}^{\infty}dk_1A_{\lambda_1}(k_1)B_{\lambda_1'}(k_1)$$
$$=+\frac{1}{4\pi^2}\frac{1}{\sqrt{\lambda_1\lambda_1'}}\int_0^\infty du\,du_1\frac{\exp[i\lambda_1F(u+i\epsilon)-i\lambda_1'F(u_1-i\epsilon)]}{(u-u_1+i\epsilon)^2},\quad \epsilon>0 \ . \tag{20b}$$

The $B_\lambda(k)$ coefficient factorizes in the product of a two-dimensional massive coefficient of effective mass M and two δ

functions. $N(\lambda,\lambda')$ and $R(\lambda,\lambda')$ also factorize in this way but, because of boundary conditions (3), they are independent of the mass of the field. Conversely, given $N(\lambda,\lambda')$ we reconstruct the mapping, i.e.,

$$f(u')=f(u_0')\exp\left[-4\pi i\,\mathrm{Re}\int_0^\infty \frac{d\lambda}{\lambda}e^{i\lambda u'}[\sqrt{\lambda\lambda'}N(\lambda,\lambda')]_{\lambda'=0}\right],\quad y'=y,\quad z'=z\ , \tag{21}$$

where $f(u_0')$ is an integration constant (scale factor of the transformation).

From Eq. (21) we get the relation

$$\frac{1}{4\pi}\frac{d}{du'}\ln f(u')=\mathrm{Re}\int d\lambda\, e^{i\lambda u'}[\sqrt{\lambda\lambda'}N(\lambda,\lambda')]_{\lambda'=0}\ . \tag{22}$$

III. UNIQUENESS OF THE EXPONENTIAL MAPPING AND THERMAL PROPERTIES OF THE ACCELERATED FRAMES

From the above results, we prove the following theorem: each one of the following statements implies the two others.

(i) The functions $N(\lambda,\lambda')$ and $R(\lambda,\lambda')$ have the form

$$N(\lambda,\lambda')=N_v(\lambda_1)\delta(\lambda_1-\lambda_1')\delta(\lambda_2-\lambda_2')\delta(\lambda_3-\lambda_3')\ ,$$

$$R(\lambda,\lambda')=0\ .$$

(ii) The Bogoliubov transformation can be decomposed as a two-term one:

$$C_\lambda=[1+N_v(\lambda_1)]^{1/2}\tilde{C}_{\lambda(+)}-[N_v(\lambda_1)]^{1/2}\tilde{C}_{\lambda(-)}\ .$$

(iii) The accelerated space is

$$f(u')=e^{2\pi T u'},\quad \text{i.e.,}\quad F(u)=\frac{1}{2\pi T}\ln u\ , \tag{23}$$

where

$$T=[\lambda_1 N_v(\lambda_1)]_{\lambda_1=0}\ . \tag{24}$$

The equivalence of statements (i) and (iii) follows from Eq. (16).

The equivalence of statements (i) and (ii) follows from the relation

$$A_\lambda(k)=\left[\frac{1+N_v(\lambda_1)}{N_v(\lambda_1)}\right]^{1/2}B_\lambda(k)$$

which is necessary and sufficient condition for the Bogoliubov transformation being decomposable. This condition allows us to define a basis

$$\tilde{C}_{\lambda(+)}=\int_{-\infty}^{\infty}d^3k\frac{A_\lambda(k)}{[1+N_v(\lambda_1)]^{1/2}}a_k\ ,$$

$$\tilde{C}_{\lambda(-)}=\int_{-\infty}^{\infty}d^3k\frac{B_\lambda^*(k)}{[N_v(\lambda_1)]^{1/2}}a_k^\dagger\ ,$$

such that

$$[\tilde{C}_{\lambda(+)},\tilde{C}^\dagger_{\lambda'(+)}]=\int_{-\infty}^{\infty}d^3k\,A_\lambda^*(k)A_{\lambda'}(k)=[1+N_v(\lambda_1)]\delta^{(3)}(\lambda-\lambda')\ , \tag{25a}$$

$$[\tilde{C}_{\lambda(-)},\tilde{C}^\dagger_{\lambda'(-)}]=\int_{-\infty}^{\infty}d^3k\,B_\lambda^*(k)B_{\lambda'}(k)=N_v(\lambda_1)\delta^{(3)}(\lambda-\lambda')\ , \tag{25b}$$

$$[\tilde{C}_{\lambda(+)},\tilde{C}^\dagger_{\lambda'(-)}]=\int_{-\infty}^{\infty}d^3k\,A_\lambda(k)B_{\lambda'}(k)=0=[\tilde{C}^\dagger_{\lambda(+)},\tilde{C}_{\lambda'(-)}]\ . \tag{25c}$$

We see that Eqs. (25a) and (25b) give statement (i). It should be noted that the theorem defines $f(u')$ as given by Eq. (21) up to a bilinear transformation such that

$$\left[\frac{u-u_-}{u_+-u}\right]=e^{2\pi T u'},\quad u_-\le u\le u_+\ . \tag{26}$$

y' and z' are defined up to a linear transformation on t', namely,

$$y=y'+\sigma t',\quad z=z'+\gamma t'\ ,$$

i.e., up to a drifting (or uniform rotation) in y and z; σ and γ are constants.

A corollary of the theorem is the following: If $N(\lambda,\lambda')$ satisfies the statement (i), then $N_v(\lambda)$ is given by

$$N_v(\lambda)=\frac{1}{(e^{\lambda/T}-1)}\ , \tag{27}$$

but the converse is not true.

We see that the parameter T as defined by Eq. (24) plays the role of a temperature. For any of the statements (i), (ii), (iii), Eq. (22) is equal to a constant of value $2\pi T$. The theorem characterizes a situation of global thermal equilibrium over the whole accelerated space. This situation implies the presence of events horizons. The Rindler frame has one event horizon [Eq. (23)] or two event horizons at the same temperature [Eq. (26)]. Note that the presence of event horizons is a necessary (but not a sufficient) condition for global thermal equilibrium.

A local (or asymptotic) thermal equilibrium situation is described by the class of mappings $f(u')$ such that

$$\lim_{u'\to\pm\infty}f(u')=e^{2\pi T_\pm u'}\ , \tag{28a}$$

or equivalently by

$$\lim_{\lambda\to\lambda'}N(\lambda,\lambda')=N_v(\lambda_1)\delta^{(3)}(\lambda-\lambda')\ ,\qquad \lim_{\lambda\to\lambda'}R(\lambda,\lambda')=0\ . \tag{28b}$$

The accelerated spaces corresponding to Eqs. (28a) or (28b) have nonuniform acceleration but for $u'\to\pm\infty$ the acceleration becomes uniform, i.e, the systems become of Rindler type. For all these spaces $N_v(\lambda)$ is given by Eq. (27), or more generally by

$$N_v(\lambda)=\frac{1}{2}\left[\frac{1}{(e^{\lambda/T_-}-1)}+\frac{1}{(e^{\lambda/T_+}-1)}\right]. \tag{29}$$

The asymptotic temperature $T_\pm$ are given by

$$T_\pm=\lim_{u'\to\pm\infty}\int_0^\infty d\lambda\cos\lambda u'[\sqrt{\lambda\lambda'}N(\lambda,\lambda')]_{\lambda'=0} \tag{30}$$

or equivalently by

$$T\pm=\frac{1}{2\pi}\frac{d}{du'}[\ln f(u')]_{u'=\pm\infty}=\frac{1}{2\pi}(\sqrt{g_{00}}\,a)\ , \tag{31}$$

where

$$a=\frac{1}{[\Lambda(x',t')]^{1/2}}\partial_{x'}\ln\Lambda(x',t')$$

is the acceleration. A typical nonthermal situation is described by mappings $f(u')$ such that

$$\lim_{u'\to\pm\infty}f(u')=\frac{\alpha}{u'}+\beta u' \tag{32}$$

or equivalently by

$$\begin{aligned}&\lim_{\lambda\to\lambda'}N(\lambda,\lambda')=N(\lambda_1)\delta(\lambda_2-\lambda_2')\delta(\lambda_3-\lambda_3')\ ,\\&\lim_{\lambda\to\lambda'}R(\lambda,\lambda')=R(\lambda_1)\delta(\lambda_2+\lambda_2')\delta(\lambda_3+\lambda_3')\ ,\end{aligned} \tag{33}$$

where $N(\lambda_1)$ and $R(\lambda_1)$, are nonvanishing finite functions of λ_1. For these spaces, the acceleration is nonuniform and particle production takes place in a nonthermal situation and is confined within a finite volume of the space (the total vacuum energy is finite). All the spaces of this class have

$$T_\pm=0 \quad \text{and} \quad N_v(\lambda_1)=0\ . \tag{34}$$

IV. THE ENERGY-MOMENTUM TENSOR AND ITS RENORMALIZATION

When normalization with respect to some particular state which has been defined to be empty is not appropriate, it has become a practice in this field to introduce a complete renormalization of the theory, essentially by extending techniques developed for empty Minkowski space. Whatever the problems implied by that approach, we will look, for the present, at the properties of some of the schemes adopted to carry out the combined regularization and renormalization of the theory, with an emphasis on the point-separation scheme. This has the advantage for the task at hand that not only is it defined entirely in and on the original manifold (as is also ζ function regularization) but the divergences can actually be obtained from the divergent terms of the effective action given by the vacuum-to-vacuum amplitude. Thus, in a sense, the renormalization is immediate, with both infinite and finite subtractions being given by those terms in the effective action which lead to divergences. Dimensional regularization involves an extension of the space-time with extra flat dimensions and may not be very appropriate in curved spaces where differences from other schemes have occurred for some higher spin fields. Our explicit representation of the fields means that point separation is an appropriate regularization scheme to discuss here.

We first consider the Green's function $G(P_1,P_2)$ and the renormalization of $\langle 0|\Psi^2|0\rangle$. Inertial and accelerated observers define the same Green's function

$$G(P_1,P_2)=\langle 0|[\Psi(P_1),\Psi(P_2)]_+|0\rangle \tag{35}$$

for the free fields. Inertial observers express G in terms of the modes φ_k; accelerated observers express G in terms of the modes ϕ_λ:

$$\begin{aligned}G(P_1,P_2)&=\int d^3k\,\varphi_k(P_1)\varphi_k^*(P_2)\\&=\int d^3\lambda\,\phi_\lambda(P_1)\phi_\lambda^*(P_2)+2\,\mathrm{Re}\int\int d^3\lambda\,d^3\lambda'[N(\lambda,\lambda')\phi_\lambda(P_1)\phi_{\lambda'}^*(P_2)+R(\lambda,\lambda')\phi_\lambda(P_1)\phi_{\lambda'}(P_2)]\ .\end{aligned} \tag{36}$$

It can be noted that

$$\begin{aligned}G'(P_1,P_2)&=\langle 0'|[\Psi(P_1),\Psi(P_2)]_+|0'\rangle\\&=\int d^3\lambda\,\phi_\lambda(P_1)\phi_\lambda^*(P_2)\end{aligned} \tag{37}$$

is the "false" Green's function defined with respect to the accelerated state $|0'\rangle$. G' is not translationally invariant. As is well known, when one attempts to calculate $\langle 0|\Psi^2|0\rangle$, divergences appear:

$$\begin{aligned}\langle 0|\Psi^2|0\rangle&=\int_{-\infty}^{\infty}d^3k\,|\varphi_k|^2\\&=\mathcal{F}+\langle 0'|\Psi^2|0'\rangle\ ,\end{aligned} \tag{38}$$

where

$$\mathcal{F}=2\,\mathrm{Re}\int\int d^3\lambda\,d^3\lambda'[N(\lambda,\lambda')\phi_\lambda\phi_{\lambda'}^*+R(\lambda,\lambda')\phi_\lambda\phi_{\lambda'}]\ , \tag{39a}$$

$$\langle 0|\Psi^2|0\rangle=\int d^3\lambda\,|\phi_\lambda|^2\ . \tag{39b}$$

The left-hand side (LHS) of (38) is divergent. In the RHS the first term ($\mathcal{F}$) is finite whereas the second also has a divergent part. In fact, in the accelerated state, the divergences of $\langle 0'|\Psi^2|0'\rangle$ are exactly those which appear in $\langle 0|\Psi^2|0\rangle$; that is, they are independent of the particular state chosen. This divergent behavior is general for the expectation value of any composite operator. With respect to the eigenmodes of the accelerated state, $\langle 0|\Psi^2|0\rangle$ separates into a finite term plus a term which can be recast as the expectation value on the state $|0'\rangle$. This term contains the infinite part. The problem is then to separate the divergent and finite parts of the term given by Eq. (39b) and to justify discarding the divergences. Although usually thought of as resulting from the short-distance behavior, the divergences in Eq. (38) or in the integral (39b) are actually governed by the properties of the mode functions in the asymptotic regions of the space-time. The apparent divergence dependence on the asymptotic region comes precisely from the fact that, whatever the asymptotic region, we always choose fields which, asymptotically, are "free" fields in that region. Thus we will always have the divergence appearing exactly in the way that it occurs for the vacuum in Minkowski space. This allows us to subtract the divergences in the accelerated frame by following a similar procedure to that in the inertial frames. In the inertial frame, with some regulator ϵ, we would have

$$\langle 0 | \Psi^2(\epsilon) | 0 \rangle = \langle 0 | \Psi^2(\epsilon) | 0 \rangle_{\rm IF} + \langle 0 | \Psi^2(\epsilon) | 0 \rangle_{\rm ID} ,$$

where

$$\lim_{\epsilon\to 0} \langle 0 | \Psi^2(\epsilon) | 0 \rangle_{\rm IF} < \infty ,$$

$$\lim_{\epsilon\to 0} \langle 0 | \Psi^2(\epsilon) | 0 \rangle_{\rm ID} = \infty .$$

ID and IF stand for the inertial divergent and finite parts, respectively. By subtracting $\langle 0 | \Psi^2(\epsilon) | \, \rangle_{\rm ID}$ and letting $\epsilon \to 0$, one obtains the (inertial) renormalized quantity

$$\langle 0 | \Psi^2 | 0 \rangle_{\rm Iren} = \lim_{\epsilon\to 0} [\langle 0 | \Psi^2(\epsilon) | 0 \rangle - \langle 0 | \Psi^2(\epsilon) | 0 \rangle_{\rm ID}] = 0 . \tag{40a}$$

In the accelerated frame we follow a similar procedure, but the inertial (ϵ) and accelerated (ϵ') regulators need not necessarily be identical:

$$\langle 0 | \Psi^2(\epsilon') | 0 \rangle = \mathcal{F} + \langle 0' | \Psi^2(\epsilon') | 0' \rangle$$

with

$$\langle 0' \Psi^2(\epsilon') | 0' \rangle = \langle 0' | \Psi^2(\epsilon') | 0' \rangle_{\rm AF} + \langle 0' | \Psi^2(\epsilon') | 0' \rangle_{\rm AD} ,$$

$$\lim_{\epsilon'\to 0} \langle 0' | \Psi^2(\epsilon') | 0' \rangle_{\rm AF} = \infty, \quad \lim_{\epsilon'\to 0} \langle 0' | \Psi^2(\epsilon') | 0' \rangle = \infty .$$

Whence, we subtract $\langle 0' | \Psi^2(\epsilon') | 0' \rangle_{\rm AD}$ to obtain the (accelerated) renormalized quantity

$$\langle 0' | \Psi^2 | 0' \rangle_{\rm Aren} = \lim_{\epsilon\to 0} [\langle 0' | \Psi^2(\epsilon') | 0' \rangle - \langle 0' | \Psi^2(\epsilon') | 0' \rangle_{\rm AD}] \tag{40b}$$

since in the Minkowski vacuum $\langle 0 | \Psi^2 | 0 \rangle_{\rm Iren} = 0$. Here AF and AD stand for the accelerated finite and divergent parts, respectively; Aren stands for the renormalized value in the accelerated state. Clearly, renormalization in flat (Minkowski) space can be made easier than it would be in general, since vacuum expectation values of $\hat{\Psi}^2$ (and $\hat{T}_{\mu\nu}$) are zero in the inertial vacuum, and our observations here are also sufficient for a discussion of the renormalization in the interacting case.

From Eqs. (40a) and (40b) we see that whereas field quantization lead to indentical divergences, renormalization has assigned different finite values to the same operator in different quantum states. This is irrespective of the adopted regularization scheme. Thus $\langle \hat{T}_{\mu\nu} \rangle_{\rm reg}$ need not be covariant (e.g., the introduction of a cutoff in the high momenta, or ordinary "point splitting" are not) but $\langle T_{\mu\nu} \rangle_{\rm ren}$ should be. We refer to Ref. 4 for a critical discussion of various proposals of regularization. For later use we note the possibility of evaluating $\langle 0 | \Psi^2 | 0 \rangle$ as the $\lim_{P_1 \to P_2} G(P_1, P_2)$ in a space-time representation rather than in a mode sum. However, for arbitrary mappings we will need to use the modes ϕ_λ in calculating the divergent part [Eq. (39b)] in order to obtain a finite $\langle 0' | \Psi^2 | 0' \rangle_{\rm ren}$. It is only the convenient representation of quantized fields asymptotically in some particular coordinate system which has sometimes lead to the incorrect notion that the states associated with those fields are coordinate dependent: of course they are not.

We now consider the energy-momentum operator (for a minimally coupled field) given by

$$\hat{T}_{\mu\nu} = \partial_\mu \hat{\psi} \partial_\nu \hat{\psi} - \tfrac{1}{2} g_{\mu\nu} (\partial^\sigma \hat{\psi} \partial_\sigma \hat{\psi} + m^2 \hat{\psi}^2) .$$

With respect to eigenmodes for the accelerated state, the vacuum expectation value $\langle 0 | \hat{T}_{\mu\nu} | 0 \rangle$ can be expressed as

$$\langle 0 | \hat{T}_{\mu\nu} | 0 \rangle = 2\,{\rm Re} \int\int d^3\lambda\, d^3\lambda' [N(\lambda,\lambda') T_{\mu\nu}(\phi_\lambda, \phi^*_{\lambda'}) + R(\lambda,\lambda') T_{\mu\nu}(\phi_{\lambda'}, \phi_{\lambda'})] + \langle 0' | \hat{T}_{\mu\nu} | 0' \rangle , \tag{41}$$

where

$$\langle 0' | \hat{T}_{\mu\nu} | 0' \rangle = \int d^3\lambda\, T_{\mu\nu}(\phi_\lambda, \phi^*_\lambda) \tag{42}$$

and

$$T_{\mu\nu}(\phi,\varphi) = \partial_\mu \phi \partial_\nu \varphi - \tfrac{1}{2} g_{\mu\nu} (g^{\sigma\rho} \partial_\sigma \phi \partial_\rho \varphi + m^2 \phi\varphi) .$$

Derivatives and indices generally will refer to primed (accelerated) coordinates. In our coordinates defined by Eq. (1), $T_{\mu\mu}$ and T_{0i} components can be concisely expressed as

$$T_{\mu\mu}(\phi,\varphi) = \partial_{t'} \phi \partial_{t'} \varphi + \alpha_\mu \partial_{x'} \phi \partial_{x'} \varphi + \Lambda \beta_\mu (\partial_{y'} \phi \partial_{y'} \varphi + \alpha_\mu \partial_{z'} \phi \partial_{z'} \varphi + \gamma_\mu m^2 \phi\varphi), \quad T_{0i}(\phi,\varphi) = \partial_{t'} \phi \partial_{t'} \varphi ,$$

where

$$\left.\begin{array}{r} \alpha_0 = \beta_0 = \gamma_0 \\ \alpha_1 = -\beta_1 = \gamma_1 \\ -\alpha_2 = \beta_2 = -\gamma_2 \\ -\alpha_3 = -\beta_3 = \gamma_3 \end{array}\right\} = 1 .$$

The vacuum energy and momentum densities are given by $\langle 0' | \hat{T}_{00} | 0' \rangle$ and $\langle 0' | -\hat{T}_{0i} | 0' \rangle$, respectively. By using Eqs. (19c) and (19d) and performing the integrations in λ'_2 and λ'_3 we see that the diagonal 00 and $0i$ components can be written directly in terms of their two-dimensional counterparts with the modified M^2 and the 22 and 33 in terms of a modified operator which we can write for all cases as

$$T^{(2)}_{\mu\mu}(\phi,\varphi;M_\mu)=\partial_{t'}\phi\partial_{t'}\varphi+\alpha_\mu\partial_{x'}\phi\partial_{x'}\varphi+\Lambda M_\mu{}^2\phi\varphi$$

with

$$M_\mu{}^2=\beta_\mu(\lambda_2{}^2+\alpha_\mu\lambda_3{}^2+\gamma_\mu m^2) \quad (M_0{}^2\equiv M^2) .$$

Then

$$\langle 0 | \hat{T}^{(4)}_{\mu\mu}(x',t';m) | 0\rangle = \int\int_{-\infty}^{\infty} d\lambda_2 d\lambda_3 \langle 0 | \hat{T}^{(2)}_{\mu\mu}(x',t';M_\mu) | 0\rangle ,$$

where

$$\langle 0 | \hat{T}^{(2)}_{\mu\mu}(x',t';M_\mu) | 0\rangle = 2\,\mathrm{Re}\int\int_0^\infty d\lambda_1 d\lambda_1' [N(\lambda_1,\lambda_1')T^{(2)}_{\mu\mu}(\phi_{\lambda_1},\phi^*_{\lambda_1'};M_\mu)+R(\lambda_1,\lambda_1')T^{(2)}_{\mu\mu}(\phi_{\lambda_1},\phi_{\lambda_1'};M_\mu)] + \int_0^\infty d_{\lambda_1} T^{(2)}_{\mu\mu}(\phi_{\lambda_1},\phi^*_{\lambda_1'};M_\mu) .$$

The equation is written analogously for $\langle 0 | \hat{T}_{0i} | 0\rangle$; $\langle \hat{T}_{00}(2)\rangle$ is just the two-dimensional energy density with M^2 instead of m^2.

For the purpose of computation, it is useful to introduce the quantity

$$G_{\mu\nu}=\langle 0 | \partial_\mu\hat{\psi}\partial_\nu\hat{\psi} | 0\rangle = 2\,\mathrm{Re}\int\int d^3\lambda\, d^3\lambda' [N(\lambda,\lambda')\partial_\mu\psi_\lambda\partial_\nu\psi^*_{\lambda'}+R(\lambda,\lambda')\partial_\mu\psi_\lambda\partial_\nu\psi_{\lambda'}]+\int d^3\lambda\, \partial_\mu\psi_\lambda\partial_\nu\psi^*_\lambda . \tag{43}$$

and give explicitly

$$H=\langle 0 | \hat{T}_{00} | 0\rangle = \tfrac{1}{2}[G_{00}+G_{11}+\Lambda(G_{22}+G_{33}+m^2G)] ,$$

$$P_i=\langle 0 | \hat{T}_{0i} | 0\rangle = -G_{0i} ,$$

$$\langle 0 | \hat{T}_{11} | 0\rangle = \tfrac{1}{2}[G_{00}+G_{11}-\Lambda(G_{22}+G_{33}+m^2G)] ,$$

$$\langle 0 | \hat{T}_{22} | 0\rangle = \frac{1}{2\Lambda}[G_{00}-G_{11}+\Lambda(G_{22}-G_{33}-m^2G)] ,$$

$$\langle 0 | \hat{T}_{33} | 0\rangle = \frac{1}{2\Lambda}[G_{00}-G_{11}-\Lambda(G_{22}-G_{33}+m^2G)] ,$$

where $G=\langle 0 | \hat{\Psi}^2 | 0\rangle$ is given by Eq. (36).

According to the renormalization prescription given by Eq. (40), we have

$$\langle 0 | \hat{T}_{\mu\nu} | 0\rangle_{\mathrm{Aren}} = \langle 0 | \hat{T}_{\mu\nu} | 0\rangle - \langle 0' | \hat{T}_{\mu\nu} | 0'\rangle_{\mathrm{AD}}$$
$$= 2\,\mathrm{Re}\int\int d^3\lambda\, d^3\lambda' [N(\lambda,\lambda')T_{\mu\nu}(\phi^*_\lambda,\phi_{\lambda'})+R(\lambda\lambda')T_{\mu\nu}(\phi_\lambda,\phi_{\lambda'})]+\left[\int d^3\lambda\, T_{\mu\nu}(\phi^*_\lambda,\phi_{\lambda'})\right]_{\mathrm{AF}} . \tag{44}$$

It is implied here that a regulator (ϵ') is introduced and the $\epsilon'\rightarrow 0$ limit is taken after subtraction.

A. Applications

By way of application, we consider the explicit evaluation of our formalism for the example of Rindler space, which, as we have shown, has unique thermal properties globally. The mapping which gives rise to Rindler space is given by

$$f(x'\pm t')=le^{\alpha(x'\pm t')}$$

(l and α^{-1} are unrelated length scales) so that

$$\Lambda=l^2f'(x'+t')f'(x'-t')=l^2\alpha^2e^{2\alpha x'} .$$

The explicit solution of (8) satisfying boundary conditions (9) can be found by evaluating (10), or in this case, directly: i.e.,

$$\psi_\lambda=\frac{1}{2\pi}e^{i(\lambda_2 y'+\lambda_3 z')}\phi_{\lambda_1}(x',t') , \tag{45a}$$

$$\phi_{\lambda_1}(x',t')=\frac{(lM/2)^{-i\lambda_1/\alpha}e^{-i\lambda_1 t'}}{\sqrt{\pi\lambda_1}\Gamma(-i\lambda_1/\alpha)}K_{-i\lambda_1/\alpha}(lM\,e^{\alpha x'}) , \tag{45b}$$

where $K_\gamma(z)$ is a modified Bassel function. Then, from Eqs. (19) and (20) [or equivalently (19) and (18)] we find

$$B_{\lambda_1}(k,M)=\frac{1}{2\pi\alpha}\left[\frac{\lambda_1}{E}\right]^{1/2}\Gamma\left[-\frac{i\lambda_1}{\alpha}\right]e^{-\pi\lambda_1/2\alpha}\left[\frac{k_1+E}{2}\right]^{i\lambda_1/\alpha}, \tag{46}$$

$$A_{\lambda_1}(k,M)=e^{\lambda\alpha\pi}B_{\lambda_1}(k,M)\ , \tag{47a}$$

$$N(\lambda,\lambda')=\frac{\delta^{(3)}(\lambda-\lambda')}{(e^{2\pi\lambda_1/\alpha}-1)},\ \ R(\lambda,\lambda')=0\ . \tag{47b}$$

Since the full field is given by (45a) we have immediately $\partial_{t'}\psi_\lambda=-i\lambda_1\psi_\lambda$, $\partial_{y'}\psi_\lambda=-i\lambda_2\psi_\lambda$, $\partial_{z'}\psi_\lambda=i\lambda_3\psi_\lambda$. Thus, for example, in G_{02} we have

$$2\,\mathrm{Re}\int\int d^3\lambda\, d^3\lambda' \frac{\delta^{(3)}(\lambda-\lambda')}{e^{2\pi\lambda_1/\alpha}-1}(i\lambda_1\psi^*_\lambda)(_i\lambda_2'\psi\lambda')$$
$$=2\,\mathrm{Re}\int_0^\infty d\lambda_1\int_{-\infty}^{\infty}d\lambda_2 d\lambda_3\frac{(-\lambda_1\lambda_2)}{e^{2\pi\lambda_1\alpha}-1}\psi^*_\lambda\psi_\lambda$$
$$=2\,\mathrm{Re}\int_0^\infty d\lambda_1\int_{-\infty}^{\infty}d\lambda_2 d\lambda_3\frac{(-\lambda_1\lambda_2)}{8\pi^4\alpha}e^{-\pi\lambda_1/\alpha}K_{i\lambda_1/\alpha}(M\,e^{\alpha x'})K_{-i\lambda_1/\alpha}(M\,e^{\alpha x'})=0\ .$$

This last result follows since the λ_1 and λ_3 integrals are finite and the λ_2 integral is odd; recall from (8) that M depends on λ_2 and λ_3. Similarily, we find that the corresponding term in the expressions for G_{03} and G_{23} also vanish. And, since $K_\nu=K_{-\nu}$ we have in G_{01}

$$2\,\mathrm{Re}\int\int d\lambda\, d\lambda'\frac{\delta(\lambda-\lambda')}{(e^{2\pi\lambda_1/\alpha}-1)}(i\lambda_1\psi^*_\lambda)(\partial_{x'}\psi_{\lambda'})$$
$$=2\,\mathrm{Re}\int d\lambda\frac{i\lambda_1}{e^{2\pi\lambda_1/\alpha}-1}(\partial_{x'}\psi_\lambda)\psi_\lambda=0\ ,$$

where the integrals are all finite but the integration result is purely imaginary. The same result follows for this term in G_{12} and G_{13}. Thus we have shown in particular that each $\langle\hat{T}_{0i}\rangle$ (component of the Poynting vector) is zero, in accordance with the global thermal equilibrium properties enjoyed by Rindler space-time. The same is true for the vacuum expectation value of angular momentum operator

$$\hat{L}^{\mu\nu}=\int_\Sigma dv\,(x^\mu\hat{T}^{0\nu}-x^\nu\hat{T}^{0\mu})\ .$$

This cannot be considered, however, as characterizing an isotropic thermal radiation for the Rindler vacuum. Rindler observers have a preferred direction, namely, their spatial direction of acceleration. In Rindler space, $\langle\hat{T}_{00}\rangle$ is a constant up to a dependence in x' through $\sqrt{g_{00}}=\alpha e^{\alpha x'}$. One could regard the gradient $\partial_{x'}\langle\hat{T}_{00}\rangle\check{e}_{x'}$ as defining a preferred direction of the thermal vacuum. On the other hand, the response of a uniformly accelerated detector model with a directional discrimination has been found to be nonisotropic.[5–7] It is not possible to construct an accelerated flat space-time for which the vacuum is spherically symmetric and in global thermal equilibrium. The mapping

$$r\pm t=e^{\alpha(r'\pm t')},\ \ \varphi=\varphi',\ \ \theta=\theta' \tag{48}$$

(r,φ,θ, being spherical type coordinates) yields to

$$ds^2=\alpha^2e^{2\alpha r'}(dr'^2-dt'^2)+e^{2\alpha r'}\cosh^2\alpha t'\,d\Omega^2(\theta,\varphi)\ .$$

The vacuum spectrum $N_v(\lambda)$ is Planckian with asymptotic temperature $\alpha/2\pi$, but there is an asymptotic (rather than global) thermal equilibrium situation in this case.

It is useful at this point to consider the case of two dimensions, for which the solutions to the wave equation and Bogoliubov coefficients are given directly by Eqs. (45b) and (46) with $M=m$; N and R are given by Eqs. (47) with $\lambda\equiv\lambda_1$. Thus, we have for G

$$G=\left[\int_0^\infty d\lambda_1\coth\frac{\pi\lambda_1}{\alpha}\phi^*_{\lambda_1}\phi_{\lambda_1}\right]_{\mathrm{AF}}$$
$$=\left[\int_0^\infty\frac{d\lambda_1}{\alpha\pi^2}\cosh\frac{\pi\lambda_1}{\alpha}K_{i\lambda_1/\alpha}(m\,e^{\alpha x'})\times K_{-i\lambda_1/\alpha}(m\,e^{\alpha x'})\right]_{\mathrm{AF}}\ . \tag{49}$$

The point of reconsidering the massive two-dimensional case is the following: we find that a naive regularization scheme suggests a renormalization which (surprisingly) leads to a sensible physical result. We can examine this outcome rigorously only in the two-dimensional massless case but particular computations for four dimensions show that a class of mappings exists for which a similar result might be proved. The special features we have used in two dimensions apparently need only be partially present in four dimensions. A simple evaluation of $\langle\hat{T}_{00}\rangle$ for any $f(g)$ in the massless case can be obtained from a space-time representation rather than from a mode sum for G. Point separation is a natural choice of regularization in that case. For the purpose of this section, we will use ordinary point splitting; a fully covariant regularization is given in the next section.

The same kind of separation in Eq. (38) between finite and infinite terms occurring in the accelerated frame can be observed if we consider $\langle 0|\Psi^2|0\rangle$ as $\lim_{P_1\to P_2}G(P_1,P_2)$ in a space-time representation. The square length

$$\sigma^2=(x_1-x_2)^2-(t_1-t_2)^2\equiv\Delta u\,\Delta v$$
$$(\Delta u\equiv u_1-u_2,\ \ \Delta v\equiv v_1-v_2)$$

is expressed in terms of the mapping f,g as

$$\sigma^2=[f(u_1')-f(u_2')][g(v_1')-g(v_2')]\ .$$

In the inertial frame, when $\Delta u\to 0$, $\Delta v\to 0$, σ^{-2} diverges as $(\Delta u\Delta v)^{-1}$. In the accelerated frame, when $\Delta u'\to 0$, $\Delta v'\to 0$ ($\Delta u'\equiv u_1'-u_2'$, $\Delta v'\equiv v_1'-v_2'$),

$$f(u_1')=f(u_2')+f'(u_2')\Delta u'+f''(u_2')\frac{\Delta u'^2}{2!}+f'''\frac{\Delta u'^3}{3!}+\cdots\ ,$$

$$g(v_1')=g(v_2')+g'(v_2')\Delta v'+g''(v_2')\frac{\Delta v'^2}{2!}+g'''\frac{\Delta v'^3}{3!}+\cdots\ ,$$

and then

$$\sigma^{-2}=\frac{1}{\Delta u'\Delta v'f'g'}(1-a_f\Delta u'+b_f\Delta u'^2)\times(1-a_g\Delta v'+b_g\Delta v'^2)\ ,$$

where

$$a_f=\frac{1}{2}\frac{f''}{f'},\ \ b_f=\frac{1}{2}\left[\frac{1}{2}\left[\frac{f''}{f'}\right]^2-\frac{1}{3}\frac{f'''}{f'}\right],$$

$$a_g=\frac{1}{2}\frac{g''}{g'},\ \ b_g=\frac{1}{2}\left[\frac{1}{2}\left[\frac{g''}{g'}\right]^2-\frac{1}{3}\frac{g'''}{g'}\right].$$

In the limit $\Delta u'\to 0$, $\Delta v'\to 0$, σ^{-2} separates in a finite plus an infinite term as

$$\sigma^{-2}=\frac{1}{f'g'}\left[\frac{1}{\Delta u'\Delta v'}-\frac{af}{\Delta v'}-\frac{ag}{\Delta u'}\right]+\frac{1}{f'g'}\left[b_f\frac{\Delta u'}{\Delta v'}+bg\frac{\Delta v'}{\Delta u'}+a_fa_g\right].$$

Even to have a well-determined finite part here, one must be careful to specify how the limits $\Delta u'=(\Delta x'-\Delta t')\to 0$, $\Delta v'=(\Delta x'+\Delta t')\to 0$ should be taken. But this simple calculation shows how a purely divergent quantity acquires a different finite term depending on the "renormalization" scheme chosen (here expressed in curvilinear coordinates). Particularly illustrative of this is the evaluation of $\langle 0|\hat{T}_{\mu\nu}|0\rangle$ in the massless case. In two dimensions we have

$$G(P_1,P_2)=-\frac{1}{4\pi}\ln[(u_1-u_2)(v_1-v_2)-i\epsilon]=-\frac{1}{4\pi}\ln[(f_1-f_2)(g_1-g_2)-i\epsilon] \tag{50}$$

and

$$\mathscr{D}_{12}G=(\partial_{u_1'}\partial_{u_2'}+\partial_{v_1'}\partial_{v_2'})G=-\frac{1}{4\pi}\left[\frac{f_1'f_2'}{(f_1-f_2)^2}+\frac{g_1'g_2'}{(g_1-g_2)^2}\right]. \tag{51}$$

Here $f_i\equiv f(u_i')$, $g_i\equiv g(v_i')$, $i=1,2$. In the limit $u_1'\to u_2'$, we have

$$\frac{f_1'f_2'}{(f_1-f_2)^2}=\frac{1}{\Delta u'^2}+\frac{1}{6}\left[\frac{f'''}{f'}-\frac{3}{2}\left[\frac{f''}{f'}\right]^2\right].$$

Then

$$\mathscr{D}_{12}G=-\frac{1}{4\pi}\left\{\frac{1}{\Delta u'^2}+\frac{1}{\Delta v'^2}+\frac{1}{6}\left[\frac{f'''}{f'}-\frac{3}{2}\left[\frac{f''}{f'}\right]^2\right]+\frac{1}{6}\left[\frac{g'''}{g'}-\frac{3}{2}\left[\frac{g''}{g'}\right]^2\right]\right\}+O(\Delta u')+O(\Delta v')\ . \tag{52}$$

On the other hand,

$$\mathscr{D}_{12}G'=\frac{1}{4\pi}\int_0^\infty\frac{d\lambda}{\lambda}(e^{i\lambda\Delta u'}+e^{i\lambda\Delta v'})=-\frac{1}{4\pi}\left[\frac{1}{\Delta u'^2}+\frac{1}{\Delta v'^2}\right] \tag{53}$$

and we see that the divergent part appearing above is the whole of the operator for the accelerated state. Thus a subtraction of the divergent part is equivalent to a normalization with respect to the accelerated state. So we have

$$\langle 0|\hat{T}_{00}|0\rangle_{\rm AF}\equiv\langle 0|\hat{T}_{00}|0\rangle-\langle 0'\hat{T}_{00}|0'\rangle=-\frac{1}{24\pi}\left\{\left[\frac{f'''}{f'}-\frac{3}{2}\left[\frac{f''}{f'}\right]^2\right]+\left[\frac{g'''}{g'}-\frac{3}{2}\left[\frac{g''}{g'}-\frac{3}{2}\left[\frac{g''}{g'}\right]^2\right]\right]\right\}. \tag{54}$$

In the Rindler case $f=e^{\alpha u'}$, $g=e^{\alpha v'}$, each $f(g)$ term gives $-\alpha^2/2$, so

$$\langle 0|\hat{T}_{00}|0\rangle_{\rm AF}=+\frac{\alpha^2}{24\pi}=\frac{\pi T^2}{6}$$

corresponding to the energy density of a Planckian gas at temperature $T=\alpha/2\pi$. We can also show how this result emerges from a mode sum representation. If we take point separation in the time direction for simplicity, we obtain

$$\langle 0|\hat{T}_{00}|0\rangle = \frac{1}{2\pi}\left[\frac{\alpha}{\pi}\right]^2 \int_0^\infty d\lambda\,\lambda\, e^{i\lambda\epsilon'\alpha/\pi}\coth\lambda$$

$$= \frac{1}{2\pi}\left[\frac{\alpha}{\pi}\right]^2\left[\frac{1}{2}\xi\left[2,-\frac{i\alpha\epsilon'}{\pi}\right]+\frac{\pi^2}{\alpha^2\epsilon'^2}\right],$$

where

$$\xi\left[2,-\frac{i\alpha\epsilon'}{2\pi}\right]=\sum_{n=0}^{\infty}\frac{1}{(n-i\alpha\epsilon'/2\pi)^2}$$

$$=-\frac{4\pi^2}{\alpha^2\epsilon'^2}+\xi(2)\,.$$

Then

$$\langle 0|\hat{T}_{00}|0\rangle = -\frac{1}{2\pi}\frac{1}{\epsilon'^2}+\frac{\alpha^2}{24\pi}$$

and

$$\langle 0|\hat{T}_{00}|0\rangle_{\rm AF}=\frac{\pi T^2}{6}\,.$$

The same calculation in four dimensions starting from

$$G(P_1,P_2)=\frac{1}{4\pi}\frac{1}{(\sigma^2+\Delta y^2+\Delta z^2)}=\frac{1}{4\pi}\frac{1}{\Delta},$$

$$\Delta=\Delta f\Delta g+\Delta y^2+\Delta z^2\,,$$

$$\Delta f\Delta g=(f_1-f_2)(g_1-g_2),$$

$$\Delta y=(y_1-y_2),\quad \Delta z=(z_1-z_2)\,,$$

leads to

$$\mathscr{D}_{12}G=\frac{1}{2\pi^2\Delta^3}(\Delta g^2 f'_1 f'_2+\Delta f^2 g'_1 g'_2)+\frac{\Lambda}{\pi^2\Delta}\left[\Delta y^2+\Delta z^2-\frac{1}{4\Delta^2}\right].$$

Now

$$\mathscr{D}_{12}\equiv\partial_{u'_1}\partial_{u'_2}+\partial_{v'_1}\partial_{v'_2}+\frac{\Lambda}{2}(\partial_{y_1}\partial_{y_2}+\partial_{z_1}\partial_{z_2})$$

and

$$\langle\hat{T}_{00}\rangle=\Lambda\lim_{1\to 2}\mathscr{D}_{12}G\,.$$

Taking the limit $\Delta y'\to 0$, $\Delta z'\to 0$ first:

$$\mathscr{D}_{12}G=\frac{1}{2\pi^2}\frac{1}{\Delta f\Delta g}\left[\frac{f'_1 f'_2}{\Delta f^2}+\frac{g'_1 g'_2}{\Delta g^2}\right]-\frac{\Lambda}{4\pi^2}\frac{1}{(\Delta f\Delta g)^2}, \tag{55}$$

and then $u'_1\to u'_2$ ($v'_1\to v'_2$), straightforward but lengthy calculations [expansions to the fourth and fifth derivatives of $f(g)$ must be now included], give

$$\langle 0|T_{00}|0\rangle_{\rm AD}=-\frac{1}{2\pi^2}\frac{1}{\Delta u'^4}[1+(a-\tilde{B}-\tilde{A}^2)\Delta u'^2-(b+a\tilde{A}+\tilde{C}-\tilde{A}\tilde{B})\Delta u'^3]+(\text{same term in }\Delta v')\,,$$

$$\langle 0|T_{00}|0\rangle_{\rm AF}=\frac{1}{2\pi^2}\left[\left[a+\frac{\tilde{B}}{2}\right]\tilde{B}+\left[b-\frac{\tilde{C}}{2}\right]\tilde{A}+(c-\tilde{D})\right], \tag{56}$$

where

$$a=\frac{1}{6}\left[\frac{f'''}{f'}-\frac{3}{2}\left[\frac{f''}{f'}\right]^2\right],\quad b=\frac{1}{(12)^2}\left[\frac{f^{\rm IV}}{f'}-4\frac{f''f'''}{f^{12}}+g\left[\frac{f''}{f'}\right]^3\right],$$

$$c=\frac{1}{(24)^2}\left[\frac{f^{\rm V}}{f'}-3\frac{f''f^{\rm IV}}{f'^2}+21\left[\frac{f''}{f'}\right]^2\frac{2f'''}{f'}-\left[\frac{f'''}{f'}\right]^2\right],$$

$$\tilde{A}=A,\quad \tilde{B}=(A^2-B),\quad \tilde{C}=2AB-C,\quad \tilde{D}=(B^2-2AC-D)\,,$$

$$A=\frac{1}{2}\left[\frac{g''}{g'}-\frac{f''}{f'}\right],\quad B=\frac{1}{6}\left[\frac{g'''}{g'}+\frac{f'''}{f'}-\frac{3}{2}\frac{f''g''}{f'g'}\right],\quad C=\frac{1}{24}\left[\frac{g^{\rm IV}}{g'}-\frac{f^{\rm IV}}{f'}-2\left[\frac{f''g'''-g''f'''}{f'g'}\right]\right],$$

$$D=\frac{1}{120}\left[\frac{g^{\rm V}}{g'}+\frac{f^{\rm V}}{f'}-\frac{5}{2}\left[\frac{f^{\rm IV}}{f'}-\frac{g^{\rm IV}}{g'}\right]\left[\frac{f''}{f'}-\frac{g''}{g'}\right]+\frac{10}{3}\frac{f'''g'''}{f'g'}\right].$$

In the Rindler case, these equations give

$$a=\alpha^2/12,\quad b=\alpha^3/3,\quad c=\alpha^4/32\ ,$$
$$A=0,\quad B=\alpha^2/12,\quad C=0,\quad D=\tfrac{2}{45}\alpha^4\ ,$$
$$\tilde{A}=0,\quad \tilde{B}=-\alpha^2/12,\quad \tilde{C}=0,\quad \tilde{D}=\tfrac{3}{80}\alpha^4\ ,$$
$$\langle 0|\hat{T}_{00}|0\rangle_{\rm AF}=\frac{1}{2\pi^2}(\tfrac{3}{2}\tilde{B}^2+c-\tilde{D})$$
$$=\frac{1}{2\pi^2}\frac{\alpha^4}{240}=\frac{\pi^2T^4}{30}\ , \tag{57}$$

corresponding to a Planckian density energy at temperature $T=\alpha/2\pi$ in four dimensions.

In the two-dimensional case, the finite part of $D_{12}\,G$ is

$$-\frac{1}{24\pi}\left[\frac{f'''}{f'}-\frac{3}{2}\left[\frac{f''}{f'}\right]^2\right]-\frac{1}{24\pi}\left[\frac{g'''}{g'}-\frac{3}{2}\left[\frac{g''}{g'}\right]^2\right],$$

where each factor is related to a Schwarzian derivative which, like the measure in Eq. (20) is invariant under homographic transformations: $f\to(af+b)(cf+d)^{-1}$. But, from Davies and Fulling,[8] or our covariant point-splitting result later in this paper, this is just the negative of

$$\langle 0'|\hat{T}_{00}|0'\rangle_{\rm Aren}=\langle 0'|\hat{T}_{00}|0'\rangle-\langle 0|\hat{T}_{00}|0\rangle\ .$$

Thus we know that the subtracted divergence is exactly an expectation value with respect to some state, which happens to be (homographically equivalent to) the accelerated state, and that we have therefore normalized the Minkowski vacuum with respect to this state. In the four-dimensional case we stumble upon a similar result for the exponential mapping without being able to find the general class for which it is true.

B. ν dimensions

For massive fields, calculation by dimensional regularization is often very instructive. We shall first calculate $\langle\hat{T}_{\mu\lambda}\rangle$ in generic space-time dimensions and then consider dimensionalities of interest. We analyze now $\langle\hat{T}_{\mu\lambda}\rangle$ as a function of the space-time dimension.

The Green's function in ν dimensions is given by

$$G(X)=\int\frac{d^\nu k}{(2\pi)^\nu}\frac{e^{ikX}}{k^2+m^2+i\epsilon}$$
$$=\frac{m^{(\nu/2)-1}}{(2\pi)^{\nu/2}}\frac{K_{(\nu/2)-1}[m(X^2+i\epsilon)^{1/2}]}{[(X^2+i\epsilon)^{1/2}]^{(\nu/2)-1}}\ , \tag{58}$$

where

$$X^2=-X_0{}^2+X_1{}^2+\cdots+X_{\nu-1}{}^2$$

and

$$K_\lambda(z)=\frac{\pi}{2}\frac{I_{-\lambda(z)}+I_{\lambda(z)}}{\sin\pi\lambda\,{\rm Re}\lambda>0}\ .$$

By using

$$I_\lambda(z)=\frac{1}{\Gamma(\lambda+1)}\left[\frac{z}{2}\right]^\lambda\left[1+\frac{1}{(\lambda+1)}\left[\frac{z}{2}\right]^2+\frac{1}{2(\lambda+1)(\lambda+2)}\left[\frac{z}{2}\right]^4+O(z^6)\right]$$

we have

$$G(X)=\frac{m^{\nu/2}}{4\pi^{\nu/2}}\Gamma\left[1-\frac{\nu}{2}\right]\left[1+\frac{m^2X^2}{2\nu}+O(X^4)\right]+\frac{m^{2-\nu}}{4\pi^{\nu/2}}\Gamma\left[\frac{\nu}{2}-1\right]\left[1+\frac{m^2X^2}{2(4-\nu)}+\frac{m^4X^4}{8(4-\nu)(6-\nu)}+O(X^6)\right]. \tag{59}$$

By computing

$$\partial_\mu\partial_\lambda G(X)=\frac{m^\nu}{(4\pi)^{\nu/2}}\Gamma\left[1-\frac{\nu}{2}\right]\frac{g_{\mu\lambda}}{\nu}+\frac{\Gamma\left[\frac{\nu}{2}-1\right]}{4\pi^{\nu/2}}(x^2)^{-\nu/2}$$
$$\times\left[(2-\nu)g_{\mu\lambda}-\nu(2-\nu)\frac{X_\mu X_\lambda}{X^2}+\frac{m^2}{2}[X^2g_{\mu\lambda}+(2-\nu)X_\mu X_\lambda]+\frac{m^4}{8}\left[\frac{X^4g_{\mu\lambda}}{(4-\nu)}+X^2X_\mu X_\lambda\right]\right],$$
$$\partial_\mu\partial^\mu G(X)=\frac{m^\nu}{(4\pi)^{\nu/2}}\Gamma\left[1-\frac{\nu}{2}\right]+\frac{\Gamma\left[\frac{\nu}{2}-1\right](X^2)^{1-\nu/2}}{4\pi^{\nu/2}}m^2\left[1+\frac{m^2X^2}{2(4-\nu)}+O(X^4)\right]$$

we obtain

$$\langle T_{\mu\lambda}(X)\rangle=\lim_{X\to 0}\left[\partial_\mu\partial_\lambda-\frac{g_{\mu\lambda}}{2}(\partial_\alpha\partial^\alpha-m^2)\right]G(X)=\langle T_{\mu\lambda}(X=0)\rangle+\langle\tilde{T}_{\mu\lambda}(X)\rangle\ . \tag{60}$$

Here

$$\langle T_{\mu\lambda}(X=0)\rangle=\frac{m^{\nu}}{(4\pi)^{\nu/2}}\Gamma\left[1-\frac{\nu}{2}\right]\left[\frac{1}{\nu}-1\right]g_{\mu\lambda}\,, \tag{61}$$

$$\langle \tilde{T}_{\mu\lambda}(X)\rangle=\frac{(X^2)^{1-\nu/2}}{4\pi^{\nu/2}}\Gamma\left[\frac{\nu}{2}-1\right]\left[\frac{2-\nu}{X^2}\left[g_{\mu\lambda}-\nu\frac{X_\mu X_\lambda}{X^2}\right]\right.$$
$$\left.-\frac{m^2}{2}\left[g_{\mu\lambda}-(2-\nu)\frac{X_\mu X_\lambda}{X^2}-\frac{m^2}{4}X_\mu X_\lambda+\frac{3}{4}\frac{m^2X^2}{(4-\nu)}g_{\mu\lambda}+\frac{m^2X^4}{8(4-\nu)(6-\nu)}g_{\mu\lambda}\right]\right]. \tag{62}$$

Now let $\nu\to d=2,4$. We have

$$\nu=d+(\nu-d),\quad \frac{X^{-\nu}}{4\pi^{\nu/2}}=\frac{\beta^{d-\nu}}{4\pi^{d/2}}\frac{1}{X^d}\left[1-(\nu-d)\ln\frac{\sqrt{\pi}X}{\beta}+O(\nu-d)^2\right],$$

$$\Gamma\left[1-\frac{\nu}{2}\right]=(-1)^{1-d/2}\left[\frac{2}{d-\nu}-c\right]+O(\nu-d),\quad \left[\frac{1}{\nu}-1\right]=-\frac{1}{d}\left[d-1+\frac{1}{d}(\nu-d)+O(\nu-d)^2\right].$$

We get

$$\langle T_{\mu\lambda}(X=0)\rangle_{\nu\to2}=\frac{\beta^{\nu-2}}{4\pi}\frac{m^2}{2}g_{\mu\lambda}\left[\frac{2}{\nu-2}+\ln\left[\frac{ma}{2\sqrt{\pi}\beta}\right]^2\right],\quad a=e^{(1/2)(1+c)}\,, \tag{63a}$$

$$\langle T_{\mu\lambda}(X=0)\rangle_{\nu\to4}=\frac{\beta^{\nu-4}}{(4\pi)^2}\frac{m^4}{2}g_{\mu\lambda}\left[-\frac{6}{\nu-4}-\ln\left[\frac{m\beta}{2\sqrt{\pi}b}\right]^6\right],\quad b=e^{-1/12} \tag{63b}$$

and

$$\langle \tilde{T}_{\mu\lambda}(X)\rangle_{\nu\to2}=\frac{\beta^{\nu-2}}{4\pi}\frac{m^2}{2}g_{\mu\lambda}\left[-\frac{2}{\nu-2}+\ln a\pi\left[\frac{x}{\beta}\right]^2\right]-\frac{\beta^{\nu-2}}{4\pi}\left[2\frac{g_{\mu\lambda}}{X^2}-4\frac{X_\mu X_\lambda}{X^4}+m^2\frac{X_\mu X_\lambda}{X^2}\right], \tag{64a}$$

$$\langle \tilde{T}_{\mu\lambda}(X)\rangle_{\nu\to4}=\frac{\beta^{\nu-4}}{(4\pi)^2}\frac{m^4}{4}g_{\mu\lambda}\left[\frac{6}{\nu-4}-\ln\left[\frac{\sqrt{\pi}X}{\beta}\right]^6\right]-\frac{\beta^{4-\nu}}{(4\pi)^2}\left[\left[\frac{8}{X^4}+\frac{2m^2}{X^2}\right]g_{\mu\lambda}\left[\frac{32}{X^b}+\frac{4m^2}{X^4}+\frac{m^4}{2X^2}\right]X_\mu X_\lambda\right]. \tag{64b}$$

Now, if ν is used as a regulator, we take ν negative and $X=0$. Thus, in dimensional regularization $\langle T_{\mu\lambda}\rangle$ is given by Eq. (61) and its expressions for $\nu\to2,4$ are Eqs. (63a) and (63b). If we use X as a regulator, then $\langle T_{\mu\lambda}(X)\rangle$ is given by Eq. (60) and its expressions for $\nu\to2,4$ are

$$\langle T_{\mu\lambda}(X)\rangle_{\nu\to2}=\frac{\beta^{2-\nu}}{4\pi}\left[\frac{m^2}{2}g_{\mu\lambda}\ln\left[\frac{mXa}{2}\right]^2-2\frac{g_{\mu\lambda}}{X^2}+4\frac{X_\mu X_\lambda}{X^4}-m^2\frac{X_\mu X_\lambda}{X^2}\right], \tag{65a}$$

$$\langle T_{\mu\lambda}(X)\rangle_{\nu\to4}=\frac{\beta^{4-\nu}}{(4\pi)^2}\left[-\frac{m^4}{4}g_{\mu\lambda}\ln\left[\frac{mX}{2b}\right]^6-8\frac{g_{\mu\lambda}}{X^4}+\frac{32X_\mu X_\lambda}{X^b}-\frac{2m^2g_{\mu\lambda}}{X^2}+4m^2\frac{X_\mu X_\lambda}{X^4}+\frac{m^4}{X^2}\frac{X_\mu X_\lambda}{X^2}\right]. \tag{65b}$$

The poles at $\nu=2,4$ appearing in the $X=0$ and $X\neq0$ terms (63) and (64) mutually cancel.

It can be noted that Eq. (61) is only valid for $m\neq0$. $\langle T_{\mu\lambda}\rangle$ as given by dimensional regularization does not include the $m=0$ limit. This is so because for $X=0$, the only term contributing to $G(X)$ comes from $I_{\nu/2-1}(X)$. The $m=0$ contribution is contained in the $I_{(\nu/2)-1}(X)$ term which vanishes at $X=0$. To include the $m=0$ term, $\langle T_{\mu\lambda}\rangle$ must be defined for $X\neq0$ and point splitting is required.

V. COVARIANT POINT-SPLITTING REGULARIZATION

There is a very natural form of the regularized operator which suggests itself here. In keeping with the efforts of other authors, it embodies numerous properties expected from the classical definition. It can be easily built up piece by piece to reflect the symmetry, conservation, and conformal tracelessness of the classical expression and by construction has the property of being parallelly transportable between the separated points which give rise to its definition. Somewhat surprisingly, in one way or another, it differs from the corresponding quantity used by other authors in the field, but only by terms which are odd in the separation, i.e., whose sign changes under parallel transport. Whatever the value of using it for a renormalization may be, it would seem worthwhile to record our regularized tensor here and to give some arguments for its adoption.

A. Constructing the tensor

From the point-splitting point of view, just as the Green's function is the fundamental object from which to construct quantities such as $\langle \psi^2 \rangle$, we seek some analogous object for $\langle T_{\mu\nu} \rangle$ which preserves the Lorentz covariance and, where applicable, the conformal invariance: also, we seek an object which is formally conserved in the coincidence limit, the final result automatically being conserved, since renormalization takes place in the Lagrangian. However, it is a little difficult to implement conservation for our regularized tensor unless the tensor itself is regular in the coincidence limit.

The classical expression for the stress-energy tensor of a scalar field has terms such as $a_{\mu\nu}\psi(X)\psi(X)$, $\psi_{;\mu}(X)\psi_{;\nu}(X)$, and $\psi_{;\mu\nu}(X)\psi(X)$ occurring in it, where the (regular) $a_{\mu\nu}$ may be $g_{\mu\nu}$, $R_{\mu\nu}$, $Rg_{\mu\nu}$ (contractions of these terms also appear). In the quantum theory, since these terms represent products of field operators (or their derivatives) at a point, they are divergent (ill defined). However, terms such as $\psi(X)\psi(X')$, $\psi_{;\mu}(X)\psi_{;\nu'}(X')$, $\psi_{;\mu'}(X')\psi(X)$, $\psi_{;\mu\nu}(X)\psi(X')$, $\psi_{;\mu'\nu'}(X')\psi(X)$ are clearly well defined and will be suitable for use in building up a regularized quantity. In addition, there is an (essentially) unique object, the geodetic bivector of parallel displacement, $g_{\mu\nu'}$, which allows the indices expressing dependence at one point to be translated to the other, so that a properly tensorial quantity can be defined. Whatever other properties the stress energy may have, since this parallel transport is needed for the definition of our regularized tensor, it would seem most useful to have a definition which is invariant under such transport between the separated points: i.e.,

$$\hat{\mathscr{T}}_{\mu\nu}(X,X')=g_\mu{}^{\mu'}g_\nu{}^{\nu'}\hat{\mathscr{T}}_{\mu'\nu'}(X,X') \,.$$

We can think of this as a "symmetry" under $X\leftrightarrow X'$; terms lacking this symmetry would vanish in the classical expression obtained from the coincidence limit. Thus

$$\psi_{;\mu}(X)\psi_{;\nu}(X)\rightarrow \tfrac{1}{2}[g_\mu{}^{\mu'}\psi_{;\mu'}(X')\psi_{;\nu}(X) + g_\nu{}^{\nu'}\psi_{;\mu}(X)\psi_{;\nu'}(X')] \,,$$

$$\psi_{;\mu\nu}(X)\psi(X)\rightarrow \tfrac{1}{2}[\psi_{;\mu\nu}(X)\psi(X') + g_\mu{}^{\mu'}g_\nu{}^{\nu'}\psi_{;\mu'\nu'}(X')\psi(X)] \,,$$

and

$$a_{\mu\nu}(X)\psi(X)\psi(X)\rightarrow \tfrac{1}{2}[a_{\mu\nu}(X)+g_\mu{}^{\mu'}g_\nu{}^{\nu'}a_{\mu'\nu'}(X')]\psi(X)\psi(X').$$

In the first two cases, the symmetry under $\mu\leftrightarrow\nu$ is preserved; in the third case no symmetry whatever is used in the construction. The contraction terms in the regularized tensor follow unambiguously from these definitions, which lead to an explicit preservation of the vanishing of the trace in the conformally invariant theory. Thus, in a curved space-time, with

$$\mathscr{G}^{\mu\nu}=R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R, \quad G=[\psi(X_1),\psi(X_2)]_+$$

and conformal coupling ξ, we will consider

$$\hat{\mathscr{T}}^{\mu\nu}=(\tfrac{1}{2}-\xi)(g^\nu{}_{\nu'}G^{;\mu\nu'}+g^\mu{}_{\mu'}G^{;\nu\mu'})+2(\xi-\tfrac{1}{4})g^{\mu\nu}g_{\rho\sigma'}G^{;\rho\sigma'}-\xi(G^{;\mu\nu}+g^\mu{}_{\mu'}g^\nu{}_{\nu'}G^{;\mu'\nu'})$$
$$+\xi g^{\mu\nu}(G_\rho{}^\rho+G_{\rho'}{}^{\rho'})+\tfrac{1}{2}\xi(\mathscr{G}^{\mu\nu}+g^\mu{}_{\mu'}g^\nu{}_{\nu'}\mathscr{G}^{\mu'\nu'})G-\tfrac{1}{2}m^2g^{\mu\nu}G \,. \qquad (66)$$

In the expression given by Brown and Ottewill,[9] the operators occurring in the first and third terms of Eq. (66) are not distinguished. In Ref. 10, Wald does not apply point splitting to the Einstein tensor. Dowker and Critchley[11] split the Ricci curvature and $\nabla^\mu\nabla^\nu$, but not the scalar curvature and $\nabla^\mu\nabla^{\nu'}$. Christensen[12] uses the conformally coupled field equations

$$\psi(X)_{;\mu}{}^{\mu}=[\xi R(X)+m^2]\psi(X),$$

$$\psi(X')_{;\mu'}{}^{\mu'}=[\xi R(X')+m^2]\psi(X) \,,$$

to partially remove these second-order operators (so that tracelessness in the conformally invariant case becomes independent of the field equations), but he makes no distinction between the scalar curvature at the two points. Candelas[13] fully removes the $\Box G$ (and $\Box' G$) terms. Davies and Fulling[7] in fact remark that some asymmetry between X and X' may be introduced into the regularized tensor by use of the field equations. In his examination of the scheme proposed by Adler, Lieberman, and Ng,[14] Wald shows that a certain regular, boundary-dependent Green's function used in their analysis, is not symmetric and does not satisfy the wave equation in both X and X'. Using his proposal for the regularized energy-momentum tensor, he finds that their renormalized tensor is not conserved and suggests a modification to correct this fault at the expense of introducing an anomalous trace. Although our proposal alters a number of the features used by Wald in his analysis, it will not spoil his results concerning conservation since he applies his regularized operator only to a nondivergent quantity, and it turns out that the difference we would introduce vanishes identically in the coincidence limit. However, our proposal introduces additional direction-dependent finite terms which would in general lead to additional $\Box R$ terms in the trace (even with respect to Wald's work, but not that of Christensen) and would spoil conservation if there is anyone who has genuinely established conservation for his renormalized result. Ultimately, the residual effect of our proposal partly depends on one's handling of direction dependent terms.

In flat space-time, with $\xi=0$, from Eq. (66) we have

$$\hat{\mathcal{T}}^{\mu\nu}=\tfrac{1}{2}(g^{\nu}{}_{\nu'}G^{;\mu\nu'}+g^{\mu}{}_{\mu'}G^{;\mu'\nu}-g^{\mu\nu}g_{\rho\sigma'}G^{;\rho\sigma'}-m^2g^{\mu\nu}G)\ .$$

In the coordinate system defined by Eq. (1), we have

$$g_{\mu\nu}=(-\Lambda,\Lambda,1,1)\ ,$$

$$g_{\mu\nu'}=\begin{bmatrix}-r & s & & \\ -s & r & & \\ & & 1 & \\ & & & 1\end{bmatrix}\ ,$$

$$g^{\mu}{}_{\nu'}=\frac{1}{\Lambda_1}\begin{bmatrix}r & -s & & \\ -s & r & & \\ & & \Lambda_1 & \\ & & & \Lambda_1\end{bmatrix}\ ,$$

$$g^{\mu\nu'}=\frac{1}{\Lambda_1\Lambda_2}\begin{bmatrix}-r & -s & & \\ s & r & & \\ & & \Lambda_1\Lambda_2 & \\ & & & \Lambda_1\Lambda_2\end{bmatrix}\ ,$$

$$g^{\nu'}{}_{\mu}=\frac{1}{\Lambda_2}\begin{bmatrix}r & s & & \\ s & r & & \\ & & \Lambda_2 & \\ & & & \Lambda_2\end{bmatrix}\ ,$$

with

$$r=\tfrac{1}{2}(f'_1g'_2+g'_1f'_2)\quad \Lambda_1 f'_1 g'_1,\quad f_i\equiv f(u'_i)\ ,$$
$$s=\tfrac{1}{2}(f'_1g'_2-g'_1f'_2),\quad \Lambda_2=f'_2g'_2,\quad g_i\equiv g(v'_i)\ ,$$
$$i=1,2\ .$$

Then

$$\hat{\mathcal{T}}_{00}=\frac{1}{2\Lambda_2}[r(G_{00'}+G_{11'})+s(G_{01'}+G_{10'})+\Lambda_1\Lambda_2(G_{22'}+G_{33'}+m^2G)]\ .$$

Of course, we could take the expectation of this with respect to any state. If we choose, say, the Minkowski vacuum $|0\rangle$, the operator G would become just the Minkowski Green's function. If we choose an accelerated state $|0'\rangle$, G would become the Green's function for that state.

As an example, we consider the two-dimensional massless case referred to earlier (as then, we will for convenience use the Feymann Green's function rather than the symmetric one). For the accelerated state we have

$$G'=-\frac{1}{4\pi}\ln(\Delta u'\Delta v')\quad (\Delta u'=u'_1-u'_2,\Delta v'=v'_1-v'_2)$$

and then

$$\langle 0'|\hat{\mathcal{T}}_{00}|0'\rangle=-\frac{1}{4\pi}\left[\frac{f'_1}{f'_2}\frac{1}{(\Delta u')^2}+\frac{g'_1}{g'_2}\frac{1}{(\Delta v')^2}\right]\ .$$

Using $\partial/\partial u=(1/f')(\partial/\partial u')$, we expand

$$\frac{f'_1}{f'_2}=f'_1\left\{\left[\frac{1}{f'}+\frac{1}{f'}\partial_{u'}\left[\frac{1}{f'}\right]\Delta u+\frac{1}{f'}\partial_{u'}\left[\frac{1}{f'}\partial u'\left[\frac{1}{f'}\right]\right]\frac{\Delta u^2}{2!}\right\}_{f'=f'_1}=1-\frac{f''_1}{f'^2_1}\Delta u+\left[-\frac{f'''_1}{f'^3_1}+3\frac{f''^2_1}{f'^4_1}\right]\frac{\Delta u^2}{2!}$$

and

$$\frac{1}{\Delta u'}=\frac{1}{\Delta u}\frac{(f_2-f_1)}{\Delta u'}=\frac{1}{\Delta u}\left[f'_1+f''_1\frac{\Delta u'}{2!}+f'''_1\frac{(\Delta u')^2}{3!}\right]=\frac{f'_1}{\Delta u}\left[1+\frac{f''_1}{2f'^2_1}\Delta u+\left[\frac{1}{6}\frac{f'''_1}{f'^3_1}-\frac{1}{4}\frac{f''^2_1}{f'^4_1}\right](\Delta u)^2\right]$$

to obtain

$$\langle 0'|\hat{\mathcal{T}}_{00}|0'\rangle=-\frac{1}{4\pi}\left\{\frac{f'^2_1}{\Delta u^2}+\frac{g'^2_1}{\Delta v^2}+\left[-\frac{1}{6}\frac{f'''_1}{f'_1}+\frac{1}{4}\left[\frac{f''_1}{f'_1}\right]^2\right]+\left[-\frac{1}{6}\frac{g'''_1}{g'_1}+\frac{1}{4}\left[\frac{g}{g'_1}\right]^2\right]\right\}\ .$$

From our previous expression

$$\sigma^2=(f_2-f_1)(g_2-g_1)\ ,$$
$$\partial_{u'_1}\sigma^2=-f'_1\Delta v,\quad \partial_{v'_1}\sigma^2=-g'_1\Delta u\ ,$$

the divergent part can be written as

$$-\frac{1}{4\pi}\left[\frac{(\partial_{u'_1}\sigma^2)^2+(\partial_{v'_1}\sigma^2)^2}{(\sigma^2)^2}\right]$$

which can be subtracted by a renormalization of the cosmological constant to give

$$\langle 0'|\hat{\mathcal{T}}_{00}|0'\rangle_{\text{ren}}=-\frac{1}{4\pi}\left\{\left[-\frac{1}{6}\frac{f'''_1}{f'_1}+\frac{1}{4}\left[\frac{f''_1}{f'_1}\right]^2\right]+\left[-\frac{1}{6}\frac{g'''_1}{g'_1}+\frac{1}{4}\left[\frac{g''_1}{g'_1}\right]^2\right]\right\}\ .$$

As mentioned earlier, this is the negative of the result of our naive subtraction scheme for the Minkowski vacuum.

Using this covariant method instead, for the Minkowski vacuum, and expressing the result with respect to accelerated coordinates we would have

$$\langle 0 | \hat{\mathscr{T}}_{00} | 0 \rangle = -\frac{1}{4\pi} \left[\frac{f_1'^2}{\Delta u^2} + \frac{g_1'^2}{\Delta v^2} \right] ,$$

and obtain

$$\langle 0 | \hat{\mathscr{T}}_{00} | 0 \rangle_{\text{ren}} = 0 .$$

VI. CONCLUSIONS

We have extended, from two to four dimensions, a previous description of accelerated frames using analytic mappings. For a treatment of the divergent operator products arising in quantum field theory, we show that in a normalization with respect to a natural ground state specified in terms of these accelerated frames, the coordinate mapping (and its inverse) relating the accelerated frames to global inertial coordinates, plays a distinguished role. As in two dimensions, the exponential mapping is uniquely singled out as giving rise to a situation indicating global thermal equilibrium, and asymptotic properties of mappings are directly related to thermal properties in the asymptotic regions. We do not attempt to settle the question whether normalization may be more appropriate in curved spaces, but as a unique way of indicating which state to use for a normalization has not yet been developed, it seems that to discuss the back-reaction problem a renormalization may be essential. In this context we propose a new regularized energy-momentum tensor using covariant point splitting, and discuss its properties and relation to previous proposals.

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