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Classification of the static vacuum metric with Ricci-flat compactification

M. Yoshimura
National Laboratory for High Energy Physics (KEK), Oho-machi, Tsukuba-gun, Ibaraki-ken, 305 Japan
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We explicitly construct all spherically symmetric vacuum solutions of the higher-dimensional Einstein equation in which the size of Ricci-flat compact internal manifolds varies with three-dimensional distance, subject to an asymptotic condition, flat M_4 times a compact manifold. Besides the usual Schwarzschild and the trivial vacuum solutions, a variety of new solutions are found, all of which contain a curvature singularity not hidden by an event horizon (naked singularity) unless the extra compact space admits an isometry.

Considerable attention has been focused on theories in more than four spacetime dimensions in which extra dimensions are compactified to a small size, apparently beyond our present ability of experimental detection. The basic idea is traced back to the old Kaluza-Klein theory,[1] but recent interest in higher-dimensional theories has been sparked by the discovery of anomaly-free superstring theories.[2] Spacetime structure in these theories is of a special kind: it must asymptotically approach a direct-product space of flat Minkowski space times a compact manifold, away from any local lump of matter. The size of the extra compact manifold may however differ from a ground-state value in the vicinity of dense matter.

In this paper we ask what kind of metric, in these theories, generalizes the Schwarzschild solution in ordinary relativity, and find a complete answer in the special case in which the compact manifold is Ricci flat and does not admit an isometry. Examples of Ricci-flat compact spaces are a torus that has an isometry, and Calabi-Yau spaces[3] in which the Riemann curvature tensor does not vanish. Calabi-Yau spaces are of great interest in superstring theories since in this case an unbroken supersymmetry in four dimensions may solve the gauge-hierarchy problem.[4]

At the outset one should note a peculiar feature of Ricci-flat space. A spacetime-dependent size $b(x)$ of extra manifolds can be viewed as a scalar field in four spacetime dimensions. When the manifold is Ricci flat, this scalar field $[\sim \ln b(x)]$ is massless and a long-range force may exist. This raises a serious question and casts a doubt on the uniqueness of the Schwarzschild solution, the Birkoff theorem in four spacetime dimensions. Indeed, examples that violate this theorem were already found.[5–7] Here we advance a step further and explicitly construct all static vacuum solutions with three-dimensional spherical symmetry. In reality, the extra space may not be completely Ricci flat and the long-range force may then be cut off at a distance scale of m^{-1}.

A variety of new solutions thus obtained are expected to have a great impact on formations of black holes if the hole's mass M obeys an inequality for the gravitational radius, $2GM < m^{-1}$. In superstring theories the inverse curvature m may well coincide with the mass scale of supersymmetry breaking. Thus, if the hole's mass is smaller than a value of 10^{11} g(1 TeV/m), solutions presented here are very important.

In all previous approaches[7] to this problem the assumption of a toroidal isometry for the extra space was crucial in constructing explicit solutions. In this paper we shall instead assume that there exists no isometry as occurs in the case of Calabi-Yau spaces. This makes an important difference in our final conclusions: our case allows no regular wormhole solution in sharp contrast to the situation in the toroidal compactification.[8] Another technical advantage of our assumption is that we can construct all solutions in any dimensions, while in the case of the toroidal compactification only partial solutions are known[7] for more than five dimensions.

Consider now a general form of a three-dimensional spherically symmetric metric:

$$ds^2 = e^{\sigma} dt^2 - e^{\omega} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - b^2 \tilde{g}_{\alpha\beta}(y) dy^\alpha dy^\beta \,. \tag{1}$$

The functions σ, ω, and b depend on three-dimensional distance r. We shall assume throughout this paper that the Ricci tensor $\tilde{R}_{\alpha\beta}(y)$ formed out of $\tilde{g}_{\alpha\beta}$ alone vanishes.

The static, three-dimensional spherical symmetry actually allows, in addition to terms written in (1), mixed terms of the form, $\lambda_\alpha(r) dy^\alpha dt$. This degree of freedom corresponds to an "electric"-type gauge field. As shown later, however, the Einstein equation demands that the "electric" field should be absent unless the extra compact manifold admits an isometry. For the time being we shall hence ignore this degree of freedom.

The Einstein equation takes a convenient form if one uses a quantity

$$u \equiv \tfrac{1}{2} r (1 - e^{-\omega}) \,. \tag{2}$$

In ordinary relativity this quantity coincides with half the gravitational radius of the vacuum Schwarzschild solution. The vacuum Einstein equation in $d+4$ dimensions is then equivalent to

$$\left[1-\frac{2u}{r}\right]\left[\frac{b''}{b}-\left[\frac{b'}{b}\right]^2\right]+\frac{2}{r}\left[1-\frac{u}{r}\right]\frac{b'}{b}=0\ , \tag{3}$$

$$u'-\frac{d}{2}\left[1+\frac{d}{2}r\frac{b'}{b}\right]^{-1}\frac{b'}{b}\left[\tfrac{1}{2}(d+1)\frac{b'}{b}r(r-2u)-u\right]=0\ , \tag{4}$$

$$\left[1+\frac{d}{2}r\frac{b'}{b}\right](\sigma'+\omega')-dr\frac{b''}{b}=0\ . \tag{5}$$

The prime means the derivative d/dr.

The simplest solution is obtained by assuming that $u=u_0$ (constant). The first equation (3) is then solved with $b'/b=C[r(r-2u_0)]^{-1}$. This is consistent with (4) only if the constant $C=0$ or $C=(d+1)^{-1}2u_0$. The former case yields the usual Schwarzschild solution, while the latter gives[6]

$$b=b_0\left[1-\frac{2u_0}{r}\right]^{1/(d+1)}\ , \tag{6a}$$

$$e^{\omega}=\left[1-\frac{2u_0}{r}\right]^{-1}\ , \tag{6b}$$

$$e^{\sigma}=\left[1-\frac{2u_0}{r}\right]^{-(d-1)/(d+1)}\ . \tag{6c}$$

A more systematic way to integrate the Einstein equation completely, (3)–(5), is to use a variable w given by

$$w=1+\frac{d}{2}r\frac{b'}{b}\ . \tag{7}$$

From (3) the quantity u can be expressed in terms of w as

$$(r-2u)^{-1}=w'(1-w)^{-1}\ . \tag{8}$$

Inserting this into (4), one finds after a little algebra that

$$\frac{d}{dx}\left[w^{-1}(1-w)^{-2}\frac{dw}{dx}\right]=2w^{-1}(1-w)^{-2}\times\frac{dw}{dx}\left[\frac{d+1}{d}w-\frac{d+2}{d}+\frac{d+2}{2d}w^{-1}\right]\ ,$$

with $x\equiv\ln r$. This differential equation can be integrated with two integration constants, $c\neq 0$, $r_0>0$, as

$$r\frac{dw}{dr}=-c(w-1)(w-w_1)(w-w_2)\ , \tag{9}$$

$$\left[\frac{r}{r_0}\right]^2=|(w-1)^{-2}(w-w_1)^{(p-1)/p}(w-w_2)^{(p+1)/p}|\ , \tag{10}$$

where, with $w_1<w_2$,

$$w_i{}^2-\left[1-\frac{2(d+1)}{dc}\right]w_i-\frac{d+2}{dc}=0\ , \tag{11}$$

$$p=\left[\left[1+\frac{2(d+1)}{dc}\right]^2-\frac{4}{c}\right]^{1/2}\left[1+\frac{2(d+1)}{dc}\right]^{-1}\ . \tag{12}$$

In the special case of $c=0$ these relations read as

$$\frac{r}{r_0}=\left|(w-1)^{-1}\left[w-\frac{d+2}{2(d+1)}\right]\right|\ . \tag{13}$$

It is straightforward to express all metric components in terms of w with the help of (2) and (7)–(9):

$$b=\text{const}\times|(w-w_1)(w-w_2)^{-1}|^{2/(dcq)}\ , \tag{14}$$

$$e^{\omega}=c(w-w_1)(w-w_2)\ , \tag{15}$$

$$e^{\sigma}=\text{const}\times|(w-w_1)(w-w_2)^{-1}|^{[1-2(d-1)/(dc)]/q}\ ,\quad q\equiv\left[\left[1+\frac{2(d+1)}{dc}\right]^2-\frac{4}{c}\right]^{1/2}\ . \tag{16}$$

The constants in (14) and (16) are readily determined by the asymptotic condition, $b\to b_0$ and $\sigma\to 0$ as $r\to\infty$. In the case of $c=0$ it can be shown that the metric coincides with the previous one (6).

The asymptotic value of w as $r\to\infty$ must be 1 and cannot be w_i, irrespective of the parameter p in the exponent of (10), because at w_i the metric (15) vanishes and does not yield a sensible spacetime. As r decreases, the parameter w monotonically changes according to one of the following patterns:

$c>0$,

$w=1$ to $+\infty$ as $r=\infty$ to r_0 ,

$w=1$ to w_2 as $r=\infty$ to 0 ,

$c<0$,

$w=1$ to w_2 as $r=\infty$ to 0 ,

$w=1$ to w_1 as $r=\infty$ to 0 .

Note that $[(d+2)(d+1)^{-1}/2]^{1/2}<p<1$ for $c>0$ and $p<-1$ for $c<0$. It will be shown later how the metric can be extended beyond the end point at r_0.

The nature of the solutions is characterized by the presence and type of singularities. Curvature singularities are determined by direct computation of invariant Riemann curvatures squared:

$$\begin{aligned}R_{ABCD}R^{ABCD}=\ &e^{-2\omega}(\sigma''+\tfrac{1}{2}\sigma'^2-\tfrac{1}{2}\sigma'\omega')^2+2r^{-2}e^{-2\omega}(\sigma')^2+2r^{-2}e^{-2\omega}(\omega')^2+4r^{-4}(1-e^{-\omega})^2\\&+4de^{-2\omega}(b^{-1}b''-\tfrac{1}{2}b^{-1}b'\omega')^2+de^{-2\omega}(b^{-1}b'\sigma')^2+8dr^{-2}e^{-2\omega}(b^{-1}b')^2\\&+2d(d-1)e^{-2\omega}(b^{-1}b')^4+b^{-4}\tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta}\ .\end{aligned} \tag{17}$$

The r derivative here is readily converted to the w derivative by (9). Straightforward computations demonstrate that curvature singularities appear at w_i, typically behaving like $|w-w_i|^{2(1-2|p|)/|p|}$, and that $w=1$ and ∞ are regular points.

Solutions thus explicitly constructed are classified according to type of singularities present in these solutions. We shall discuss these in sequence. When we count the number of parameters in the solution, N, we shall exclude the asymptotic value b_0 of the scale factor, which is not fixed in the case of Ricci flat compactification. Hence the maximum number of integration constants is equal to $N=2$.

(1) Schwarzschild solution. The size of the extra space is constant everywhere. The curvature singularity at $r=0$ is shielded by the event horizon at $r=2GM$; $N=1$. This solution is unique among all solutions, in the sense that only this has an event horizon.

(2) Singular solution. A naked singularity, not covered by event horizon, appears at $r=0$ (that is, at $w=w_i$), or at a finite positive $r,2u_0$ in (6). $N=2$ when $c\neq 0$, and $N=1$ when $c=0$.

(3) Pseudoregular solution. When $c>0$ and w increases from 1 ($r=\infty$), there is no curvature singularity at $w=\infty$ ($r=r_0$), yet the extended metric hits a singularity at w equal to w_1. $N=2$ in general. To uncover the global structure of this metric, it is convenient to use a new coordinate u equal to w^{-1}. The region around $r=r_0$ corresponds to that around $u=0$;

$$r-r_0 \sim K r_0 u^2, \quad K>0 \; .$$

The space defined by $1\geq u \geq {w_1}^{-1}$ has a curvature singularity at ${w_1}^{-1}$ corresponding to another infinity in the r coordinate. This metric has a wormholelike structure in the sense that the r space is doubly covered, but it actually contains naked singularities.

How do particles move in the metric thus obtained? For simplicity we shall restrict ourselves to radial trajectories for which coordinates ϕ, θ, and y^α remain constant. Standard methods yield a once-integrated equation for the geodesic:

$$e^{\omega-2\sigma}\left[\frac{dr}{dt}\right]^2 - e^{-\sigma} = -E \; . \tag{18}$$

The constant $E=0$ for massless particles and $E>0$ for massive particles. At large distances far away from the center, a Newtonian potential for a massive nonrelativistic particle,

$$\tfrac{1}{2}\sigma \sim -\frac{k}{r} \; ,$$

$$k=\pm\left[\frac{c}{2}-\frac{d-1}{d}\right] r_0 \, |1-w_1|^{(p-1)/2p} \times |1-w_2|^{(p+1)/2p} \; ,$$

follows by expanding ω and σ around zero in the asymptotic region. Note that at large distances the force can be either attractive or repulsive, depending on the sign of k.

Near singularities particles dynamically behave in bizarre fashions. Let us study in detail geodesics near $w \sim w_1^-$ in the pseudoregular solution. In this case $0<p<1$ and from (10)

$$w_1 - w \propto r^{-2p/(1-p)} \; .$$

From (14)—(16) one then determines leading behaviors of metric components. Combined with the geodesic equation (18), it is not difficult to show that infinitely strong forces, attractive or repulsive, act depending on the following cases: ∞ attraction for $E=0$ (massless), or $E>0$ (massive) and $c>2(d-1)/d$, ∞ repulsion for $E>0$ and $0<c<2(d-1)/d$. The fact that massive particles always experience strong repulsion near singularities is very intriguing and invites novel physical applications. Similar peculiar behaviors are observed near $r=0$ for $c>0$.

Our solution involves a complicated relation (10) between r and w, which is difficult to invert. In the special limit case of $|p|\to\infty$, however, one can explicitly invert this relation. From (11) and (12),

$$w_i = 1 \pm [d(d+1)^{-1}/2]^{1/2} \; ,$$

and the relation (10) yields, with $r_0=1$,

$$w-1=\pm\left[\frac{d}{2}(d+1)^{-1}\right]^{1/2}(1+r^2)^{-1/2} \; .$$

From (7), (8), and (5) one obtains, after some calculation,

$$b=b_0[(1+r^{-2})^{1/2}-r^{-1}]^{-\sqrt{2/d(d+1)}} \; , \tag{19a}$$

$$e^{\omega}=(1+r^{-2})^{-1} \; , \tag{19b}$$

$$e^{\sigma}=[(1+r^{-2})^{1/2}-r^{-1}]^{2\sqrt{2d/(d+1)}} \; . \tag{19c}$$

This adds, besides (6), to a list of new solutions for which metric components are explicitly written in terms of the original coordinate r.

Finally, let us discuss whether the electric gauge field term of $2\lambda_\alpha(r)dy^\alpha dt$ is permissible or not. Viewed as an effective four-dimensional field theory, the gauge potential λ_α is only allowed to be a function of the three-dimensional distance. In the presence of this mixed term the inverse of metric components, g^{AB}, contains nontrivial dependence on the internal coordinate y^α. For example,

$$g^{tt}=e^{-\sigma}\Sigma, \quad g^{t\alpha}=b^{-2}e^{-\sigma}\Sigma\tilde{g}^{\alpha\beta}\lambda_\beta \; ,$$

$$g^{\alpha\beta}=-b^{-2}\tilde{g}^{\alpha\beta}+b^{-4}e^{-\sigma}\Sigma\tilde{g}^{\alpha\gamma}\tilde{g}^{\beta\delta}\lambda_\gamma\lambda_\delta \; ,$$

$$\Sigma^{-1}=1+b^{-2}e^{-\sigma}\tilde{g}^{\alpha\beta}\lambda_\alpha\lambda_\beta \; ,$$

where $\tilde{g}^{\alpha\beta}(y)$ is the inverse of $\tilde{g}_{\alpha\beta}(y)$. Additional contributions to the Einstein equation caused by λ_α terms introduce nontrivial y dependence which is in general impossible to be obeyed. This means that one has $\lambda_\alpha=0$ as the only consistent solution of Einstein equations. The situation, however, changes if the internal metric $\tilde{g}_{\alpha\beta}$ is somewhat trivial and the space admits an isometry. For instance, when the compact space is a torus, $\tilde{g}_{\alpha\beta}(y)=\delta_{\alpha\beta}\times$const and extra electric contributions yield a sensible equation for the electric field λ_α. Indeed, in previous approaches[7] it was demonstrated that in the case of this toroidal compactification one can construct regular wormhole solutions in the presence of mixed electric terms. It appears that the loss of an isometry deprives the electric degree of freedom and renders a regular solution singular.

In summary, we have constructed and classified all static vacuum solutions with three-dimensional spherical symmetry when the extra compact manifold is Ricci flat and does not admit an isometry. The Schwarzschild solution is the only solution that possesses the event horizon. A more general class of solutions has naked singularities. These solutions are expected to have a great impact on formation of primordial black holes in the early Universe if the hole's mass is small enough.

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