Test-particle motion in Einstein's unified field theory. III. Magnetic monopoles and charged particles

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(Received 2 December 1985)

In a previous paper (paper I), we developed a method for finding the exact equations of structure and motion of multipole test particles in Einstein's unified field theory—the theory of the nonsymmetric field. In that paper we also applied the method and found in Einstein's unified field theory the equations of structure and motion of neutral pole-dipole test particles possessing no electromagnetic multipole moments. In a second paper (paper II), we applied the method and found in Einstein's unified field theory the exact equations of structure and motion of charged test particles possessing no magnetic monopole moments. In the present paper (paper III), we apply the method and find in Einstein's unified field theory the exact equations of structure and motion of charged test particles possessing magnetic monopole moments. It follows from the form of these equations of structure and motion that in general in Einstein's unified field theory a test particle possessing a magnetic monopole moment in a background electromagnetic field must also possess spin.

I. INTRODUCTION

In a previous paper[1] (paper I), we developed a method for finding the exact equations of structure and motion of multipole test particles[2] in Einstein's unified field theory—the theory of the nonsymmetric field.[3] In that paper, we also applied the method and found in Einstein's unified field theory the exact equations of structure and motion of neutral pole-dipole test particles possessing no electromagnetic multiple moments. In a second paper[4] (paper II), we used the method to find in Einstein's unified field theory the exact equations of structure and motion of charged test particles possessing no magnetic monopole moments. In the present paper (paper III), we shall use the method to find in Einstein's unified field theory the exact equations of structure and motion of charged test particles possessing magnetic monopole moments.

A knowledge of the exact equations of structure and motion of test particles possessing magnetic monopole moments is of interest for the following reasons. First, under many circumstances a test particle can be considered a good model for a physical particle, and it is possible that particles which possess magnetic monopole moments may exist in nature. Until now, in Einstein's unified field theory only approximate equations of structure and motion have been found for particles possessing magnetic monopole moments and these approximate equations are only applicable to particles moving in weak background fields.[5] Thus, a knowledge of the exact equations of structure and motion of test particles possessing magnetic monopole moments will allow one for the first time to investigate in Einstein's theory the motion of particles possessing magnetic monopole moments in background fields which are not necessarily weak. In addition to this obvious reason for an interest in the exact equations of structure and motion of test particles possessing magnetic monopole moments, there is an additional reason for an interest in such equations which is particular to Einstein's theory.

In Einstein's unified field theory it has been shown that if each particle in a set of particles possesses a magnetic monopole moment proportional to its charge, and if the masses, charges, and magnetic monopole moments of the particles in the set are of the order of magnitude one expects to find associated with elementary particles, then in a weak-field approximation which under these conditions should be valid over macroscopic interaction distances (laboratory and astronomical distances) the interaction among these particles is found to be independent of their magnetic monopole moments.[5] In spite of the presence of magnetic monopole moments on the particles and as long as the field acting on each of the particles is not excessively strong, the particles interact over macroscopic distances as if they had no magnetic monopole moments. In fact, it has been shown that in Einstein's unified field theory they interact over laboratory and moderate astronomical distances through the conventional classical electromagnetic interaction as if they had no magnetic monopole moments.[5]

The above results have important physical consequences. They mean that if Einstein's unified field theory is correct, one cannot at the present time rule out the possibility that the charged particles which have so far been observed in nature (which interacting over macroscopic distances seem to have no magnetic monopole moments) actually possess magnetic monopole moments proportional to their charge. Only through the study in Einstein's theory of the interaction of such particles over microscopic distances (that is, atomic and molecular distances or smaller, interaction distances over which the weak-field equations of structure and motion cannot be considered reliable[5]) or through the study of the motion of such particles in strong macroscopic fields (again where the weak-field equations of structure and motion cannot be considered reliable) can one determine whether the existence of such particles allowed by Einstein's theory is in agreement or disagreement with experiment. Finding the exact equations of structure and motion of charged test particles possessing magnetic monopole moments is a first step in such an investigation.[1]

The equations of structure and motion of a test particle possessing a magnetic monopole moment in Einstein's theory will be found to have an interesting property. It will follow from the form of the equations of structure and motion that in general in Einstein's unified field theory a test particle possessing a magnetic monopole moment in a background electromagnetic field must also possess spin.

II. SIMPLE CHARGED TEST PARTICLES POSSESSING MAGNETIC MONOPOLE MOMENTS

As mentioned in the Introduction, in this paper we shall find in Einstein's unified field theory the exact equations of structure and motion of charged test particles possessing magnetic monopole moments. By the condition that a particle possess a magnetic monopole moment we mean that the electromagnetic moment e^M associated with the particle does not vanish.[6] In addition, we shall confine our investigation in this paper to particles which when isolated and possessing no spin can be represented through a time-independent spherically symmetric solution to Einstein's field equations in which the symmetric part of the fundamental field is flat at infinity.[7] By the above statement we do not mean that the particles cannot possess spin. What we mean is that we shall restrict our study to particles which when isolated possess only those multipole moments consistent with spherical symmetry with but one exception—and that exception is that the particles may possess spin. We shall also restrict our study in this paper to particles which under interaction develop the minimum number of higher multipole moments consistent with Einstein's field equations. We shall call the particles we shall be studying simple particles. Simple charged particles possessing negligible spins have been studied in considerable detail in two previous papers.[5,8]

From the investigations found in these two previous papers,[5,8] and from earlier work,[9] and using the definitions of $\gamma^*_{[\mu\nu]}$ and $\gamma_{(\mu\nu)}$ given in paper I, it follows that in a harmonic coordinate system and keeping only terms linear in the multipole moments which characterize a particle, one finds for the fields $\gamma^*_{[\mu\nu]}$ and $\gamma_{(\mu\nu)}$ associated with an isolated simple charged particle,[10]

$$\gamma^*_{[\mu\nu]}=\gamma^{*E}_{[\mu\nu]}+\gamma^{*M}_{[\mu\nu]}\ , \tag{2.1}$$

$$\gamma^{*E}_{[\mu\nu]}=\gamma^{E}_{\mu,\nu}-\gamma^{E}_{\nu,\mu},\quad \gamma^{*M}_{[\mu\nu]}=\epsilon_{\mu\nu\rho\sigma}\gamma^{M\sigma,\rho}\ , \tag{2.2}$$

where

$$\gamma^E_\mu=l[qu_\mu(r_\rho u^\rho)^{-1}+\tfrac{1}{2}(q/l^2)r_\mu]_A\ , \tag{2.3}$$

$$\gamma^M_\mu=l[q_M u_\mu(r_\rho u^\rho)^{-1}]_A\ , \tag{2.4}$$

and

$$\gamma_{(\mu\nu)}=4[(mu_\mu u_\nu+\tfrac{1}{2}\dot{s}_{\mu\rho}u^\rho u_\nu+\tfrac{1}{2}\dot{s}_{\nu\rho}u^\rho u_\mu)(r_\rho u^\rho)^{-1}]_A$$
$$+4[(\tfrac{1}{2}s_{\mu\rho}u_\nu+\tfrac{1}{2}s_{\nu\rho}u_\mu)(r_\rho u^\rho)^{-1}]_A{}^{,\rho}\ . \tag{2.5}$$

In (2.3)—(2.5), we are using the notation

$$r^\mu=x^\mu-\xi^\mu,\quad r_\mu=\eta_{\mu\rho}r^\rho,\quad u^\mu=\frac{d\xi^\mu}{d\tau}\ ,$$
$$u_\mu=\eta_{\mu\rho}u^\rho,\quad \dot{s}_{\mu\nu}=\frac{ds_{\mu\nu}}{d\tau}\ , \tag{2.6}$$

and

$$[f]_A=a_{\rm ret}[f]_{\rm ret}+a_{\rm adv}[f]_{\rm adv}\ ,$$
$$a_{\rm ret}+a_{\rm adv}=1\ . \tag{2.7}$$

The points ξ^μ form the world line of the particle and are parametrized by a quantity τ defined through the equation

$$d\tau^2=\eta_{\mu\nu}d\xi^\mu d\xi^\nu\ . \tag{2.8}$$

The quantities $a_{\rm ret}$ and $a_{\rm adv}$ in (2.7) are constants and can be regarded as characterizing the structure of the particle. The subscript ret indicates that in the expression in brackets those quantities which are associated with the particle are to be evaluated at the "retarded point,"

$$(r_\rho r^\rho)=0,\quad r^4>0\ ,$$

while the subscript adv indicates that the expression in brackets is to be multiplied by -1 and then in the expression in brackets those quantities associated with the particle are to be evaluated at the "advanced point,"

$$(r_\rho r^\rho)=0,\quad r^4<0\ .$$

The quantities l, q, q_M, and $s_{\mu\nu}$ in (2.3)—(2.5) characterize the particle. The quantity l is a constant and a universal length—the same for each particle.[11] The quantity q is a constant and represents the charge of the particle,[12] the quantity q_M is a constant and represents the magnetic monopole moment of the particle,[13] the quantity m represents the mass of the particle,[14] and the quantity $s_{\mu\nu}$ represents the spin of the particle.[9]

From their definitions given in Sec. V of paper I, we find in a harmonic coordinate system for the quantities $i^{\rm kin}_\mu$ and $s^{\rm kin}_\mu$ associated with an isolated particle in Einstein's theory[1]

$$i^{\rm kin}_\mu=\tfrac{1}{6}\eta_{\mu\lambda}\epsilon^{\rho\sigma\kappa\lambda}\gamma^{*L}_{[\rho\sigma,\kappa]}\ , \tag{2.9}$$

$$s^{\rm kin}_\mu=\Box^2\gamma^{*L}_{[\nu\mu]}{}^{,\nu}\ , \tag{2.10}$$

and for the quantity $t^{\rm kin}_{\mu\nu}$ associated with the particle[1]

$$t^{\rm kin}_{\mu\nu}=\Box^2\gamma^L_{(\mu\nu)}\ . \tag{2.11}$$

The fields $\gamma^{*L}_{[\mu\nu]}$ and $\gamma^L_{(\mu\nu)}$ appearing in (2.9)—(2.11) are those parts of $\gamma^*_{[\mu\nu]}$ and $\gamma_{(\mu\nu)}$, respectively, which are linear in the multipole moments characterizing the particle.

Making use of Eqs. (2.1)—(2.5) in Eqs. (2.9)—(2.11), we find associated with a simple charged particle[15]

$$i^{\rm kin}_\mu=4\pi l\int q_M u_\mu\delta(x-\xi)d\tau\ , \tag{2.12}$$

$$s_\mu^{\text{kin}}=(4\pi/l)\left[\int qu_\mu\delta(x-\xi)d\tau + \int ql^2(\eta_{\mu\lambda}u_\kappa-\eta_{\kappa\lambda}u_\mu) \times\delta^{,\kappa\lambda}(x-\xi)d\tau\right] , \quad (2.13)$$

$$t_{\mu\nu}^{\text{kin}}=16\pi\int[mu_\mu u_\nu+\tfrac{1}{2}\dot{s}_{\mu\rho}u^\rho u_\nu+\tfrac{1}{2}\dot{s}_{\nu\rho}u^\rho u_\mu]\delta(x-\xi)d\tau + 16\pi\int[\tfrac{1}{2}s_{\mu\rho}u_\nu+\tfrac{1}{2}s_{\nu\rho}u_\mu]\delta^{,\rho}(x-\xi)d\tau . \quad (2.14)$$

Comparing Eqs. (2.12)–(2.14) with Eqs. (5.12)–(5.14) of paper I, we see that a simple charged particle is characterized by the electromagnetic monopole moments e^M and e^E, an electromagnetic quadrupole moment $e^E_{[\mu\nu],\lambda}$, a mass monopole moment m^G, and a spin $S^G_{\mu\nu}$, where[16]

$$e^M=c^2lq_M , \quad (2.15)$$

$$e^E=(c^2/l)q , \quad (2.16)$$

$$e^E_{[\mu\nu]\lambda}=(c^2/l)ql^2(\eta_{\mu\lambda}u_\kappa-\eta_{\kappa\lambda}u_\mu) , \quad (2.17)$$

$$m^G=4mc^2 , \quad (2.18)$$

$$S^G_{\mu\nu}=4s_{\mu\nu}c^2 . \quad (2.19)$$

Making use of the relationship of $\mathbf{i}^\mu$ to i_μ^{kin}, of $\mathbf{s}^\mu$ to s_μ^{kin}, and of $\mathbf{T}^{\mu\nu}$ to $t_{\mu\nu}^{\text{kin}}$ (these relationships are discussed in Sec. V of paper I), we see that in the test-particle limit[2] a simple charged particle will be associated with an electromagnetic current density $\mathbf{i}^\mu$ of the form

$$\mathbf{i}^\mu=\int\tilde{i}^\mu(x)\delta(x-\xi)ds , \quad (2.20)$$

an electromagnetic current density $\mathbf{s}^\mu$ of the form

$$\mathbf{s}^\mu=\int\tilde{s}^\mu(x)\delta(x-\xi)ds+\int[\tilde{s}^{\mu\kappa}(x)\delta(x-\xi)]_{;\kappa}ds + \int[\tilde{s}^{\mu\kappa\lambda}(x)\delta(x-\xi)]_{;\kappa\lambda}ds , \quad (2.21)$$

and an energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ of the form

$$\mathbf{T}^{\mu\nu}=\int\tilde{T}^{(\mu\nu)}(x)\delta(x-\xi)ds + \int[\tilde{T}^{(\mu\nu)\kappa}(x)\delta(x-\xi)]_{;\kappa}ds . \quad (2.22)$$

The notation used in (2.20)–(2.22) is the same as that used in paper I. In particular, the world line of the particle has been parametrized by a quantity s defined through the equation

$$ds^2=a_{\mu\nu}d\xi^\mu d\xi^\nu , \quad (2.23)$$

and the displacement field in (2.20)–(2.22) is given through the Christoffel symbol $\{{}^{\rho}_{\mu\nu}\}$, where

$$\left\{\begin{matrix}\rho\\ \mu\nu\end{matrix}\right\}=\tfrac{1}{2}a^{\rho\sigma}(a_{\sigma\mu,\nu}+a_{\sigma\nu,\mu}-a_{\mu\nu,\sigma}) , \quad (2.24)$$

$$a_{\mu\rho}a^{\nu\rho}=\delta^\nu_\mu,\ \ a_{\mu\nu}=a_{\nu\mu} . \quad (2.25)$$

The background metric tensor fields $a_{\mu\nu}$ and $a^{\mu\nu}$ are defined in paper I.

We have retained dipole terms in (2.2) although such terms are not present in (2.13). The reason we have retained such terms is that we wish at this stage of our analysis to leave open the possibility that through interaction with the background field the test particle might develop such terms. We do know, however, that no multipole terms higher than those present in (2.20)–(2.22) will be generated through interaction with the background field. This follows from the general form of Eqs. (5.29)–(5.31) in paper I.

In summary, in the test-particle limit a simple charged particle will be associated with an electromagnetic current density $\mathbf{i}^\mu$ of the form (2.20), an electromagnetic current density $\mathbf{s}^\mu$ of the form (2.21), and an energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ of the form (2.22). From the analysis of Einstein's field equations contained in paper I, we also know that the current densities $\mathbf{i}^\mu$ and $\mathbf{s}^\mu$ and the energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ are subject to the equations[17]

$$\mathbf{i}^\mu{}_{;\mu}=0 , \quad (2.26)$$

$$\mathbf{s}^\mu{}_{;\mu}=0 , \quad (2.27)$$

$$\mathbf{T}^{\mu\nu}{}_{;\nu}=\mathbf{f}^\mu , \quad (2.28)$$

where

$$\mathbf{f}^\mu=a^{\mu\rho}\gamma^*_{[\nu\rho]}\mathbf{s}^\nu+(-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa})\mathbf{i}_\nu + (-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]}{}_{;\lambda}+\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^\sigma_{[\lambda\tau]})\mathbf{i}_{\nu;\kappa}-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]}\mathbf{i}_{\nu;\kappa\lambda} , \quad (2.29)$$

and

$$\mathbf{i}_\nu=a_{\nu\rho}\mathbf{i}^\rho . \quad (2.30)$$

The fields $R^\kappa{}_{\mu\nu\lambda}$, $R_{\mu\nu}$, $\Gamma^\sigma_{[\lambda\tau]}$, $\gamma^*_{[\mu\nu]}$, $\gamma^{[\mu\nu]}$, $a_{\mu\nu}$, and $a^{\mu\nu}$ appearing in (2.29) and (2.30) are background fields in the vicinity of the test particle. The fields $R^\kappa{}_{\mu\nu\lambda}$, $R_{\mu\nu}$, and $\Gamma^\sigma_{[\lambda\tau]}$ are, respectively, the curvature tensor, contracted curvature tensor, and the torsion tensor of spacetime in Einstein's unified field theory and are defined in paper I. The fields $\gamma^*_{[\mu\nu]}$ and $\gamma^{[\mu\nu]}$, which are also defined in paper I, are an oriented tensor and a tensor, respectively, and are related through the equations

$$\gamma^*_{[\mu\nu]}=\tfrac{1}{2}(-a)^{1/2}\epsilon_{\mu\nu\rho\sigma}\gamma^{[\sigma\rho]} , \quad (2.31)$$

$$\gamma^{[\mu\nu]}=\tfrac{1}{2}(-a)^{-1/2}\epsilon^{\mu\nu\rho\sigma}\gamma^*_{[\sigma\rho]} , \quad (2.32)$$

where a is the determinant of $a_{\mu\nu}$. Where covariant differentiation is indicated in (2.26)–(2.29), the displacement field is given through the Christoffel symbol $\{{}^{\rho}_{\mu\nu}\}$ defined in (2.24).[18] We shall use Eqs. (2.26)–(2.30), along with Eqs. (2.20)–(2.22) and Eqs. (2.12)–(2.14), to find the equations of structure and motion of simple charged test particles in Einstein's unified field theory.

III. EQUATIONS OF MOTION

A. Electromagnetic current densities

We first investigate the constraints that Eqs. (2.26) and (2.27) place on a test particle characterized by an electromagnetic current density $\mathbf{i}^{\mu}$ of the form (2.20) and an electromagnetic current density $\mathbf{s}^{\mu}$ of the form (2.21). In doing this, we obtain the quantities which characterize the electromagnetic structure of the test particle, and we also obtain the equations of structure satisfied by these quantities.

Making use of the same method of analysis described in detail in Sec. III A of paper II, one finds from (2.20) and (2.26) that the electromagnetic current density $\mathbf{i}^{\mu}$ can always be written in the form

$$\mathbf{i}^{\mu}=\int e_M U^{\mu}\delta(x-\xi)ds \ , \tag{3.1}$$

where

$$\frac{de_M}{ds}=0 \ , \tag{3.2}$$

and one finds from (2.21) and (2.27) that the electromagnetic current density $\mathbf{s}^{\mu}$ can always be written in the form

$$\begin{aligned}\mathbf{s}^{\mu}=&\int eU^{\mu}\delta(x-\xi)ds\\ &+\int\left[\tilde{s}^{[\mu\kappa]}+\tilde{s}^{\mu\rho\sigma}\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}-\tilde{s}^{\kappa\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\kappa}(x-\xi)ds\\ &+\int\tilde{s}^{\mu\kappa\lambda}\delta_{,\kappa\lambda}(x-\xi)ds \ ,\end{aligned} \tag{3.3}$$

where

$$\frac{de}{ds}=0 \ , \tag{3.4}$$

$$\tilde{s}^{(\mu\kappa)\lambda}+\tilde{s}^{(\kappa\lambda)\mu}+\tilde{s}^{(\lambda\mu)\kappa}=0 \ . \tag{3.5}$$

The quantities e_M, e, $\tilde{s}^{[\mu\kappa]}$, and $\tilde{s}^{\mu\kappa\lambda}$ which appear in (3.1) and (3.3) characterize the electromagnetic structure of the test particle.[19] The quantity e_M is a scalar, e is an oriented scalar, and $\tilde{s}^{[\mu\kappa]}$ and $\tilde{s}^{\mu\kappa\lambda}$ are oriented tensors. Both e_M and e are constants, but $\tilde{s}^{[\mu\kappa]}$ and $\tilde{s}^{\mu\kappa\lambda}$ may be functions of s. The notation

$$U^{\mu}=\frac{d\xi^{\mu}}{ds} \ , \tag{3.6}$$

is being used in (3.1) and (3.3).

B. Energy-momentum tensor density

We shall next investigate the constraints that Eqs. (2.28)–(2.30) place on a test particle associated with an electromagnetic current density $\mathbf{i}^{\mu}$ of the form (3.1) where (3.2) is valid, an electromagnetic current density $\mathbf{s}^{\mu}$ of the form (3.3)–(3.5), and an energy-momentum tensor density of the form (2.22). In the process of doing this we shall find the quantities which characterize the gravitational structure of the test particle, the equations of structure satisfied by these quantities, and the equations of motion satisfied by the particle.

If we make use of the definition of covariant differentiation found in Sec. IV of paper I (and discussed in Sec. II of the present paper) and also make use of the identities (6.8) and (6.9) of paper I, Eqs. (2.22) can be put into the form

$$\begin{aligned}\mathbf{T}^{\mu\nu}=&\int[\tilde{T}^{(\mu\nu)\kappa}]\delta_{,\kappa}(x-\xi)ds\\ &+\int\left[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\\ &\times\delta(x-\xi)ds \ ,\end{aligned} \tag{3.7}$$

where the quantities in the square brackets in (3.7) are evaluated along the world line ξ^{μ} of the test particle and are functions of s. Making use of the definition of $\mathbf{T}^{\mu\nu}{}_{;\nu}$ given in paper I, one finds from (3.7) that

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}=&\int[\tilde{T}^{(\mu\nu)\kappa}]\delta_{,\kappa\nu}(x-\xi)ds+\int\left[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\rho\sigma)\nu}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\nu}(x-\xi)ds\\ &+\int\left[\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}+2\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}\right]\delta(x-\xi)ds \ ,\end{aligned} \tag{3.8}$$

where the quantities in square brackets in (3.8) are functions of s.

From (3.3)–(3.5), again making use of the identities (6.8) and (6.9) of paper I, and also making use of the identity (3.1) of paper II, one finds

$$\begin{aligned}\gamma^{*[\mu}{}_{\nu]}\mathbf{s}^{\nu}=&\int[\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\nu\kappa}]\delta_{,\kappa\nu}(x-\xi)ds+\int\Bigg[\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{[\rho\nu]}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{s}^{\rho\sigma\nu}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{s}^{\rho\nu\sigma}\\ &\qquad+\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\sigma\kappa}\begin{Bmatrix}\nu\\ \sigma\kappa\end{Bmatrix}-\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\nu\sigma\kappa}\begin{Bmatrix}\rho\\ \sigma\kappa\end{Bmatrix}\Bigg]\delta_{,\nu}(x-\xi)ds\\ &+\int\left[e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{s}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu],\kappa\lambda}\tilde{s}^{\nu\kappa\lambda}+\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{s}^{\kappa\sigma\lambda}\begin{Bmatrix}\nu\\ \sigma\lambda\end{Bmatrix}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{s}^{\nu\sigma\lambda}\begin{Bmatrix}\kappa\\ \sigma\lambda\end{Bmatrix}\right]\delta(x-\xi)ds \ .\end{aligned} \tag{3.9}$$

We are using the definition

$$\gamma^{*[\mu}{}_{\nu]}=a^{\mu\rho}\gamma^{*}_{[\rho\nu]}\ . \tag{3.10}$$

From (3.1) and (3.2), identities (6.8) and (6.9) of paper I, identity (3.1) of paper II, and the definition of covariant derivative found in Sec. IV of paper I, one also finds

$$\begin{aligned}
&(-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa})\mathbf{i}_{\nu}\\
&\quad+(-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]}{}_{;\lambda}+\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]})\mathbf{i}_{\nu;\kappa}-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\nu]}\mathbf{i}_{\nu;\kappa\lambda}\\
&\quad=\int e_M[\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}U_{\nu}]\delta_{,\kappa\lambda}(x-\xi)ds\\
&\quad+\int e_M\left[-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}+\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\rho]}U_{\kappa}\begin{Bmatrix}\nu\\ \lambda\rho\end{Bmatrix}+\tfrac{1}{2}a^{\nu\lambda}\gamma^{[\kappa\rho]}U_{\kappa}\begin{Bmatrix}\mu\\ \lambda\rho\end{Bmatrix}+\tfrac{1}{2}a^{\lambda\rho}\gamma^{[\kappa\nu]}U_{\kappa}\begin{Bmatrix}\mu\\ \lambda\rho\end{Bmatrix}\right]\delta_{,\nu}(x-\xi)ds\\
&\quad+\int e_M\Bigg[\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}+\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_{\nu}-\tfrac{1}{2}a_{\rho\sigma}a^{\alpha\lambda}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}\begin{Bmatrix}\mu\\ \alpha\kappa\end{Bmatrix}\\
&\qquad-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_{\nu}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_{\nu}\\
&\qquad-\tfrac{1}{2}a^{\rho\kappa}\gamma^{[\nu\lambda]}U_{\nu}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}_{,\lambda}+a^{\rho\kappa}\gamma^{[\nu\lambda]}U_{\nu}\begin{Bmatrix}\sigma\\ \rho\lambda\end{Bmatrix}\begin{Bmatrix}\mu\\ \kappa\sigma\end{Bmatrix}\Bigg]\delta(x-\xi)ds\ ,
\end{aligned} \tag{3.11}$$

where

$$U_{\mu}=a_{\mu\nu}U^{\nu}\ , \tag{3.12}$$

so that making use of (3.8), (2.29), and (3.11) one has

$$\begin{aligned}
\mathbf{T}^{\mu\nu}{}_{;\nu}-\mathbf{f}^{\mu}=&\int[\tilde{T}^{(\mu\nu)\kappa}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\nu\kappa}-\tfrac{1}{2}e_M a^{\mu\nu}\gamma^{[\lambda\kappa]}U_{\lambda}]\delta_{,\kappa\nu}(x-\xi)ds\\
&+\int\Bigg[\tilde{T}^{(\mu\nu)}+\tilde{T}^{(\mu\rho)\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\nu\rho)\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\rho\sigma)\nu}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\\
&\qquad+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\mu}{}_{\rho],\sigma}\tilde{S}^{\rho\nu\sigma}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\sigma\kappa}\begin{Bmatrix}\nu\\ \sigma\kappa\end{Bmatrix}-\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\rho\\ \sigma\kappa\end{Bmatrix}\\
&\qquad+e_M\Bigg[\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}-\tfrac{1}{2}a^{\mu\lambda}\gamma^{[\kappa\rho]}U_{\kappa}\begin{Bmatrix}\nu\\ \rho\lambda\end{Bmatrix}-\tfrac{1}{2}a^{\nu\lambda}\gamma^{[\kappa\rho]}U_{\kappa}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}\\
&\qquad-\tfrac{1}{2}a^{\lambda\rho}\gamma^{[\kappa\nu]}U_{\kappa}\begin{Bmatrix}\mu\\ \rho\lambda\end{Bmatrix}\Bigg]\Bigg]\delta_{,\nu}(x-\xi)ds\\
&+\int\Bigg[\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}+2\tilde{T}^{(\rho\sigma)\kappa}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}\begin{Bmatrix}\mu\\ \sigma\lambda\end{Bmatrix}+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{[\nu\kappa]}+\gamma^{*[\mu}{}_{\nu],\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}\\
&\qquad+\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{\kappa\sigma\lambda}\begin{Bmatrix}\nu\\ \sigma\lambda\end{Bmatrix}-\gamma^{*[\mu}{}_{\nu],\kappa}\tilde{S}^{\nu\sigma\lambda}\begin{Bmatrix}\kappa\\ \sigma\lambda\end{Bmatrix}\\
&\qquad+e_M\Bigg[\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}+\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_{\nu}+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}\\
&\qquad+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_{\nu}-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}\\
&\qquad-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_{\nu}+\tfrac{1}{2}a_{\rho\sigma}a^{\alpha\lambda}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}\begin{Bmatrix}\mu\\ \alpha\kappa\end{Bmatrix}\\
&\qquad+\tfrac{1}{2}a^{\rho\kappa}\gamma^{[\nu\lambda]}U_{\nu}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}_{,\lambda}-a^{\rho\kappa}\gamma^{[\nu\lambda]}U_{\nu}\begin{Bmatrix}\sigma\\ \rho\lambda\end{Bmatrix}\begin{Bmatrix}\mu\\ \kappa\sigma\end{Bmatrix}\Bigg]\Bigg]\delta(x-\xi)ds\ .
\end{aligned} \tag{3.13}$$

Using the fact that (2.28)–(2.30) must be satisfied, one can show from (3.13) that there is no loss in generality in choosing $\widetilde{T}^{(\mu\nu)\kappa}$ to be of the form[20]

$$\widetilde{T}^{(\mu\nu)\kappa}=\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{\rho\nu\kappa}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\widetilde{S}^{\rho\mu\kappa}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{\rho\kappa\nu}$$
$$-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\widetilde{S}^{\rho\kappa\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\widetilde{S}^{\rho\nu\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\widetilde{S}^{\rho\mu\nu}-\tfrac{1}{2}e_M a^{\mu\nu}\gamma^{[\kappa\lambda]}U_{\lambda}\ , \tag{3.14}$$

where $S^{\mu\nu}$ is an antisymmetric second-rank tensor characterizing the test particle. Placing (3.14) in (3.13) and making use of the identity

$$\int [S^{\mu\kappa}U^{\nu}]\delta_{,\kappa\nu}(x-\xi)ds=\int \left[\frac{dS^{\mu\nu}}{ds}\right]\delta_{,\nu}(x-\xi)ds\ , \tag{3.15}$$

one finds

$$\mathbf{T}^{\mu\nu}{}_{;\nu}-\mathbf{f}^{\mu}=\int\Bigg[\widetilde{T}^{(\mu\nu)}+\tfrac{1}{2}\frac{DS^{\mu\nu}}{Ds}+\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{[\rho\nu]}-\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\nu}-\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\nu\sigma}$$
$$+\tfrac{1}{2}e_M a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}\Bigg]\delta_{,\nu}(x-\xi)ds$$
$$+\int\Bigg[\widetilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\gamma^{*[\rho}{}_{\nu]}\widetilde{S}^{[\nu\kappa]}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\widetilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\widetilde{S}^{\nu\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}$$
$$+\tfrac{1}{2}e_M a_{\rho\sigma}a^{\alpha\lambda}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}\begin{Bmatrix}\mu\\ \alpha\kappa\end{Bmatrix}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu];\kappa}\widetilde{S}^{[\nu\kappa]}$$
$$+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\widetilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\widetilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\widetilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}$$
$$+e_M\big(\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}+\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_{\nu}+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}$$
$$+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_{\nu}-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_{\nu}\big)\Bigg]\delta(x-\xi)ds\ . \tag{3.16}$$

The definition

$$\frac{DS^{\mu\nu}}{Ds}=\frac{dS^{\mu\nu}}{ds}+S^{\mu\rho}U^{\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}+S^{\rho\nu}U^{\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix} \tag{3.17a}$$

is being used in (3.16). The absolute derivative $DS^{\mu\nu}/Ds$ is a second-rank tensor. We are also using the notation

$$R^{*\mu}{}_{\rho\sigma\kappa}=\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}_{,\kappa}-\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}_{,\sigma}-\begin{Bmatrix}\mu\\ \lambda\sigma\end{Bmatrix}\begin{Bmatrix}\lambda\\ \rho\kappa\end{Bmatrix}+\begin{Bmatrix}\mu\\ \lambda\kappa\end{Bmatrix}\begin{Bmatrix}\lambda\\ \rho\sigma\end{Bmatrix}. \tag{3.17b}$$

The first integral on the right-hand side of (3.16) can be written in the form

$$\int [X^{(\mu\nu)}+Y^{[\mu\nu]}]\delta_{,\nu}(x-\xi)ds\ , \tag{3.18}$$

where

$$X^{(\mu\nu)}=\widetilde{T}^{(\mu\nu)}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{[\rho\nu]}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\widetilde{S}^{[\rho\mu]}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\mu}$$
$$-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\nu\sigma}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\mu\sigma}+\tfrac{1}{4}e_M a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}$$
$$+\tfrac{1}{4}e_M a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}\ , \tag{3.19}$$

$$Y^{[\mu\nu]}=\tfrac{1}{2}\frac{DS^{\mu\nu}}{Ds}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\widetilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\widetilde{S}^{[\rho\mu]}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\sigma\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\widetilde{S}^{\rho\nu\sigma}$$
$$+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\widetilde{S}^{\rho\mu\sigma}+\tfrac{1}{4}e_M a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}-\tfrac{1}{4}e_M a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}\ . \tag{3.20}$$

It can be shown that with no loss in generality one can always write[21]

$$X^{(\mu\nu)}={}^*X^{(\mu\nu)}+X^\mu U^\nu+X^\nu U^\mu+MU^\mu U^\nu\ , \tag{3.21}$$

$$Y^{[\mu\nu]}={}^*Y^{[\mu\nu]}+Y^\mu U^\nu-Y^\nu U^\mu\ , \tag{3.22}$$

where

$$ {}^*X^{(\mu\nu)}U_\nu=0,\quad X^\mu U_\mu=0\ , \tag{3.23}$$

$$ {}^*Y^{[\mu\nu]}U_\nu=0,\quad Y^\mu U_\mu=0\ . \tag{3.24}$$

This means that the first integral on the right-hand side of (3.16) can always be written in the form

$$\int[{}^*X^{(\mu\nu)}+{}^*Y^{[\mu\nu]}+X^\nu U^\mu-Y^\nu U^\mu]\delta_{,\nu}(x-\xi)ds+\int\left[\frac{d}{ds}(MU^\mu+X^\mu+Y^\mu)\right]\delta(x-\xi)ds\ , \tag{3.25}$$

so that if we make use of (3.25) in (3.16) we find

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}-\mathbf{f}^\mu=&\int[{}^*X^{(\mu\nu)}+{}^*Y^{[\mu\nu]}+X^\nu U^\mu-Y^\nu U^\mu]\delta_{,\nu}(x-\xi)ds\\
&+\int\Bigg[\frac{d}{ds}(MU^\mu+X^\mu+Y^\mu)+\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}S^{\rho\sigma}U^\kappa R^{*\mu}{}_{\kappa\rho\sigma}+e\gamma^{*[\mu}{}_{\nu]}U^\nu-\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}\\
&+\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{[\nu\kappa]}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}\\
&+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\\
&+e_M\Bigg[\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_\nu+\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_\nu+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_\nu\\
&+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_\nu-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_\nu-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_\nu\\
&+\tfrac{1}{2}a_{\rho\sigma}a^{\alpha\lambda}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\nu\begin{Bmatrix}\mu\\ \sigma\kappa\end{Bmatrix}\Bigg]\Bigg]\delta(x-\xi)ds\ .\end{aligned} \tag{3.26}$$

Since (2.28) must be satisfied, we see from (3.26) that one must have[22]

$$ {}^*X^{(\mu\nu)}+{}^*Y^{[\mu\nu]}+X^\nu U^\mu-Y^\nu U^\nu=0\ , \tag{3.27}$$

which is equivalent to the requirement

$$ {}^*X^{(\mu\nu)}=0,\quad {}^*Y^{[\mu\nu]}=0,\quad Y^\mu=X^\mu\ . \tag{3.28}$$

Making use of (3.28) in (3.26), we find

$$\begin{aligned}\mathbf{T}^{\mu\nu}{}_{;\nu}-\mathbf{f}^\mu=\int\Bigg[&\frac{D}{Ds}(MU^\mu+2X^\mu)-\tfrac{1}{2}S^{\rho\sigma}U^\kappa R^{*\mu}{}_{\kappa\rho\sigma}+e\gamma^{*[\mu}{}_{\nu]}U^\nu-\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}+\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{[\nu\kappa]}\begin{Bmatrix}\mu\\ \rho\kappa\end{Bmatrix}\\
&+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\sigma\kappa}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\\
&-\gamma^{*[\rho}{}_{\nu];\kappa}\tilde{S}^{\nu\kappa\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}+\tilde{T}^{(\rho\sigma)}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-(MU^\rho+2X^\rho)U^\sigma\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\\
&+e_M\Bigg[\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_\nu+\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_\nu+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_\nu+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_\nu\\
&-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_\nu-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_\nu\\
&+\tfrac{1}{2}a_{\rho\sigma}a^{\alpha\lambda}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\nu\begin{Bmatrix}\mu\\ \sigma\kappa\end{Bmatrix}\Bigg]\Bigg]\delta(x-\xi)ds\ ,\end{aligned} \tag{3.29}$$

where in (3.29) we have used the definition

$$\frac{DC^{\mu}}{Ds}=\frac{dC^{\mu}}{ds}+C^{\rho}U^{\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix} \tag{3.30}$$

of the absolute derivative DC^{μ}/Ds of a vector C^{μ} defined along the world line ξ^{μ}.

Making use of (3.21), (3.22), and (3.28) in (3.19) and (3.20), we find

$$\begin{aligned}\tilde{T}^{(\mu\nu)}=&MU^{\mu}U^{\nu}+X^{\mu}U^{\nu}+X^{\nu}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}\\&+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}+e_M(-\tfrac{1}{4}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}-\tfrac{1}{4}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa})\,,\end{aligned} \tag{3.31}$$

$$\begin{aligned}\frac{DS^{\mu\nu}}{Ds}=&2X^{\mu}U^{\nu}-2X^{\nu}U^{\mu}-\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}+\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}\\&+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}+e_M(-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}+\tfrac{1}{2}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa})\,,\end{aligned} \tag{3.32}$$

and from (3.32) and (3.23) one has

$$\begin{aligned}X^{\mu}=&\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}U_{\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}U_{\nu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}U_{\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}U_{\nu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}U_{\nu}\\&+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}U_{\nu}+e_M(-\tfrac{1}{4}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\nu})\,.\end{aligned} \tag{3.33}$$

Making use of (3.33) in (3.31), we find

$$\begin{aligned}\tilde{T}^{(\mu\nu)}=&MU^{\mu}U^{\nu}+\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{1}{2}\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\nu}\\&+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\mu}-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_{\rho}U^{\nu}-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\nu]}U_{\rho}U^{\mu}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}\\&+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\nu}\\&-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\mu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_{\rho}U^{\nu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\nu\kappa}U_{\rho}U^{\mu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_{\rho}U^{\nu}\\&+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\nu}U_{\rho}U^{\mu}+e_M(-\tfrac{1}{4}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}-\tfrac{1}{4}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}\\&-\tfrac{1}{4}a^{\alpha\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\alpha}U^{\nu}-\tfrac{1}{4}a^{\alpha\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\alpha}U^{\mu})\,,\end{aligned} \tag{3.34}$$

while making use of (3.33) in (3.32) gives as the equations of structure satisfied by $S^{\mu\nu}$,

$$\frac{DS^{\mu\nu}}{Ds}-\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}=N^{\mu\nu}\,, \tag{3.35}$$

where

$$\begin{aligned}N^{\mu\nu}=&-\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}+\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\nu}-\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\mu}-\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_{\rho}U^{\nu}\\&+\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\nu]}U_{\rho}U^{\mu}+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}+\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}-\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\nu}\\&+\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\mu}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\nu}+\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\mu}\\&+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_{\rho}U^{\nu}-\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\nu\kappa}U_{\rho}U^{\mu}+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_{\rho}U^{\nu}-\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\nu}U_{\rho}U^{\mu}\\&+e_M(-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}+\tfrac{1}{2}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}-\tfrac{1}{2}a^{\alpha\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\alpha}U^{\nu}\\&+\tfrac{1}{2}a^{\alpha\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\alpha}U^{\mu})\,.\end{aligned} \tag{3.36}$$

Making use of (3.31) and (3.33) in (3.29) one finds

$$\mathbf{T}^{\mu\nu}{}_{;\nu}-\mathbf{f}^{\mu}=\int\left[\frac{D}{Ds}\left\{MU^{\mu}+\frac{DS^{\mu\rho}}{Ds}U_{\rho}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}-\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_{\rho}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}\right.\right.$$
$$+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_{\rho}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_{\rho}$$
$$\left.-\tfrac{1}{2}e_{M}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\nu}\right\}-\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\rho\sigma}+e\gamma^{*[\mu}{}_{\nu]}U^{\nu}-\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}$$
$$+\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}-\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}+\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}$$
$$+e_{M}(\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}+\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_{\nu}+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}$$
$$\left.+\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_{\nu}-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_{\nu})\right]\delta(x-\xi)ds\ . \qquad (3.37)$$

Since (2.28) must be satisfied, we see from (3.37) that the test particle will obey the equations of structure and motion

$$\frac{DP^{\mu}}{Ds}+\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\sigma\rho}=F^{\mu}\ , \qquad (3.38)$$

where

$$P^{\mu}=MU^{\mu}+\frac{DS^{\mu\rho}}{Ds}U_{\rho}+\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}-\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_{\rho}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}$$
$$+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_{\rho}-\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}+\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_{\rho}-\tfrac{1}{2}e_{M}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\nu}\ , \qquad (3.39)$$

$$F^{\mu}=-e\gamma^{*[\mu}{}_{\nu]}U^{\nu}+\gamma^{*[\mu}{}_{\nu];\kappa}\tilde{S}^{[\nu\kappa]}-\gamma^{*[\mu}{}_{\nu];\kappa\lambda}\tilde{S}^{\nu\kappa\lambda}+\gamma^{*[\rho}{}_{\nu]}\tilde{S}^{\nu\kappa\lambda}R^{*\mu}{}_{\lambda\kappa\rho}-\tfrac{2}{3}\gamma^{*[\mu}{}_{\nu]}\tilde{S}^{\rho\kappa\lambda}R^{*\nu}{}_{(\rho\kappa)\lambda}$$
$$+e_{M}(-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}-\tfrac{1}{2}a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_{\nu}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}-\tfrac{1}{2}a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_{\nu}$$
$$+\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}+\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_{\nu})\ . \qquad (3.40)$$

If we place (3.14) and (3.34) in (3.7), we find that the energy-momentum tensor density $\mathbf{T}^{\mu\nu}$ associated with the test particle is given by

$$\mathbf{T}^{\mu\nu}=\int[\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\nu\kappa}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{\rho\mu\kappa}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{\rho\kappa\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{\rho\kappa\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{S}^{\rho\kappa\mu}$$
$$+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{S}^{\rho\mu\nu}-\tfrac{1}{2}e_{M}a^{\mu\nu}\gamma^{[\kappa\nu]}U_{\lambda}]\delta_{,\kappa}(x-\xi)ds$$
$$+\int\left[MU^{\mu}U^{\nu}+\frac{1}{2}\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{1}{2}\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\nu]}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\mu]}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\nu}\right.$$
$$+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{S}^{[\rho\sigma]}U_{\sigma}U^{\mu}-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\mu]}U_{\rho}U^{\nu}-\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma]}\tilde{S}^{[\sigma\nu]}U_{\rho}U^{\mu}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\nu}$$
$$+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\mu}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\nu\sigma}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\mu\sigma}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\nu}$$
$$-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\sigma\kappa}U_{\kappa}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho];\sigma}\tilde{S}^{\rho\kappa\sigma}U_{\kappa}U^{\mu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\mu\kappa}U_{\rho}U^{\nu}$$
$$+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\nu\kappa}U_{\rho}U^{\mu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\mu}U_{\rho}U^{\nu}+\tfrac{1}{2}\gamma^{*[\rho}{}_{\sigma];\kappa}\tilde{S}^{\sigma\kappa\nu}U_{\rho}U^{\mu}+\tfrac{1}{2}S^{\mu\rho}U^{\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}$$
$$+\tfrac{1}{2}S^{\nu\rho}U^{\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\mu}{}_{\kappa]}\tilde{S}^{\kappa\rho\sigma}\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}-\gamma^{*[\nu}{}_{\kappa]}\tilde{S}^{\kappa\rho\sigma}\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}$$

$$+e_M\Bigg[-\tfrac{1}{4}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa-\tfrac{1}{4}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa$$

$$-\tfrac{1}{4}a^{\alpha\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa U_\alpha U^\nu-\tfrac{1}{4}a^{\alpha\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa U_\alpha U^\mu$$

$$-\tfrac{1}{2}a^{\mu\rho}\gamma^{[\sigma\lambda]}U_\lambda\begin{Bmatrix}\nu\\ \rho\sigma\end{Bmatrix}-\tfrac{1}{2}a^{\nu\rho}\gamma^{[\sigma\lambda]}U_\lambda\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}\Bigg]\Bigg]\delta(x-\xi)ds\ . \qquad (3.41)$$

We see that the gravitational structure of the test particle is characterized by the quantities M and $S^{\mu\nu}$. The quantity M is constrained by the equations of structure and motion (3.38)–(3.40), and the quantity $S^{\mu\nu}$ obeys the equations of structure (3.35) and (3.36). The form of the equations of motion and the form of the equations of structure allow us to identify M with the mass of the particle and $S^{\mu\nu}$ with its spin.

C. Simple charged test particles

In Sec. II we found that if one neglects its interaction with other particles, a simple charged particle is characterized by the electromagnetic monopole moments e^M and e^E, an electromagnetic quadrupole moment $e^E_{[\mu\nu]\lambda}$, a mass monopole moment m^G, and a spin $S^G_{\mu\nu}$, where

$$e^M=(c^2l)q_M\ , \qquad (3.42)$$

$$e^E=(c^2/l)q\ , \qquad (3.43)$$

$$e^E_{[\mu\nu]\lambda}=(c^2/l)ql^2(\eta_{\mu\lambda}u_\nu-\eta_{\nu\lambda}u_\mu)\ , \qquad (3.44)$$

$$m^G=4mc^2\ , \qquad (3.45)$$

$$S^G_{\mu\nu}=4s_{\mu\nu}c^2\ . \qquad (3.46)$$

There are no additional multipole moments associated with the particle. We thus have associated with the particle

$$i^{\text{kin}}_\mu=\frac{4\pi}{c^2}\int e^M u_\mu\delta(x-\xi)d\tau\ , \qquad (3.47)$$

$$s^{\text{kin}}_\mu=\frac{4\pi}{c^2}\int e^E u_\mu\delta(x-\xi)d\tau+\frac{4\pi}{c^2}\int e^E_{[\mu\kappa]\lambda}\delta^{,\kappa\lambda}(x-\xi)d\tau\ , \qquad (3.48)$$

and

$$t^{\text{kin}}_{\mu\nu}=\frac{4\pi}{c^2}\int[m^G u_\mu u_\nu+\tfrac{1}{2}\dot S^G_{\mu\rho}u^\rho u_\nu+\tfrac{1}{2}\dot S^G_{\nu\rho}u^\rho u_\mu]\delta(x-\xi)d\tau+\frac{4\pi}{c^2}\int[\tfrac{1}{2}S^G_{\mu\rho}u_\nu+\tfrac{1}{2}S^G_{\nu\rho}u_\mu]\delta^{,\rho}(x-\xi)d\tau\ , \qquad (3.49)$$

where the quantities e^M, e^E, $e^E_{[\mu\nu]\lambda}$, m^G, and $S^G_{\mu\nu}$ in (3.47)–(3.49) are given by (3.42)–(3.46). Making use of Eqs. (3.1), (3.3), and (3.41) and the relationship of $\mathbf{i}^\mu$ to i^{kin}_μ, of $\mathbf{s}^\mu$ to s^{kin}_μ, and of $\mathbf{T}^{\mu\nu}$ to $t^{\text{kin}}_{\mu\nu}$ (those relationships are discussed in Sec. V of paper I), we see that in an external field a simple charged test particle will be characterized by the electromagnetic monopole moments e_M and e, an electromagnetic quadrupole moment $\tilde{s}^{\mu\nu\lambda}$, a mass M, and a spin $S^{\mu\nu}$, where

$$e_M=4\pi lq_M\ , \qquad (3.50)$$

$$e=(2\pi/l)q\ , \qquad (3.51)$$

$$\tilde{s}^{\mu\nu\lambda}=(2\pi/l)ql^2(a^{\mu\lambda}U^\nu-a^{\nu\lambda}U^\mu)\ , \qquad (3.52)$$

$$M=8\pi(m+\Delta m)\ , \qquad (3.53)$$

$$S^{\mu\nu}=8\pi s^{\mu\nu}\ . \qquad (3.54)$$

The particle will possess no additional multipole moments in the external field. In (3.50)–(3.52) the length l is a universal constant, and the quantities q^M and q represent, respectively, the magnetic monopole moment and the charge of the particle. The quantities m and $s^{\mu\nu}$ in (3.53) and (3.54) represent, respectively, the mass and the spin of the particle. The quantity Δm is at this point of our study arbitrary and represents a certain freedom one always has in defining the mass of a particle in the presence of an external field.

In arriving at (3.52) as the exact expression for $\tilde{s}^{\mu\nu\lambda}$ we have assumed that in a locally inertial coordinate system one finds

$$\tilde{s}^{\mu\nu\lambda}=(2\pi/l)ql^2(\eta^{\mu\lambda}u^\nu-\eta^{\nu\lambda}u^\mu)\ ; \qquad (3.55)$$

that is, $\tilde{s}^{\mu\nu\lambda}$ takes its flat-space value. We are here defining a locally inertial coordinate system at point x^μ as a coordinate system such that at x^μ

$$a_{\mu\nu}=\eta_{\mu\nu},\quad a_{\mu\nu,\lambda}=0\ . \qquad (3.56)$$

It is always possible to introduce a locally inertial coordinate system at any given point x^μ. At that point and in such a coordinate system the acceleration of a neutral pole test particle possessing no electromagnetic multipole moments will vanish[23]—this is, of course, the reason for the name locally inertial coordinate system. That in a locally inertial coordinate system $\tilde{s}^{\mu\nu\lambda}$ takes its flat-space value is a natural condition to place on the structure of a charged test particle, and we shall restrict our study from now on to charged test particles satisfying this condition.

At this point, we therefore place an additional restriction on what we mean by a simple charged test particle in Einstein's unified field theory. In addition to such a particle being the test-particle limit of what we have called a simple charged test particle, we also require that the electromagnetic multipole moments e_M, e, $\tilde{s}^{[\mu\nu]}$, and $\tilde{s}^{\mu\nu\lambda}$ associated with the particle take their flat-space value in a locally inertial coordinate system.

We first investigate the equations of spin satisfied by such a test particle. Making use of (3.2) and (3.4), we see from (3.50) and (3.51) that both q_M and q are constant even in the presence of an external field. Placing (3.52) in (3.36), and making use of (3.50), and the fact that $\tilde{s}^{[\mu\nu]}=0$, and that associated with the background field $\Gamma^{\nu}_{[\mu\nu]}=0$, we find from (3.36)

$$N^{\mu\nu}=2\pi l q_M[-(a^{\mu\lambda}-U^{\mu}U^{\lambda})a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}+(a^{\nu\lambda}-U^{\nu}U^{\lambda})a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}] . \tag{3.57}$$

This means that the equations of spin (3.35) satisfied by a simple charged test particle in Einstein's unified field theory take the form

$$\frac{DS^{\mu\nu}}{Ds}-\frac{DS^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{DS^{\nu\rho}}{Ds}U_{\rho}U^{\mu}=N^{\mu\nu} , \tag{3.58}$$

where $N^{\mu\nu}$ is given by (3.57).

We next investigate the equations of mass and motion satisfied by the particle. Placing (3.50)–(3.54) in (3.30)–(3.40) and making use of the fact that $s^{[\mu\nu]}=0$, we find as the equations of mass and motion of a simple charged test particle in Einstein's unified field theory

$$\frac{DP^{\mu}}{Ds}+\tfrac{1}{2}S^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\sigma\rho}=F^{\mu} , \tag{3.59}$$

where

$$P^{\mu}=8\pi\left[(m+\Delta m)U^{\mu}+\frac{Ds^{\mu\rho}}{Ds}U_{\rho}+\left[\frac{1}{4l}\right]ql^2(\gamma^{*[\rho\sigma]}{}_{;\sigma}U_{\rho}U^{\mu}-\gamma^{*[\mu\sigma]}{}_{;\sigma})+\left[\frac{1}{4l}\right]q_M l^2(-a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\nu})\right] , \tag{3.60}$$

$$F^{\mu}=(2\pi/l)[q\gamma^{*[\nu\mu]}U_{\nu}+ql^2(a^{\kappa\lambda}\gamma^{*[\mu\nu]}{}_{;\kappa\lambda}U_{\nu}-\gamma^{*[\mu\lambda]}{}_{;\kappa\lambda}U^{\kappa}-\gamma^{*[\rho\sigma]}R^{*\mu}{}_{\rho\kappa\sigma}U^{\kappa}-a^{\mu\kappa}\gamma^{*[\rho\sigma]}R^{*}_{\kappa\rho}U_{\sigma})$$
$$+q_M l^2(-a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}-a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_{\nu}-a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_{\nu}-a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_{\nu}$$
$$+a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_{\nu}+a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_{\nu})] . \tag{3.61}$$

We have made use of the definitions

$$R^{*}_{\mu\nu}=R^{*\kappa}{}_{\mu\nu\kappa},\quad \gamma^{*[\mu\nu]}=a^{\nu\rho}\gamma^{*[\mu}{}_{\rho]} . \tag{3.62}$$

Equations (3.59)–(3.61) take a simpler form if we choose

$$\Delta m=-\left[\frac{1}{4l}\right]ql^2\gamma^{*[\rho\sigma]}{}_{;\sigma}U_{\rho} , \tag{3.63}$$

and introduce the notation

$$p^{\mu}=\frac{1}{8\pi}P^{\mu}+\left[\frac{1}{4l}\right]ql^2\gamma^{*[\mu\rho]}{}_{;\rho}+\left[\frac{1}{4l}\right]q_M l^2a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U_{\nu} , \tag{3.64}$$

$$f^{\mu}=\frac{1}{8\pi}F^{\mu}+\left[\frac{1}{4l}\right]ql^2\gamma^{*[\mu\rho]}{}_{;\rho\sigma}U^{\sigma}$$
$$+\left[\frac{1}{4l}\right]q_M l^2\left[a_{\rho\sigma}\gamma^{[\kappa\mu]}{}_{;\nu}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U^{\lambda}U^{\nu}+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}{}_{;\nu}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U^{\lambda}U^{\nu}\right.$$
$$\left.+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\nu}U_{\kappa}U^{\lambda}U^{\nu}+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}\left[U_{\kappa}\frac{DU^{\lambda}}{Ds}+\frac{DU_{\kappa}}{Ds}U^{\lambda}\right]\right] . \tag{3.65}$$

Using (3.63)–(3.65), we find the equations of mass and motion (3.59)–(3.61) can be written in the form

$$\frac{Dp^{\mu}}{Ds}+\tfrac{1}{2}s^{\rho\sigma}U^{\kappa}R^{*\mu}{}_{\kappa\sigma\rho}=f^{\mu} , \tag{3.66}$$

where

$$p^{\mu}=mU^{\mu}+\frac{Ds^{\mu\rho}}{Ds}U_{\rho} , \tag{3.67}$$

$$f^{\mu}=\left[\frac{1}{4l}\right]\Big[q\gamma^{*[\nu\mu]}U_\nu+ql^2\gamma^{*[\mu\nu];\kappa}{}_{;\kappa}U_\nu+ql^2(-\gamma^{*[\rho\sigma]}R^{*\mu}{}_{\kappa\rho\sigma}U^\kappa-\gamma^{*[\mu\rho]}R^{*}_{\rho\lambda}U^\lambda-\gamma^{*}_{[\rho\sigma]}R^{*\mu\rho}U^\sigma)$$
$$+q_Ml^2\Big[-a^{\mu\kappa}a^{\rho\lambda}R^{\nu}{}_{\rho\kappa\lambda}U_\nu-a^{\mu\kappa}a^{\rho\nu}R_{\rho\kappa}U_\nu-a^{\mu\kappa}\gamma^{[\rho\lambda]}R^{\nu}{}_{\rho\kappa\lambda}U_\nu-a^{\mu\kappa}\gamma^{[\rho\nu]}R_{\rho\kappa}U_\nu$$
$$+a_{\rho\sigma}a^{\mu\lambda}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}U_\nu+a_{\rho\sigma}a^{\mu\lambda}\gamma^{[\nu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}U_\nu$$
$$+a_{\rho\sigma}\gamma^{[\kappa\mu]}{}_{;\nu}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa U^\lambda U^\nu+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}{}_{;\nu}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa U^\lambda U^\nu$$
$$+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\nu}U_\kappa U^\lambda U^\nu+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa\frac{DU^\lambda}{Ds}+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}\frac{DU_\kappa}{Ds}U^\lambda\Big]\Big]\ . \tag{3.68}$$

In (3.68) we are using the definition

$$R^{*\mu\nu}=a^{\mu\rho}a^{\nu\sigma}R^{*}_{\rho\sigma}\ . \tag{3.69}$$

One finds from (3.68) that

$$f^{\mu}U_\mu=\left[\frac{1}{4l}\right]q_Ml^2\Big[-a^{\rho\sigma}R^{\mu}{}_{\rho\lambda\sigma}-a^{\rho\mu}R_{\rho\lambda}-\gamma^{[\rho\sigma]}R^{\mu}{}_{\rho\lambda\sigma}-\gamma^{[\rho\mu]}R_{\rho\lambda}+a_{\rho\sigma}\gamma^{[\mu\kappa]}\gamma^{[\rho\tau]}{}_{;\kappa}\Gamma^{\sigma}_{[\lambda\tau]}$$
$$+a_{\rho\sigma}\gamma^{[\mu\kappa]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau];\kappa}+a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}\frac{DU_\kappa}{Ds}\Big]U^\lambda U_\mu\ . \tag{3.70}$$

Although the universal length l, the charge q, and the magnetic monopole moment q_M associated with a simple test particle are constant, we see from (3.66)—(3.68), making use of (3.70), and from (3.54), (3.57), and (3.58) that the mass m and spin $s^{\mu\nu}$ associated with the particle are in general not constant. They obey the equations of structure

$$\frac{dm}{ds}=\frac{DU_\rho}{Ds}\frac{D}{Ds}(s^{\rho\mu}U_\mu)+f^{\mu}U_\mu\ , \tag{3.71}$$

$$\frac{Ds^{\mu\nu}}{Ds}-\frac{Ds^{\mu\rho}}{Ds}U_\rho U^\nu+\frac{Ds^{\nu\rho}}{Ds}U_\rho U^\mu=n^{\mu\nu}\ , \tag{3.72}$$

where $f^{\mu}U_\mu$ is given in (3.70), and

$$n^{\mu\nu}=\left[\frac{1}{4l}\right]q_Ml^2[-(a^{\mu\lambda}-U^\mu U^\lambda)a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa$$
$$+(a^{\nu\lambda}-U^\nu U^\lambda)a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}$$
$$\times\Gamma^{\sigma}_{[\lambda\tau]}U_\kappa]\ . \tag{3.73}$$

We see from (3.72) and (3.73) that in general when in a background field possessing torsion a test particle possessing a magnetic monopole moment must also possess spin.

IV. ELECTROMAGNETIC CURRENT DENSITIES AND ENERGY-MOMENTUM TENSOR DENSITY

We have found that in Einstein's unified field theory a simple charged test particle is characterized by a mass m, a spin $s^{\mu\nu}$, a charge q, a magnetic monopole moment q_M, and a universal length l. In this section we wish to give the electromagnetic current densities $\mathbf{i}^{\mu}$ and $\mathbf{s}^{\mu}$ and the energy-momentum tensor $\mathbf{T}^{\mu\nu}$ associated with these particles.

Making use of (3.50), we find from (3.1) that

$$\mathbf{i}^{\mu}=4\pi l\int q_MU^\mu\delta(x-\xi)ds\ . \tag{4.1}$$

Making use of (3.51), (3.52), and the fact that $\tilde{s}^{[\mu\nu]}$ vanishes, we find from (3.3) that

$$\mathbf{s}^{\mu}=(2\pi/l)\int qU^\mu\delta(x-\xi)ds+(2\pi/l)\int ql^2\left[a^{\rho\sigma}U^\kappa\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-2a^{\kappa\rho}U^\sigma\begin{Bmatrix}\mu\\ \rho\sigma\end{Bmatrix}-a^{\rho\sigma}U^\mu\begin{Bmatrix}\kappa\\ \rho\sigma\end{Bmatrix}\right]\delta_{,\kappa}(x-\xi)ds$$
$$+(2\pi/l)\int ql^2[-a^{\kappa\lambda}U^\mu]\delta_{,\kappa\lambda}(x-\xi)ds\ . \tag{4.2}$$

Making use of (3.50)—(3.54), (3.63), and the fact that $\tilde{s}^{[\mu\nu]}$ vanishes, we find from (3.41) that

$$
\begin{aligned}
\mathbf{T}^{\mu\nu}=8\pi \int \Bigg[& mU^{\mu}U^{\nu}+\frac{1}{2}\frac{Ds^{\mu\rho}}{Ds}U_{\rho}U^{\nu}+\frac{1}{2}\frac{Ds^{\nu\rho}}{Ds}U_{\rho}U^{\mu}+\left\{\frac{1}{4l}\right\}ql^{2}(-a^{\mu\rho}\gamma^{*[\nu\sigma]}{}_{;\rho}U_{\sigma}-a^{\nu\rho}\gamma^{*[\mu\sigma]}{}_{;\rho}U_{\sigma}) \\
& +\tfrac{1}{2}s^{\mu\rho}U^{\sigma}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}+\tfrac{1}{2}s^{\nu\rho}U^{\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}+\left\{\frac{1}{4l}\right\}ql^{2}\Bigg[a^{\rho\sigma}\gamma^{*[\mu\kappa]}U_{\kappa}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}+a^{\rho\sigma}\gamma^{*[\nu\kappa]}U_{\kappa}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\} \\
& \qquad -\gamma^{*[\mu\rho]}U^{\sigma}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}-\gamma^{*[\nu\rho]}U^{\sigma}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\Bigg] \\
& +\left\{\frac{1}{4l}\right\}q_{M}l^{2}\Bigg[-\tfrac{1}{2}a^{\mu\lambda}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}-\tfrac{1}{2}a^{\nu\lambda}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa} \\
& \qquad -\tfrac{1}{2}a_{\rho\sigma}\gamma^{[\kappa\mu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U^{\lambda}U^{\nu}-\tfrac{1}{2}a_{\rho\sigma}\gamma^{[\kappa\nu]}\gamma^{[\rho\tau]}\Gamma^{\sigma}_{[\lambda\tau]}U_{\kappa}U^{\lambda}U^{\mu} \\
& \qquad -a^{\mu\rho}\gamma^{[\sigma\lambda]}U_{\lambda}\left\{\begin{matrix}\nu\\ \rho\sigma\end{matrix}\right\}-a^{\nu\rho}\gamma^{[\sigma\lambda]}U_{\lambda}\left\{\begin{matrix}\mu\\ \rho\sigma\end{matrix}\right\}\Bigg]\Bigg]\delta(x-\xi)ds \\
+8\pi\int\Bigg[& \tfrac{1}{2}s^{\mu\kappa}U^{\nu}+\tfrac{1}{2}s^{\nu\kappa}U^{\mu}+\left\{\frac{1}{4l}\right\}ql^{2}(a^{\mu\kappa}\gamma^{*[\nu\rho]}U_{\rho}+a^{\nu\kappa}\gamma^{*[\mu\rho]}U_{\rho}-\gamma^{*[\mu\kappa]}U^{\nu}-\gamma^{*[\nu\kappa]}U^{\mu} \\
& -a^{\mu\nu}\gamma^{*[\kappa\rho]}U_{\rho})-\left\{\frac{1}{4l}\right\}q_{M}l^{2}a^{\mu\nu}\gamma^{[\kappa\nu]}U_{\lambda}\Bigg]\delta_{,\kappa}(x-\xi)ds \;. \qquad (4.3)
\end{aligned}
$$

V. COMPARISON WITH THE RESULTS OF THE APPROXIMATION PROCEDURE

If we make use of the power-series expansion in κ,[24]

$$g_{\mu\nu}=\eta_{\mu\nu}+\sum_{k=1}^{\infty}\kappa^{k}{}_{(k)}g_{\mu\nu} \;, \qquad (5.1)$$

and regard $\gamma^{*}_{[\mu\nu]}$, $\gamma_{[\mu\nu]}$, q, and q_M as possessing only terms of odd order in κ, and $a_{\mu\nu}$ and m as possessing only terms of even order in κ where m itself is of order two in κ (that there is no loss in generality in making such assumptions is shown in paper RVIII of Ref. 9), and treat the spin $s^{\mu\nu}$ as of order four in κ [this is easily seen to be compatible with the equations of spin (3.72) and (3.73)], we find from the equations of mass and motion (3.66)–(3.68)

$$
\begin{aligned}
m\dot{u}^{\mu}=\left\{\frac{1}{4l}\right\}[& q\tilde{\gamma}^{*[\rho\mu]}u_{\rho}-ql^{2}\Box^{2}\tilde{\gamma}^{*[\rho\mu]}u_{\rho} \\
& -q_{M}l^{2}\Box^{2}\tilde{\gamma}^{[\rho\mu]}u_{\rho}]+O(\kappa^{4}) \;, \qquad (5.2)
\end{aligned}
$$

where in (5.2) all indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}=\eta^{\mu\nu}$ and

$$\tilde{\gamma}^{*}_{[\mu\nu]}=\gamma^{*}_{[\mu\nu]}=\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{g}^{[\sigma\rho]} \;, \qquad (5.3)$$

$$\tilde{\gamma}^{[\mu\nu]}=\mathbf{g}^{[\mu\nu]} \;, \qquad (5.4)$$

and

$$u^{\mu}=\frac{d\xi^{\mu}}{d\tau},\quad \dot{u}^{\mu}=\frac{du^{\mu}}{d\tau} \;, \qquad (5.5)$$

$$d\tau^{2}=\eta_{\mu\nu}d\xi^{\mu}d\xi^{\nu} \;. \qquad (5.6)$$

The fields $\tilde{\gamma}^{*[\mu\nu]}$ and $\tilde{\gamma}^{[\mu\nu]}$ appearing in (5.2) are background fields. The field $\mathbf{g}^{[\mu\nu]}$ in (5.3) and (5.4) is defined in paper I. Equations of motion (5.2) are in agreement with the equations of motion to second order of a simple charged test particle obtained using the author's approximation procedure discussed in papers RI–RVIII of Ref. 9.[25]

APPENDIX A: $\tilde{s}^{\mu\kappa\lambda}$

We shall show that there is no loss in generality in choosing the oriented tensor $\tilde{s}^{\mu\kappa\lambda}$, appearing in (3.3), to be of the form

$$\tilde{s}^{\mu\kappa\lambda}=\tilde{s}^{[\mu\kappa]\lambda} \;. \qquad (A1)$$

As has been discussed in Sec. III A, $\tilde{s}^{\mu\kappa\lambda}$ can always be written in the form

$$\tilde{s}^{\mu\kappa\lambda}=\tilde{s}'^{[\mu\kappa]\lambda}+\tilde{s}'^{(\mu\kappa)\lambda} \;, \qquad (A2)$$

where

$$\tilde{s}'^{(\mu\kappa)\lambda}+\tilde{s}'^{(\kappa\lambda)\mu}+\tilde{s}'^{(\lambda\mu)\kappa}=0 \;. \qquad (A3)$$

Making use of (A3), we see that we can always write $\tilde{s}'^{(\mu\kappa)\lambda}$ in the form

$$\tilde{s}'^{(\mu\kappa)\lambda}=\tfrac{1}{3}(\tilde{s}'^{(\kappa\mu)\lambda}-\tilde{s}'^{(\kappa\lambda)\mu}+\tilde{s}'^{(\mu\kappa)\lambda}-\tilde{s}'^{(\mu\lambda)\kappa}) \;, \qquad (A4)$$

or in the form

$$\tilde{s}'^{(\mu\kappa)\lambda}=\Delta\tilde{s}'^{[\mu\lambda]\kappa}+\Delta\tilde{s}'^{[\kappa\lambda]\mu} \;, \qquad (A5)$$

where we have defined $\Delta\tilde{s}'^{[\mu\kappa]\lambda}$ through the equations

$$\Delta\tilde{s}'^{[\mu\kappa]\lambda}=\tfrac{1}{3}(\tilde{s}'^{(\lambda\mu)\kappa}-\tilde{s}'^{(\lambda\kappa)\mu}) \;. \qquad (A6)$$

Thus, we see from (A2) and (A5) that we can always write $\tilde{s}^{\mu\kappa\lambda}$ in the form

$$\tilde{s}^{\mu\kappa\lambda}=\tilde{s}'^{[\mu\kappa]\lambda}+\Delta\tilde{s}'^{[\mu\lambda]\kappa}+\Delta\tilde{s}'^{[\kappa\lambda]\mu} . \quad (A7)$$

Placing (A7) in (3.3), one finds

$$\mathbf{s}^{\mu}=\int eU^{\mu}\delta(x-\xi)ds + \int \left[\tilde{s}^{[\mu\kappa]}+\tilde{s}^{[\mu\rho]\sigma}\left\{ {\kappa \atop \rho\sigma} \right\}-\tilde{s}^{[\kappa\rho]\sigma}\left\{ {\mu \atop \rho\sigma} \right\}\right] \times\delta_{,\kappa}(x-\xi)ds + \int \tilde{s}^{[\mu\kappa]\lambda}\delta_{,\kappa\lambda}(x-\xi)ds , \quad (A8)$$

where

$$\tilde{s}^{[\mu\kappa]\lambda}=\tilde{s}'^{[\mu\kappa]\lambda}+\Delta\tilde{s}'^{[\mu\kappa]\lambda} . \quad (A9)$$

We see that there is no loss in generality in choosing the oriented tensor $\tilde{s}^{\mu\kappa\lambda}$, which appears in (3.3), to be of the form (A1).

APPENDIX B: $\tilde{T}^{(\mu\nu)\kappa}$

We shall show that there is no loss in generality in choosing the tensor $\tilde{T}^{(\mu\nu)\kappa}$, which appears in (3.7), to be of the form (3.14).

From (3.13), since (2.28) must be satisfied, we have

$$\int C^{\mu\kappa\nu}\delta_{,\kappa\nu}(x-\xi)ds+\int C^{\mu\nu}\delta_{,\nu}(x-\xi)ds + \int C^{\mu}\delta(x-\xi)ds=0 , \quad (B1)$$

where $C^{\mu\kappa\nu}$ can be written in the form

$$C^{\mu\kappa\nu}=\tilde{T}^{(\mu\nu)\kappa}+\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\nu\kappa}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\mu\kappa} +\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\kappa\nu}+\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\kappa\mu}-\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\nu\mu} -\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\mu\nu}-\tfrac{1}{2}e_{M}a^{\mu\nu}\gamma^{[\lambda\kappa]}U_{\lambda} \equiv\tilde{T}'^{(\mu\nu)\kappa} . \quad (B2)$$

Using arguments similar to those applied to $\tilde{T}^{(\mu\nu)\kappa}$ in Appendix B of paper I, but here applied to $\tilde{T}'^{(\mu\nu)\kappa}$, we find that one can with no loss in generality always write $\tilde{T}'^{(\mu\nu)\kappa}$ in the form

$$\tilde{T}'^{(\mu\nu)\kappa}=\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu} , \quad (B3)$$

where $S^{\mu\nu}$ is an antisymmetric second-rank tensor characterizing the particle and

$$U^{\mu}=\frac{d\xi^{\mu}}{ds} . \quad (B4)$$

From (B2) and (B3) we then find

$$\tilde{T}^{(\mu\nu)\kappa}=\tfrac{1}{2}S^{\mu\kappa}U^{\nu}+\tfrac{1}{2}S^{\nu\kappa}U^{\mu}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\nu\kappa} -\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\mu\kappa}-\tfrac{1}{2}\gamma^{*[\mu}{}_{\rho]}\tilde{s}^{\rho\kappa\nu}-\tfrac{1}{2}\gamma^{*[\nu}{}_{\rho]}\tilde{s}^{\rho\kappa\mu} +\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\nu\mu}+\tfrac{1}{2}\gamma^{*[\kappa}{}_{\rho]}\tilde{s}^{\rho\mu\nu} -\tfrac{1}{2}e_{M}a^{\mu\nu}\gamma^{[\kappa\lambda]}U_{\lambda} . \quad (B5)$$

We see that there is no loss in generality in choosing the tensor $\tilde{T}^{(\mu\nu)\kappa}$, which appears in (3.7), to be of the form (3.14).

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[1]C. R. Johnson, Phys. Rev. D **31**, 1236 (1985). This paper will be referred to as paper I. There is a misprint in Appendix A of paper I. Equation (A1) in Appendix A of paper I should read $g^2=\tilde{h}^{-1}h$. There is also a misprint in the main text of paper I. In Eqs. (3.9) of paper I the superscript σ should be the superscript ρ.

[2]Multipole moments and multipole test particles are defined and discussed in Ref. 1. Treating a particle as a test particle means neglecting self-interaction terms in the particle's equations of structure and motion and neglecting the effect of the particle on the background field. If, in addition, the particle is described through a finite number of multipole moments, then it is known as a multipole test particle. For a more complete discussion see Ref. 1.

[3]A. Einstein, *The Meaning of Relativity*, 5th ed. (Princeton University Press, Princeton, New Jersey, 1955), Appendix II, pp. 133–166.

[4]C. R. Johnson, Phys. Rev. D **31**, 1252 (1985). This paper will be referred to as paper II. There are several misprints in paper II. In Eqs. (3.36) of paper II, ${}^{*}X^{[\mu\nu]}$ should be ${}^{*}Y^{[\mu\nu]}$. In Eqs. (3.62) of paper II, u^{κ} and u^{μ} should be U^{κ} and U^{μ}, respectively. In Eqs. (3.67) of paper II, $\gamma^{*[\mu\sigma]}{}_{;\rho}$ should be $\gamma^{*[\mu\sigma]}{}_{;\sigma}$. In Eqs. (3.75) of paper II, f^{κ} should be f^{μ}. In Eqs. (A21) of paper II, superscripts μ should be superscripts ρ.

[5]C. R. Johnson, Phys. Rev. D **24**, 327 (1981).

[6]For a definition of e^{M} see Sec. V of Ref. 1. Particles possessing a magnetic monopole moment have been studied in Ref. 5.

[7]Such particles are discussed in Ref. 5.

[8]C. R. Johnson and J. R. Nance, Phys. Rev. D **15**, 377 (1977); **16**, 533(E) (1977).

[9]C. R. Johnson, Phys. Rev. D **4**, 295 (1971); **4**, 318 (1971); **4**, 3555 (1971); **5**, 282 (1972); **5**, 1916 (1972); **7**, 2825 (1973); **7**, 2838 (1973); **8**, 1645 (1973). We shall refer to these papers as papers RI–RVIII, respectively.

[10]The notation used in this paper will be the same as that in Ref. 1. A harmonic coordinate system is defined in paper RI of Ref. 9.

[11]The universal length l is discussed in Refs. 5 and 8.

[12]The physical meaning of q is discussed in Refs. 5 and 8.

[13]The physical meaning of q_M is discussed in Ref. 5.

[14]The physical meaning of m is discussed in Refs. 5 and 8.

[15]The quantity $\delta(x-\xi)$ represents the four-dimensional Dirac delta function. The indices on both x^{μ} and ξ^{μ} have been suppressed.

[16]The constant c represents the speed of light.

[17]When applied to a test particle, Eqs. (2.26)–(2.30) are equivalent to Eqs. (5.29)–(5.31) of paper I. This is true because when Eqs. (5.31) of paper I are applied to a test particle, the terms

$$a^{\mu\rho}\gamma^{[\kappa\nu]}(T_{\kappa\rho}-\tfrac{1}{2}a_{\kappa\rho}a^{\lambda\tau}T_{\lambda\tau})\mathbf{i}_{\nu}$$

appearing in them, associated with self-interaction, are, by the definition of a test particle, to be neglected.

[18]For a further discussion of the notation see Ref. 1.

[19]It can be shown that there is no loss in generality in choosing $\tilde{s}^{(\mu\kappa)\lambda}=0$. See Appendix A.

[20]See Appendix B.

[21]See Appendix C of Ref. 1.

[22]In Appendix D of Ref. 1 we show that if

$$\int C^{\nu}(s)\delta_{,\nu}(x-\xi)ds+\int C(s)\delta(x-\xi)ds=0 ,$$

where $C^{\nu}U_{\nu}=0$, then $C^{\nu}=0$, $C=0$.

[23]The equations of motion of a neutral pole test particle possessing no electromagnetic multipole moments are given in Ref. 1.

[24]This power-series expansion has been used in Refs. 1, 5, and 8. It is also used and discussed in the papers of Ref. 3.

[25]See, for example, Eqs. (41) of Ref. 5. One chooses $\epsilon=1$ in Eqs. (41) of Ref. 5 if the equations are to apply to charged particles. The radiation reaction terms found in Eqs. (41) of Ref. 5 vanish in the test-particle limit and are thus not present in Eqs. (5.2).