Brans-Dicke theory in general space-time with torsion

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The Brans-Dicke theory in the general space-time endowed with torsion is investigated. Since the gradient of the scalar field as well as the intrinsic spin generate the torsion field, the interaction term of the spin-scalar field appears in the wave equation. The equations of motion are satisfied with the conservation laws.

As a modified theory of Einstein's theory of general relativity (GR), the Brans-Dicke (BD) theory[1] is based on the viewpoint of the scalar-tensor theory, say, that gravitation may be produced by a scalar field as well as the metric tensor field.[2] It is well known that the BD theory, which assumes that the reciprocal value of the gravitational "constant" G is the varying scalar field ϕ, is more relevant to Mach's principle than the absolute property of space.[3]

When the gravitational field is strong, for example, for a neutron star or in the phenomenon of gravitational collapse, the scalar field ϕ may have some effect on stellar configurations through a local modification of the gravitational constant by the matter-energy distribution of the star.[4] Moreover, most matter composing the stellar objects may have intrinsic spins which may play a dynamical role in influencing the geometry of the space-time.[5] It may be less practical to maintain the Riemannian manifold as the background space-time for those cases. The space-time should rather be a general space-time endowed with the torsion field which is taken to geometrize the intrinsic spin density of matter,[6] i.e., the Einstein-Cartan (EC) or Riemann-Cartan (RC) manifold. Allowing the space-time to carry the torsion field, it successfully incorporates the intrinsic spin. Thus if we want to investigate the problem of a strong gravitational field with spin, the background space-time of the scalar field ϕ should be extended to the EC manifold.

To investigate the BD theory in general space-time with torsion, we introduce the scalar field in a rather *ad hoc* manner into the EC manifold, like the usual BD theory on a Riemannian manifold. While Dunn[7] geometrized the scalar field on equal terms with the metric tensor field, it is in contrast with BD theory. He introduced the scalar field by defining a linear connection with nonvanishing torsion. However, we assume that the torsion field is generated by the intrinsic spin and the scalar field on equal footing. Thus we now introduce the scalar field ϕ as the fundamental field variable with tetrad field $e_a{}^{\mu}$ and spin connection field $\omega_{ab}{}^{\mu}$ independently in the gravitational action[8]

$$I_G=\int d^4x\sqrt{-g}\,(-\phi R+\gamma R^2+\omega\phi^{,\mu}\phi_{,\mu}/\phi)\ , \tag{1}$$

where ω and γ are arbitrary dimensionless parameters. The scalar curvature R is a function of $e_a{}^{\mu}$ and $\omega_{ab}{}^{\mu}$ and, just as in the EC theory,[9] is given by

$$\begin{aligned}R&=e_a{}^{\mu}e_b{}^{\nu}R^{ab}{}_{\mu\nu}\\&=e_a{}^{\mu}e_b{}^{\nu}(\omega^{ab}{}_{\mu,\nu}-\omega^{ab}{}_{\nu,\mu}+\omega^{a}{}_{c\nu}\omega^{cb}{}_{\mu}-\omega^{a}{}_{c\mu}\omega^{cb}{}_{\nu})\ .\end{aligned} \tag{2}$$

The metric is given by $g_{\mu\nu}=\eta_{ab}e^a{}_{\mu}e^b{}_{\nu}$ with $\eta_{ab}=\mathrm{diag}(-1,1,1,1)$, the Minkowski metric, and $g=\det(g_{\mu\nu})$. The torsion field is defined by

$$F^a{}_{\mu\nu}=e^a{}_{\mu,\nu}-e^a{}_{\nu,\mu}+\omega^a{}_{c\nu}e^c{}_{\mu}-\omega^a{}_{c\mu}e^c{}_{\nu}\ , \tag{3}$$

where the comma means partial differentiation. The third term is added in order to bring about a simple wave equation for ϕ. It is also invariant under the Poincaré gauge transformation, since any covariant derivative of a scalar field is identical with its partial derivative.

The mass action is given by

$$I_M=16\pi\int d^4x\sqrt{-g}\,L_M(e_a{}^{\mu},\omega_{ab}{}^{\mu};\chi_i)\ . \tag{4}$$

Here L_M is a matter Lagrangian density which is a function of $e_a{}^{\mu}$, $\omega_{ab}{}^{\mu}$, and source fields χ_i. The total action then becomes $I_T=I_G+I_M$.

The field equations for each variable can be obtained by varying the total action independently with respect to ϕ, $e_a{}^{\mu}$, and $\omega_{ab}{}^{\mu}$. The field equation for ϕ is

$$R-\omega\phi^{,\mu}\phi_{,\mu}/\phi^2+2\omega\Delta\phi/\phi-2\omega F^{\mu}{}_{\lambda\mu}\phi^{,\lambda}/\phi=0\ , \tag{5}$$

where the d'Alembertian Δ in the EC manifold is defined by

$$\Delta\phi=\phi^{,\mu}{}_{;\mu}=\phi^{,\mu}{}_{,\mu}+\Gamma^{\mu}{}_{\lambda\mu}\phi^{\lambda}\ . \tag{6}$$

Here $\Gamma^{\mu}{}_{\lambda\mu}$ denotes the linear affine connection which is written as

$$\Gamma^{\mu}{}_{\lambda\nu}=\{{}^{\mu}_{\lambda\nu}\}-K^{\mu}{}_{\lambda\nu}\ , \tag{7}$$

where the quantity $\{{}^{\mu}_{\lambda\nu}\}$, the Christoffel symbol computed from the metric tensor $g_{\mu\nu}$, is familiar and established by GR, and the contortion tensor $K^{\mu}{}_{\lambda\nu}$ is given by

$$K^{\mu}{}_{\lambda\nu}=\tfrac{1}{2}(-F^{\mu}{}_{\lambda\nu}+F_{\lambda}{}^{\mu}{}_{\nu}+F_{\nu}{}^{\mu}{}_{\lambda}) \tag{8}$$

relating $F^{\mu}{}_{\lambda\nu}$ with

$$F^{\mu}{}_{\lambda\nu}=\Gamma^{\mu}{}_{\lambda\nu}-\Gamma^{\mu}{}_{\nu\lambda}=e_a{}^{\mu}F^a{}_{\lambda\nu}\ . \tag{9}$$

The extra fourth term of Eq. (5) is added to the case of the usual BD theory as the interaction of the scalar field with torsion, since the d'Alembertian is defined by Eq. (6). If, however, the d'Alembertian is defined as the covarient derivative with the Christoffel symbol as in GR, then the fourth term is canceled for the wave equation to be the same form as that of the usual BD theory. The parameter ω couples the scalar field with torsion in the same range of the coupling with the metric.

The field equation for $e_a{}^\mu$ is

$$(-\phi+2\gamma R)G^\lambda{}_\mu+\tfrac{1}{2}\gamma\delta^\lambda_\mu R^2=-8\pi T^\lambda{}_\mu +\omega(\tfrac{1}{2}\phi_{,\tau}\phi^{,\tau}\delta^\lambda_\mu-\phi_{,\mu}\phi^{,\lambda})/\phi\ , \tag{10}$$

where the Einstein tensor $G^\lambda{}_\mu=R^\lambda{}_\mu-\frac{1}{2}\delta^\lambda_\mu R$ is asymmetric generally and the canonical energy-momentum tensor $T^\lambda{}_\mu$ is defined by

$$T^\lambda{}_\mu=\frac{1}{\sqrt{-g}}e_a{}^\lambda\frac{\delta}{\delta e_a{}^\mu}(\sqrt{-g}L_M)\ . \tag{11}$$

The canonical energy-momentum tensor $T_{\mu\nu}$ is also asymmetric, while the metric energy-momentum tensor that is obtained by varying the matter Lagrangian with respect to the metric tensor is symmetric. Thus $T_{\mu\nu}$ can be separated into two parts, a symmetric part (metric energy-momentum tensor $t_{\mu\nu}$) and an antisymmetric part (energy-momentum tensor due to spin, $\tau_{\mu\nu}$):

$$T_{\mu\nu}=t_{\mu\nu}+\tau_{\mu\nu}\ . \tag{12}$$

The field equation (10) is an extension of Einstein's equation under gravitation caused by the torsion field and the scalar field. In the usual BD theory, Einstein's field equation contains the term $(\phi_{,\mu;\nu}-g_{\mu\nu}\Delta\phi)$ which results from the presence of the second derivatives of the metric tensor in R, while Eq. (10) does not contain the term because of the independence of the tetrad and the spin connection as the fundamental variables in our case. The second term on the right-hand side is the energy-momentum tensor of the massless scalar field.

The field equation for $\omega^{ab}{}_\mu$ is given by

$$(-\phi+2\gamma R)(F^\mu{}_{\alpha\beta}+\delta^\mu_\beta F^\lambda{}_{\lambda\alpha}-\delta^\mu_\alpha F^\lambda{}_{\lambda\beta}) =-8\pi\sigma^\mu{}_{\alpha\beta}+\delta^\mu_\alpha(-\phi_{,\beta}+2\gamma R_{,\beta}) -\delta^\mu_\beta(-\phi_{,\alpha}+2\gamma R_{,\alpha})\ , \tag{13}$$

where $\sigma^\mu{}_{\alpha\beta}$ is the spin angular momentum tensor defined by

$$\sigma^\mu{}_{\alpha\beta}=\frac{1}{\sqrt{-g}}(e^a{}_\alpha e^b{}_\beta-e^a{}_\beta e^b{}_\alpha)\frac{\delta}{\delta\omega^{ab}{}_\mu}(\sqrt{-g}L_M) \tag{14}$$

and is related to $\tau_{\mu\nu}$ by the modified divergence $\nabla^*_\mu A^\mu=A^\mu{}_{;\mu}-F^\lambda{}_{\mu\lambda}A^\mu$:

$$\tau_{\mu\nu}=\nabla^*_\alpha(\sigma^\alpha{}_{\mu\nu}-\sigma_{\mu\nu}{}^\alpha-\sigma_{\nu\mu}{}^\alpha)\ . \tag{15}$$

By contracting the μ and β indices Eq. (13) is changed into

$$(-\phi+2\gamma R)F^\lambda{}_{\alpha\lambda}=4\pi\sigma^\lambda{}_{\alpha\lambda}+3(-\phi_{,\alpha}+2\gamma R_{,\alpha})/2\ . \tag{16}$$

If we substitute Eq. (16) into the field equation (13) again, it becomes

$$(-\phi+2\gamma R)F^\mu{}_{\alpha\beta}=-8\pi\Sigma^\mu{}_{\alpha\beta}+\tfrac{1}{2}\delta^\mu_\beta(-\phi_{,\alpha}+2\gamma R_{,\alpha}) -\tfrac{1}{2}\delta^\mu_\alpha(-\phi_{,\beta}+2\gamma R_{,\beta})\ , \tag{17}$$

where $\Sigma^\mu{}_{\alpha\beta}$ is given by

$$\Sigma^\mu{}_{\alpha\beta}=\sigma^\mu{}_{\alpha\beta}+\tfrac{1}{2}(\delta^\mu_\alpha\sigma^\mu{}_{\beta\lambda}-\delta^\mu_\beta\sigma^\lambda{}_{\alpha\lambda})\ . \tag{18}$$

Note that the fluctuations in the scalar curvature R and the scalar function ϕ can act as sources of the torsion field. Even if the gravitational Lagrangian density is given by $L_G=\phi R(\gamma=0)$, that is to say, in the case of no fluctuation in R, the torsion field may be generated by a nonspin term, the gradient of the scalar field. Thus in the absence of spins the torsion does not vanish; it can propagate with the scalar field. The form of the torsion field produced by the scalar field in Eq. (17) is similar to that by the tlaplon[10] which requires minimal coupling and gauge invariance for a complex scalar field. The similarity is caused by the Lagrangian (1) containing the kinetic term of ϕ. But they thought that only the tlaplon determines the torsion field, it is the same idea as Dunn's. The fact that the terms containing the parameter ω are absent in the field equation (13) means that the effects of ϕ and spin on the torsion field would be of the same order of magnitude.

By contracting the λ and μ indices in Eq. (10) the γ-containing terms are canceled and then

$$R=-8\pi T/\phi+\omega\phi_{,\tau}\phi^{,\tau}/\phi^2\ . \tag{19}$$

Substituting this into the field equation (5) gives

$$\Delta\phi=4\pi T/\omega+F^\mu{}_{\lambda\mu}\phi^{,\lambda} \tag{20}$$

while the denominator of the first term on the right-hand side in the usual BD theory is $\omega+\frac{3}{2}$. It is due to the fact that Eq. (10) does not contain the term of the second derivatives of ϕ. Here T denotes the trace of $T_{\mu\nu}$. For traceless T, for example, the vacuum or electromagnetic case, the torsion-scalar coupling term determines the wave equation of ϕ. Of course, since the torsion is described by a function of the spin and scalar, it would be the spin-scalar and scalar-scalar self-interactions. For the $\gamma=0$ case,

$$\Delta\phi=4\pi T/\omega+8\pi\Sigma^\mu{}_{\lambda\mu}\phi^{,\lambda}/\phi+3\phi_{,\lambda}\phi^{,\lambda}/2\phi \tag{21}$$

and if the magnitude of ω is large enough to make $R=0$ in Eq. (19), then the right-hand side of Eq. (21) becomes

$$8\pi\Sigma^\mu{}_{\lambda\mu}\phi^{,\lambda}/\phi+2\phi^{,\lambda}\phi_{,\lambda}/\phi\ .$$

The field equations (5), (10), and (13) are consistent with the conservation laws. Since the background spacetime of the field equations is the EC manifold, the geometrical identity which relates the curvature tensor with torsion and the Bianchi identity of EC theory[9,11] are still required to show the consistency. For convenience, consider the case of the vanishing value of γ (Ref. 12). Taking the modified divergence ∇^*_λ of both sides of Eq. (10) with $\gamma=0$,

$$\nabla^*_\lambda T^\lambda{}_\mu-F^\rho{}_{\lambda\mu}T^\lambda{}_\rho+\tfrac{1}{2}\sigma^\rho{}_{\alpha\lambda}R^{\alpha\lambda}{}_{\rho\mu}=0\ . \tag{22}$$

It is determined with the help of Eq. (5) and the identity

$$\nabla^*_\lambda G^\lambda{}_\mu=F^\rho{}_{\lambda\mu}G^\lambda{}_\rho-\tfrac{1}{2}T^\rho{}_{\alpha\lambda}R^{\alpha\lambda}{}_{\rho\mu}\ . \tag{23}$$

Also if we take the modified divergence of Eq. (13), then

$$\nabla^*_\mu \sigma^\mu{}_{\alpha\beta} - (T_{\alpha\beta} - T_{\beta\alpha}) = 0 \tag{24}$$

with the help of the identity of EC theory

$$\nabla^*_\alpha T^\alpha{}_{\beta\lambda} = G_{\beta\lambda} - G_{\lambda\beta} \, , \tag{25}$$

where $T^\lambda{}_{\mu\nu} = F^\lambda{}_{\mu\nu} + \delta^\lambda_\nu F^\rho{}_{\rho\mu} - \delta^\lambda_\mu F^\rho{}_{\rho\nu}$ is the modified torsion tensor. Both Eqs. (22) and (24) are exactly the conservation laws of energy-momentum and spin angular momentum respectively. The scalar field does not break the conservation, and it has the same base as the usual BD theory whose field equations require the conservation law of GR (Ref. 4). However, even if the torsion field vanishes, Eq. (22) does not become the usual conservation law of GR (or BD theory) since the torsion does not consist of spin only. There remains the coupling term of the scalar field and curvature.

Recently Sáez[13] did similar work on the basis of Møller's tetrad (MT) theory[14] using a different action $[R + F^2 + \phi F + (\partial\phi)^2]$. Since MT theory considered the tetrad field alone as the fundamental variable unlike our theory, the torsion field is represented in terms of the tetrad field without the spin connection field, and treating the spinless matter the spin angular momentum tensor does not appear. Therefore, the conservation law of Sáez's theory does not have the form of EC theory, Eqs. (22) and (24), but the form of GR.

It is applicable to any cosmological or astrophysical models to indicate the physical importance of the scalar field and its interaction with spin through the field equations (5), (10), and (13). Like the general BD theory[15] our calculation can be extended to the case of ϕ dependence of the parameter ω and/or γ.

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[1]C. H. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[2]E. R. Harrison, Phys. Rev. D **6**, 2077 (1971).
[3]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[4]T. Singh and L. N. Rai, Gen. Relativ. Gravit. **15**. 875 (1983).
[5]A. R. Prasanna, Phys. Rev. D **11**, 2083 (1975).
[6]F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. Nester, Rev. Mod. Phys. **48**, 393 (1976).
[7]K. A. Dunn, J. Math. Phys. **15**, 2229 (1974).
[8]This particular choice of the action is discussed in R. Rauch and H. T. Nieh, Phys. Rev. D **24**, 2029 (1983) with the Gauss-Bonnet theorem.
[9]R. T. Rauch, Phys. Rev. D **26**, 931 (1982).
[10]S. Hojman, M. Rosenbaum, M. P. Ryan, and L. C. Sheply, Phys. Rev. D **17**, 3141 (1978); G. Germàn, Class. Quantum Gravit. **2**, 455 (1985).
[11]F. W. Hehl and B. K. Datta, J. Math Phys. **12**, 1334 (1971).
[12]Although γ is not zero, the situation gives the same results.
[13]D. Sáez, Phys. Rev. D **27**, 2839 (1983); Gen. Relativ. Gravit. **17**, 579 (1985); D. Sáez and J. V. Arnau, J. Math. Phys. **26**, 2902 (1985).
[14]C. Møller, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **39**, 1 (1978).
[15]K. Nordtvedt, Jr., Astrophys. J. **161**, 1059 (1970); V. U. M. Rao and D. K. Reddy, Gen. Relativ. Gravit. **14**, 1017 (1982).