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Generalized geodesic deviation equation

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We derive a generalized geodesic deviation equation valid for any two geodesics, with arbitrary tangent vectors, not necessarily parallel. This equation is valid in any neighborhood in which the change in the curvature is small.

In 1925, Levi-Civita[1] published the equation for the second covariant derivative of the distance between two infinitesimally close geodesics, on an arbitrary n-dimensional Riemannian manifold:

$$\frac{D^2(\delta x^\alpha)}{ds^2}=R^\alpha{}_{\beta\mu\nu}u^\beta\delta x^\mu u^\nu . \tag{1}$$

Here, δx^α is the infinitesimal connecting vector between the geodesics, $u^\mu=dx^\mu[s]/ds$ is the tangent vector to the geodesics and $R^\alpha{}_{\mu\nu\delta}$ is the Riemann curvature tensor. This equation, known as the geodesic deviation equation, generalizes the classical Jacobi equation for the distance between two geodesics on a two-dimensional surface:

$$\frac{d^2y}{d\sigma^2}+Ky=0 ,$$

where y is the distance between the geodesics, σ is the arc of the base geodesic and $K[\sigma]$ is the Gaussian curvature of the surface. The hypotheses under which the geodesic deviation equation (1) was derived are the following.

(1) The two curves are geodesics:

$$\frac{Du_1^\alpha}{d\tau}=0 \text{ and } \frac{Du_2^\alpha}{d\sigma}=0 ,$$

where τ,σ are affine parameters.

(2) The law of correspondence between the points of the two geodesics—that is, the definition of the connecting vector $\delta x^\alpha[\tau]$—is such that, if $d\tau$ is an infinitesimal arc on geodesic 1 and $d\sigma$ the arc on geodesic 2 corresponding to the connecting vectors $\delta x^\alpha[\tau]$ and $\delta x^\alpha[\tau+d\tau]$, we have

$$\frac{d\sigma}{d\tau}=1+\lambda, \text{ where } \frac{d\lambda}{d\tau}=0 .$$

(3) The geodesics are infinitesimally close in a neighborhood U:

$$x_2^\alpha[\sigma]=x_1^\alpha[\tau]+\delta x^\alpha[\tau]$$

where the relative change in the curvature is small:

$$\left|\frac{\mathscr{R}_{,\alpha}\delta x^\alpha}{\mathscr{R}}\right| \ll 1 .$$

and $\mathscr{R}^{-2}$ is approximately the typical magnitude of the components of the Riemann tensor.

(4) The difference between the tangent vectors to the two geodesics is infinitesimally small in the neighborhood U:

$$\left|\frac{|\delta u^\alpha|}{|u^\alpha|}\right| \ll 1 ,$$

where

$$\delta u^\alpha \equiv u_2^\alpha[\sigma]-u_1^\alpha[\tau] .$$

(5) Equation (1) is derived neglecting terms higher than the first order (ϵ^1) in δx^α and in δu^α.

In this paper we derive a generalized geodesic deviation equation dropping condition (4) and retaining only conditions (1), (2), (3), and (5). In other words, we derive a geodesic deviation equation valid in any neighborhood U in which the change in the curvature is small, but in which the difference between the tangent vectors u_1^α and u_2^α is not necessarily infinitesimally small, i.e., u_2^α and u_1^α are completely arbitrary. For simplicity,[2] we define the connecting vector δx^α as connecting points of equal arc lengths s on the two geodesics, that is, $d\tau=d\sigma=ds$ and s satisfies

$$u_1^\alpha[s]u_{1\alpha}[s]=-1, \text{ where } u_1^\alpha[s]\equiv\frac{dx_2^\alpha[s]}{ds}$$

and

$$u_2^\alpha[s]u_{2\alpha}[s]=-1, \text{ where } u_2^\alpha[s]\equiv\frac{dx_1^\alpha[s]}{ds} .$$

Physically, s is the proper time measured by two observers comoving with two test particles following geodesic motion.

The equation of geodesic 1 is

$$\frac{Du_1^\alpha}{ds}=\frac{du_1^\alpha}{ds}+\Gamma^\alpha_{\mu\nu}[x]u_1^\mu u_1^\nu=0 \tag{2}$$

and the geodesic equation (2) is

$$\frac{Du_2^\alpha}{ds}=\frac{du_2^\alpha}{ds}+\Gamma^\alpha_{\mu\nu}[x_1+\delta x]u_2^\mu u_2^\nu$$
$$=\frac{d^2}{ds^2}(x_1^\alpha+\delta x^\alpha)+\Gamma^\alpha_{\mu\nu}[x_1+\delta x]\frac{d}{ds}(x_1^\mu+\delta x^\mu)$$
$$\times\frac{d}{ds}(x_1^\nu+\delta x^\nu)=0 . \tag{3}$$

Now, we have

$$\frac{d}{ds}(\delta x^\mu[s]) \equiv \frac{d}{ds}(x_2^\mu[s]-x_1^\mu[s])$$
$$= u_2^\mu[s]-u_1^\mu[s] \equiv \Delta u^\mu[s]$$

and contrary to the classical case, condition (4), $\Delta u^\mu[s]$ is not an infinitesimal quantity. With this notation Eq. (3) can be rewritten, using a Taylor expansion to first order in δx^α, as

$$\frac{d^2}{ds^2}(x_1^\alpha)+\frac{d^2}{ds^2}(\delta x^\alpha)+(\Gamma^\alpha_{\mu\nu}+\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho)(u_1^\mu+\Delta u^\mu)(u_1^\nu+\Delta u^\nu)=0 . \quad (4)$$

Taking the difference between Eqs. (4) and (2) we find, to first order,

$$\frac{d^2(\delta x^\alpha)}{ds^2}+\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho u_1^\mu u_1^\nu+2\Gamma^\alpha_{\mu\nu}u_1^\mu\Delta u^\nu+\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho\Delta u^\mu\Delta u^\nu+2\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho u_1^\mu\Delta u^\nu+\Gamma^\alpha_{\mu\nu}\Delta u^\mu\Delta u^\nu=0 \quad (5)$$

and using the definition $(D/ds)v^\alpha=(d/ds)v^\alpha+\Gamma^\alpha_{\mu\nu}u^\mu v^\nu$ and the expression of the Riemann tensor in terms of the Christoffel symbols and their derivatives, we have, to first order,

$$\frac{D^2(\delta x^\alpha)}{ds^2}=R^\alpha{}_{\beta\mu\nu}u_1^\beta\delta x^\mu u_1^\nu-\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho\Delta u^\mu\Delta u^\nu$$
$$-2\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho u_1^\mu\Delta u^\nu-\Gamma^\alpha_{\mu\nu}\Delta u^\mu\Delta u^\nu . \quad (6)$$

As for the classical equation (1) we expect Eq. (6) to be covariant, i.e., to have the same form in any coordinate system. However, this equation does not appear to be covariant because the terms containing $\Gamma^\alpha_{\beta\mu,\rho}$ and $\Gamma^\alpha_{\beta\mu}$ on the right side do not form a tensor. This apparent contradiction is explained if we write the transformation law of the infinitesimal quantity δx^α induced by the coordinate transformation $x^\alpha=x^\alpha[x^{\alpha'}]$:

$$\delta x^\alpha \equiv x_2^\alpha-x_1^\alpha=x^\alpha[x^{\alpha'}]\,|_2-x^\alpha[x^{\alpha'}]\,|_1$$
$$=\partial^\alpha_{\mu'}\,|_1\delta x^{\mu'}+\tfrac{1}{2}\partial^\alpha_{\mu'\nu'}\,|_1\delta x^{\mu'}\delta x^{\nu'}$$
$$+\frac{1}{3!}\partial^\alpha_{\mu'\nu'\delta'}\,|_1\delta x^{\mu'}\delta x^{\nu'}\delta x^{\delta'}+\epsilon^4[\delta x] . \quad (7)$$

δx^α is clearly a vector to first order; however, when we act on it with the operator D^2/ds^2, $D^2\delta x^\alpha/ds^2$ is no longer a vector to first order because now $d\delta x^\alpha/ds=\Delta u^\alpha$ is not an infinitesimal quantity, and terms such as

$$\tfrac{1}{2}\partial^\alpha_{\mu'\nu'}\,|_1\frac{d}{ds}(\delta x^{\mu'})\delta x^{\nu'}$$

are now of the first order.

Using the transformation law (7) for δx^α to third order in $\delta x^{\alpha'}$, after tedious calculations, it is possible to check that Eq. (6) is really covariant. However the best way to prove the covariance of Eq. (6) is to rewrite it in a manifestly covariant or tensor form. We have

$$\frac{D^2}{ds^2}(\delta x^\alpha)\equiv\frac{D}{ds}\left[\frac{d}{ds}(\delta x^\alpha)+\Gamma^\alpha_{\mu\nu}u_1^\mu\delta x^\nu\right]$$
$$=\frac{D}{ds}(\Delta u^\alpha+\Gamma^\alpha_{\mu\nu}u_1^\mu\delta x^\nu) . \quad (8)$$

Here $\Delta u^\alpha \equiv u_2^\alpha-u_1^\alpha$ is clearly not a vector; nevertheless it can be written to first order as

$$\Delta u^\alpha=Du^\alpha-\Gamma^\alpha_{\mu\nu}u_2^\mu\delta x^\nu ,$$

where $Du^\alpha\equiv u_{2T}^\alpha-u_1^\alpha$ is the vector given by the difference of the vector u_2^α parallel transported on geodesic 1 and the vector u_1^α. However in the quantity on which the operator D/ds acts, Δu^α must be written to second order in δx^α:

$$\Delta u^\alpha=Du^\alpha+\int_2^1\Gamma^\alpha_{\mu\nu}u_{2T}^\mu\delta x^\nu$$
$$=Du^\alpha-\int_1^2\Gamma^\alpha_{\mu\nu}u_{2T}^\mu\delta x^\nu , \quad (9)$$

where the integral is calculated along a line joining correspondent points $x_1^\alpha(s)$ and $x_2^\alpha(s)$ on geodesics 1 and 2. With a Taylor expansion

$$\Delta u^\alpha=Du^\alpha-\Gamma^\alpha_{\mu\nu}u_{2T}^\mu\delta x^\nu$$
$$-\tfrac{1}{2}(\Gamma^\alpha_{\mu\nu}u_{2T}^\mu)_{,\rho}\delta x^\rho\delta x^\nu+\epsilon^3[\delta x] . \quad (10)$$

Because of the way in which u_{2T}^α was constructed—parallel transported on line 1 from line 2 along δx^α—we have

$$u^\mu_{2T;\rho}=\epsilon^1[\delta x], \text{ on geodesic 1} ,$$

or

$$u^\mu_{2T,\rho}=-\Gamma^\mu_{\sigma\rho}u^\sigma_{2T}+\epsilon^1[\delta x] .$$

Replacing this expression in (10) we have

$$\Delta u^\alpha=Du^\alpha-\Gamma^\alpha_{\mu\nu}u_{2T}^\mu\delta x^\nu-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu,\rho}u_{2T}^\mu\delta x^\rho\delta x^\nu$$
$$+\tfrac{1}{2}\Gamma^\alpha_{\mu\nu}\Gamma^\mu_{\sigma\rho}u^\sigma_{2T}\delta x^\rho\delta x^\nu+\epsilon^3[\delta x] \quad (11)$$

but

$$u_{2T}^\mu=u_2^\mu+\Gamma^\mu_{\sigma\rho}u_2^\sigma\delta x^\rho$$

so that to second order

$$\Delta u^\alpha=Du^\alpha-\Gamma^\alpha_{\mu\nu}u_2^\mu\delta x^\nu-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu,\rho}u_2^\mu\delta x^\rho\delta x^\nu$$
$$-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu}\Gamma^\mu_{\sigma\rho}u_2^\sigma\delta x^\rho\delta x^\nu+\epsilon^3[\delta x] . \quad (12)$$

Replacing this expression for Δu^α in Eq. (8) we have

$$\frac{D}{ds}(Du^\alpha-\Gamma^\alpha_{\mu\nu}\Delta u^\mu\delta x^\nu-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu}\Gamma^\mu_{\sigma\rho}u^\sigma_2\delta x^\rho\delta x^\nu-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu,\rho}u^\mu_2\delta x^\rho\delta x^\nu)$$

$$=\frac{D}{ds}(Du^\alpha)-\Gamma^\alpha_{\mu\nu,\rho}u^\rho_1 Du^\mu\delta x^\nu-\Gamma^\alpha_{\sigma\rho}\Gamma^\sigma_{\mu\nu}Du^\mu\delta x^\nu u^\rho_1-\Gamma^\alpha_{\mu\nu}\frac{d^2\delta x^\mu}{ds^2}\delta x^\nu-\Gamma^\alpha_{\mu\nu}\Delta u^\mu\Delta u^\nu$$

$$-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu}\Gamma^\mu_{\sigma\rho}u^\sigma_2(Du^\rho\delta x^\nu+\delta x^\rho Du^\nu)-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu,\rho}u^\mu_2(Du^\rho\delta x^\nu+\delta x^\rho Du^\nu)$$

$$=\frac{D}{ds}(Du^\alpha)-\Gamma^\alpha_{\mu\nu,\rho}u^\rho_1 Du^\mu\delta x^\nu-\Gamma^\alpha_{\mu\nu}(-2\Gamma^\mu_{\sigma\rho}u^\sigma_1 Du^\rho-\Gamma^\mu_{\sigma\rho}Du^\sigma Du^\rho)\delta x^\nu-\Gamma^\alpha_{\mu\nu}\Delta u^\mu\Delta u^\nu$$

$$-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu}\Gamma^\mu_{\sigma\rho}(u^\sigma_1+Du^\sigma)(Du^\rho\delta x^\nu+\delta x^\rho Du^\nu)-\tfrac{1}{2}\Gamma^\alpha_{\mu\nu,\rho}(u^\mu_1+Du^\mu)(Du^\rho\delta x^\nu+\delta x^\rho Du^\nu)-\Gamma^\alpha_{\sigma\rho}\Gamma^\sigma_{\mu\nu}Du^\mu\delta x^\nu u^\rho_1$$

$$=R^\alpha{}_{\beta\mu\nu}u^\beta_1\delta x^\mu u^\nu_1-\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho Du^\mu Du^\nu-2\Gamma^\alpha_{\mu\nu,\rho}\delta x^\rho u^\mu_1 Du^\nu-\Gamma^\alpha_{\mu\nu}\Delta u^\mu\Delta u^\nu \quad (13)$$

and finally grouping the various terms, we have to first order in δx^α the tensor equation

$$\frac{D}{ds}(Du^\alpha)=R^\alpha{}_{\beta\mu\nu}u^\beta_1\delta x^\mu u^\nu_1+R^\alpha{}_{\beta\mu\nu}Du^\beta\delta x^\mu u^\nu_1+\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}Du^\beta\delta x^\mu Du^\nu+\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}u^\beta_1\delta x^\mu Du^\nu\ . \quad (14)$$

Remembering that $u^\alpha_2=u^\alpha_1+\Delta u^\alpha=u^\alpha_1+Du^\alpha+\epsilon^1[\delta x]$ this equation can be rewritten in the form

$$\frac{D}{ds}(Du^\alpha)=R^\alpha{}_{\beta\mu\nu}u^\beta_2\delta x^\mu u^\nu_1+\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}u^\beta_2\delta x^\mu Du^\nu \quad (15)$$

or, finally,

$$\frac{D}{ds}(Du^\alpha)=(Du^\alpha)_{;\sigma}u^\sigma_1=\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}u^\beta_2\delta x^\mu u^\nu_1+\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}u^\beta_2\delta x^\mu u^\nu_2\ , \quad (16)$$

where $Du^\alpha\equiv u^\alpha_{2T}-u^\alpha_1;u^\alpha_{2T}\equiv[u^\alpha_2$ parallel transported on geodesic 1]; and u^α_2,u^α_1 are the tangent vectors to geodesics 2 and 1.

From the definition of Du^α and from $Du^\alpha_1/ds=0$, Eq. (16) can also be written in the form

$$\frac{D}{ds}(u^\alpha_{2T})=(Du^\alpha)_{;\sigma}u^\sigma_1=\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}u^\beta_{2T}\delta x^\mu u^\nu_1+\tfrac{1}{2}R^\alpha{}_{\beta\mu\nu}u^\beta_{2T}\delta x^\mu u^\nu_{2T}\ . \quad (17)$$

Equations (6), (16), and (17) are different versions of the generalized geodesic deviation equation. In particular Eq. (6) gives, in nontensorial form, the second covariant derivative of the separation δx^α between the two geodesics. The tensorial equation (16) gives the first covariant derivative of the vector field Du^α and the tensorial equation (17) gives the first covariant derivative, along geodesic 1, of the vector field $u^\alpha_{2T}(s)$, that is, the covariant derivative of the vector field $u^\alpha_2(s)$ after its parallel transport on line 1 along $\delta x^\alpha(s)$.

These equations are valid for any two geodesics with arbitrary tangent vectors, in any neighborhood U in which the relative change in the curvature is small compared to one: $|\mathscr{R}_{,\alpha}\delta x^\alpha/\mathscr{R}|\ll 1$, and they are valid for any value of s within the neighborhood U.

Under the hypothesis

$$\left|\frac{|\Delta u^\alpha|}{|u^\alpha|}\right|\ll 1$$

we get the classical geodesic deviation equation (1) immediately. Equations (16) and (17) are generalized in the following sense. The classical equation (1) is valid in any neighborhood in which δx^α is small and conditions (1), (2), (4), and (5) are satisfied.

The generalized equations (16) and (17) are however valid in any neighborhood in which δx^α is small and conditions (1), (2), and (5) only are satisfied. Of course this neighborhood is not necessarily a tubular neighborhood as it was in the case of Eq. (1), but is any neighborhood in which δx^α is small, in particular, a tubular neighborhood when δu^α is small. Equation (16) is a generalized version of Eq. (1). In fact from Eq. (16) we can get immediately Eq. (1) just by imposing the particular case in which Δu^α is small, so that Eq. (1) is a particular case of Eq. (16). On the contrary from Eq. (1) we cannot get Eq. (16).

In other words, Eq. (16) is valid either when the two geodesics are "parallel" [in which case the neighborhood of condition (3) is a tubular neighborhood], or in the case in which the two geodesics are not "parallel" [in which case the neighborhood of condition (3) is not tubular]. On the contrary, Eq. (1) is valid only in the case in which the geodesics are "parallel," in which case the neighborhood is tubular.

Physically, Eqs. (6), (16), and (17) give the relative acceleration of two observers comoving with two test particles with arbitrary four-velocities (their difference Δu^α need not necessarily be small) in an arbitrary gravitational field (in an arbitrary four-dimensional Lorentz manifold). The space-time need not necessarily satisfy the Einstein field equations as long as the test particles follow geodesic motion.

Equations (6), (16), and (17) were successfully applied[3] to the problem of finding the minimum number of test particles necessary to determine the Riemann curvature tensor measuring the relative acceleration between the particles in a gravitational field. It turns out that the minimum number of test particles can be drastically reduced by using Eq. (16) instead of the classical equation (1).

This number is reduced either (a) under the hypothesis of an arbitrary four-dimensional Lorentz manifold or (b) when we have an empty region of the space-time satisfying the Einstein equations: $R_{\alpha\beta}=0$ (the calculation of the Riemann tensor reduces then to the calculation of the Weyl tensor $C^{\alpha}{}_{\beta\mu\delta}$). For the solution of this problem and for a discussion on the possibility of measuring the Riemann curvature tensor measuring the relative acceleration of artificial satellites with "laser ranging" techniques, see Refs. 3 and 4.

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APPENDIX

In this section we give a simple geometrical application and a numerical test of the generalized geodesic deviation equation. Let us consider a two-dimensional sphere S^2 with unit radius and the two geodesics on S^2:

$$\phi=\text{const} \quad \text{and} \quad \theta=\text{const}$$

(see Fig. 1). Consider then, in a neighborhood of the intersection point 0, the connecting vector δx^{α} with components $\delta x^1 \equiv \epsilon$, $\delta x^2 \equiv -\epsilon$.

Let us now solve the problem of finding the value of the second-covariant derivative of δx^{α}: $D^2\delta x^{\alpha}/ds^2$ or the values of the components of the vector $D(Du^{\alpha})/ds$ (see Fig. 1). In order to calculate $D(Du^{\alpha})/ds$ we first observe that the only independent component of the Riemann curvature tensor is, in polar coordinates,

$$R^2{}_{121}=-1$$

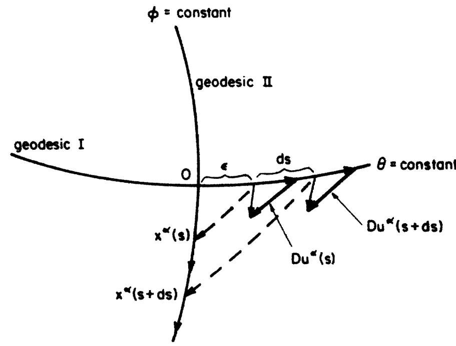

FIG. 1. Two geodesics on S^2 and the vector field Du^{α} of which Eq. (16) gives the first covariant derivative.

and using $g_{11}=r^2\equiv 1$ and $g_{22}=r^2\sin^2\theta\equiv\sin^2\theta$,

$$R^1{}_{212}=-\sin^2\theta \ .$$

The tangent vectors to the two geodesics are

$$u_{\rm I}^{\alpha}=(0,1) \quad \text{and} \quad u_{\rm II}^{\alpha}=(1,0) \ .$$

Therefore from Eq. (16) we have, immediately,

$$\frac{D}{ds}(Du^1)=\tfrac{1}{2}R^1{}_{212}u_{\rm II}^2\delta x^1 u_{\rm I}^2 + \tfrac{1}{2}R^1{}_{221}u_{\rm II}^2\delta x^2 u_{\rm II}^1=0 \qquad \text{(A1)}$$

and

$$\frac{D}{ds}(Du^2)=\tfrac{1}{2}R^2{}_{112}u_{\rm II}^1\delta x^1 u_{\rm I}^2 + \tfrac{1}{2}R^2{}_{121}u_{\rm II}^1\delta x^2 u_{\rm II}^1=\epsilon \ . \qquad \text{(A2)}$$

$D^2\delta x^{\alpha}/ds^2$ can be obtained similarly.

Equivalently the same result was obtained after a lengthier calculation: parallel transporting to geodesic I the vector $u_{\rm II}^{\alpha}$ from the two points $x^{\alpha}(s)$ and $x^{\alpha}(s+ds)$ on geodesic II, evaluating the vector $Du^{\alpha}\equiv u_{{\rm II}_T}^{\alpha}-u_{\rm I}^{\alpha}$ and finally calculating the ratio $[Du^{\alpha}(s+ds)-Du^{\alpha}(s)]/ds$ in the limit of ds very small.

[1]T. Levi-Civita, *Lezioni di calcolo differenziale assoluto* (Stock, Rome, 1925); English translation *The Absolute Differential Calculus* (Blackie, London, 1926), p. 209; see also: T. Levi-Civita, Math. Ann. **97**, 291 (1927).

[2]For simplicity we do not consider here null geodesics.

[3]I. Ciufolini and M. Demianski, following paper, Phys. Rev. D **34**, 1018 (1986).

[4]I. Ciufolini (unpublished); see also I. Ciufolini, Phys. Rev. Lett. **56**, 278 (1986).