How to measure the curvature of space-time

Ignazio Ciufolini
Center for Theoretical Physics, Center for Relativity and Center for Space Research, The University of Texas at Austin, Austin, Texas 78712

Marek Demianski
Department of Physics and Astronomy, Williams College, Williamstown, Massachusetts 01267
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We propose a new method of measuring the curvature of space-time using test particles. We show that to determine the curvature of space-time in vacuum it is necessary to use at least four test particles and in general at least six.

The Riemann curvature tensor plays an important role in relativistic theories of gravitation. The Riemann tensor $R^{\alpha}{}_{\beta\gamma\delta}$ is defined by the commutative relation of covariant derivatives. Let us denote by ∇_α the covariant derivative and by A_α an arbitrary (at least twice differentiable) covariant vector field. Then

$$(\nabla_\gamma\nabla_\delta-\nabla_\delta\nabla_\gamma)A_\beta=A_\alpha R^{\alpha}{}_{\beta\gamma\delta}\ . \tag{1}$$

When the metric tensor is covariantly constant, i.e., when $\nabla_\alpha g_{\beta\gamma}=0$, then the Riemann tensor has the following symmetry properties:

$$R^{\alpha}{}_{\beta(\gamma\delta)}=0,\quad R^{\alpha}{}_{[\beta\gamma\delta]}=0\ , \tag{2}$$

and $R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta}$, where $R_{\alpha\beta\gamma\delta}=g_{\alpha\sigma}R^{\sigma}{}_{\beta\gamma\delta}$.

In a four-dimensional space-time the Riemann tensor has 20 independent components. When the metric of space-time is subject to the Einstein equations in vacuum, $R_{\alpha\beta}=R^{\sigma}{}_{\alpha\sigma\beta}=0$, the number of independent components of the Riemann tensor is reduced to 10 and they form the Weyl tensor, which is defined by

$$C_{\alpha\beta\gamma\delta}=R_{\alpha\beta\gamma\delta}+g_{\alpha[\delta}R_{\gamma]\beta}+g_{\beta[\gamma}R_{\delta]\alpha}+\tfrac{1}{3}Rg_{\alpha[\gamma}g_{\delta]\beta}\ , \tag{3}$$

where $R=R_{\alpha\beta}g^{\alpha\beta}$.

Synge[1] in his classic book on the general theory of relativity describes a method of measuring independent components of the Riemann tensor. Synge calls his device a five-point curvature detector. The five-point curvature detector consists of a light source and four mirrors. By performing measurements of the distance between the source and the mirrors, and between mirrors one can determine the curvature of space-time.[2]

However, in order to measure all the independent components of the Riemann tensor with Synge's method, the experiment must be repeated several times with different orientations of the detector; equivalently—and when the space-time is not stationary—it is necessary to use several curvature detectors at the same time.

Here we would like to propose a different method of determining the curvature of a general space-time by measuring the relative acceleration of test particles.

The relative acceleration of two test particles moving on infinitesimally close geodesics is given by

$$\frac{D^2\delta x^\alpha}{ds^2}=R^{\alpha}{}_{\beta\gamma\delta}u^\beta\delta x^\gamma u^\delta\ , \tag{4}$$

where δx^α is the relative displacement vector of the two test particles and u^α is the tangent vector to the geodesics. Using a sufficiently large number of test particles and measuring relative accelerations one should be able to determine all 20 independent components of the Riemann tensor.[3] It is interesting to ask what is the minimal number of test particles necessary to determine all the independent components of the Riemann tensor. In order to investigate this problem we have to generalize the geodesic deviation equation to include test particles moving with different four-velocities. The geodesic deviation equation describing the relative acceleration of two test particles moving with arbitrary four-velocities on neighboring geodesics was recently derived by Ciufolini.[4]

If by Du^α we denote the difference $Du^\alpha\equiv u^\alpha_{2T}-u^\alpha_1$ (where u^α_{2T} is the parallel transported four-velocity of the second particle to the instantaneous position of the first particle), then

$$\frac{D}{ds}(Du^\alpha)=\tfrac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}u^\beta_2\delta x^\gamma u^\delta_1+\tfrac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}u^\beta_2\delta x^\gamma u^\delta_2\ . \tag{5}$$

This generalized geodesic deviation equation when written down in terms of ordinary derivatives and the Christoffel symbols assumes the form

$$\frac{d^2}{ds^2}(\delta x^\alpha)+\Gamma^\alpha_{\beta\mu,\rho}\delta x^\rho(u^\beta_1u^\mu_1+2u^\beta_1\Delta u^\mu+\Delta u^\beta\Delta u^\mu)+\Gamma^\alpha_{\beta\mu}\Delta u^\mu(2u^\beta_1+\Delta u^\beta)=0\ , \tag{6}$$

where $\Delta u^\alpha=u^\alpha_2-u^\alpha_1$. We will assume that the coordinate system is comoving with the first test particle and in the neighborhood of this geodesic[5,6] we will use the Fermi normal-coordinate system. In this coordinate system we have

$$\frac{d^2}{ds^2}(\delta x^\alpha)+\Gamma^\alpha_{\beta\mu,\rho}\delta x^\rho(u_1^\beta u_1^\mu+2u_1^\beta \Delta u^\mu+\Delta u^\beta \Delta u^\mu)=0\ . \tag{7}$$

In vacuum space-time ($R_{\alpha\beta}=0$), in order to measure the Weyl tensor let us take a test particle moving along a geodesic with a four-velocity $u_1^\alpha=(1,0,0,0)$. In the frame comoving with this particle we can freely choose orientation of spatial axes. In order to determine the ten independent components of the Riemann tensor in vacuum space-time let us introduce three other test particles. To simplify our considerations we arrange the relative positions of these three test particles so that initially u_1^α, δx_1^α, δx_2^α, and δx_3^α form an orthonormal basis.

Using the coordinate freedom and the freedom to specify the initial velocities, we assume that

$$\begin{aligned}
&\delta x^\alpha_{(1)}=(0,0,1,0),\quad \delta x^\alpha_{(2)}=(0,0,0,1)\ ,\\
&\delta x^\alpha_{(3)}=(0,1,0,0),\quad \Delta u_1^\alpha=(0,1,0,0)\ ,\\
&\Delta u_2^\alpha=(0,0,1,0),\quad \Delta u_3^\alpha=(0,0,0,1)\ .
\end{aligned} \tag{8}$$

In the Fermi normal-coordinate system, along the chosen geodesic we have[6]

$$\Gamma^\alpha_{\mu\nu,0}=0,\quad \Gamma^\alpha_{\mu 0,\nu}=R^\alpha{}_{\mu 0\nu}\ ,$$

and (9)

$$\Gamma^\mu_{ij,k}=\tfrac{1}{3}(R^\mu{}_{ijk}+R^\mu{}_{jik})\ .$$

The generalized geodesic deviation equation for the system of four test particles can be written down explicitly. We have

$$-\frac{d^2}{ds^2}\delta x^\alpha_{(1)}=R^\alpha{}_{002}+2R^\alpha{}_{102}+\tfrac{2}{3}R^\alpha{}_{112}\ , \tag{10}$$

$$-\frac{d^2}{ds^2}\delta x^\alpha_{(2)}=R^\alpha{}_{003}+2R^\alpha{}_{203}+\tfrac{2}{3}R^\alpha{}_{223}\ , \tag{11}$$

$$-\frac{d^2}{ds^2}\delta x^\alpha_{(3)}=R^\alpha{}_{001}+2R^\alpha{}_{301}+\tfrac{2}{3}R^\alpha{}_{331}\ . \tag{12}$$

Equations (10)—(12) lead to 10 independent equations. In fact the system (10)—(12) can be inverted to give explicitly the Riemann tensor components in terms of relative accelerations:[8]

$$\begin{aligned}
&R_{1020}=\delta\ddot{x}^{\,1}_{(1)};\quad R_{2030}=\delta\ddot{x}^{\,2}_{(2)}\ ,\\
&R_{2101}=3\delta\ddot{x}^{\,1}_{(1)}-\tfrac{3}{2}\delta\ddot{x}^{\,0}_{(1)},\quad R_{3202}=3\delta\ddot{x}^{\,2}_{(2)}-\tfrac{3}{2}\delta\ddot{x}^{\,0}_{(2)}\ ,\\
&R_{1030}=\delta\ddot{x}^{\,3}_{(3)},\quad R_{1303}=3\delta\ddot{x}^{\,3}_{(3)}-\tfrac{3}{2}\delta\ddot{x}^{\,0}_{(3)}\ ,\\
&R_{2031}=-\tfrac{5}{6}\delta\ddot{x}^{\,2}_{(2)}+\tfrac{1}{2}\delta\ddot{x}^{\,3}_{(1)},\quad R_{3012}=-\tfrac{5}{6}\delta\ddot{x}^{\,3}_{(3)}+\tfrac{1}{2}\delta\ddot{x}^{\,1}_{(2)}\ ,\\
&R_{2020}=-\tfrac{36}{19}\delta\ddot{x}^{\,2}_{(2)}+\tfrac{18}{19}\delta\ddot{x}^{\,0}_{(2)}-\tfrac{6}{19}\delta\ddot{x}^{\,1}_{(3)}\\
&\qquad\quad+\tfrac{54}{19}\delta\ddot{x}^{\,3}_{(3)}-\tfrac{27}{19}\delta\ddot{x}^{\,0}_{(3)}-\tfrac{9}{19}\delta\ddot{x}^{\,2}_{(1)}\ ,\\
&R_{1010}=\tfrac{90}{19}\delta\ddot{x}^{\,2}_{(2)}-\tfrac{45}{19}\delta\ddot{x}^{\,0}_{(2)}+\tfrac{15}{19}\delta\ddot{x}^{\,1}_{(3)}\\
&\qquad\quad+\tfrac{36}{19}\delta\ddot{x}^{\,3}_{(3)}-\tfrac{18}{19}\delta\ddot{x}^{\,0}_{(3)}+\tfrac{6}{19}\delta\ddot{x}^{\,2}_{(1)}\ .
\end{aligned} \tag{13}$$

In the general case to determine the 20 independent components of the Riemann tensor we have to add two more test particles, which without losing generality, we assume move in such a way that their relative velocities and relative positions are given by

$$\delta x_4^\alpha=(0,1,1,0),\quad \delta x_5^\alpha=(0,0,1,1)\ ,$$

$$\Delta u^\alpha=(a,1,1,0),\quad \Delta u^\alpha=(a,0,1,1)\ .$$

The corresponding relative accelerations are

$$\begin{aligned}
-\frac{d^2}{ds^2}\delta x_4^\alpha=&(1+2a+a^2)(R^\alpha{}_{001}+R^\alpha{}_{002})\\
&+(2+2a)(R^\alpha{}_{101}+R^\alpha{}_{102}+R^\alpha{}_{201}+R^\alpha{}_{202})\ ,
\end{aligned} \tag{14}$$

$$\begin{aligned}
-\frac{d^2}{ds^2}\delta x_5^\alpha=&(1+2a+a^2)(R^\alpha{}_{002}+R^\alpha{}_{003})\\
&+(2+2a)(R^\alpha{}_{202}+R^\alpha{}_{203}+R^\alpha{}_{302}+R^\alpha{}_{303})\ .
\end{aligned} \tag{15}$$

Now it is even more cumbersome to show that Eqs. (10), (11), (12), (14), and (15) give 20 independent relations, which determine all 20 independent components of the Riemann tensor.

We have shown that to measure the curvature of space-time in vacuum it is sufficient to use four test particles and in general space-times it is sufficient to use six test particles. It is easy to show that in vacuum, to determine the curvature, it is also necessary to use at least four test particles. With four test particles we have three independent geodesic deviation equations leading to 12 relations between the ten independent components of the Riemann tensor and the relative accelerations. In general space-times it is necessary to use at least six test particles.

Let us point out that it is possible to determine the curvature of space-time using the standard geodesic deviation equation. It turns out however that the minimal number of test particles which is required increases to 13 in general space-times and to six in vacuum.

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[1]J. L. Synge, *Relativity, the General Theory* (North-Holland, Amsterdam, 1960), p. 408.

[2]An experiment to measure the Riemann curvature tensor by measuring the rate of change of the Doppler shift between two freely falling observers is described in B. Bertotti, J. Math. Phys. **7**, 1349 (1966).

[3]F. A. E. Pirani, Acta Phys. Pol. **15**, 389 (1956).

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[5]E. Fermi, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **31**, 21 (1922); **31**, 51 (1922).

[6]F. K. Manase and C. W. Misner, J. Math. Phys. **4**, 735 (1963).

[7]For simplicity we assume here that particles 2, 3, and 4 are photons and s is the proper time of the observer comoving with the first test particle.

[8]The solution (13) was checked using the MACSYMA algebraic manipulation system.