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Fields in nonaffine bundles. I. The general bitensorially gauge-covariant differentiation procedure

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The standard covariant differentiation procedure for fields in vector bundles is generalized so as to be applicable to fields in general nonaffine bundles in which the fibers may have an arbitrary nonlinear structure. In addition to the usual requirement that the base space should be flat or endowed with its own linear connection Γ, and that there should be an ordinary gauge connection A on the bundle, it is necessary to require also that there should be an intrinsic, bundle-group-invariant connection $\hat{\Gamma}$ on the fiber space. The procedure is based on the use of an appropriate primary-field-(i.e., section-) dependent connector ω that is constructed in terms of the natural fiber-tangent-vector realization $\mathbf{A}$ of the gauge connection. The application to gauged-harmonic mappings will be described in the following article.

I. INTRODUCTION

Since at least the time of Clerk-Maxwell, or even earlier, nearly all the most successful mathematical models for the description of the physical world at a fundamental (and often also at a higher) level have been essentially based on the conceptual framework of local *field theory.* The fields in question, whose behavior is governed by local differential equations of usually not higher than second order, are generally interpretable—at a classical level—as sections of fiber bundles over some appropriate base space (which might, for example, represent ordinary four-dimensional space-time, or some higher-dimensional extension or lower-dimensional subspace or quotient space thereof).

In the most familiar and well-developed examples (including Yang-Mills theory), although the *theories* themselves may be *nonlinear* (in the sense that the field equations contain coupling terms of quadratic or higher order) the actual *fields* are *intrinsically linear* in so much as they belong to bundles whose fibers are *flat.* In the simplest cases the fiber space is actually *vectorial,* and even in the case of gauge-connection fields (e.g., of Yang-Mills type) the fiber space still has a well-defined *affine* structure, although there is no longer any preferred origin. For fields in such essentially linear (i.e., affinely fibered) bundles, the standard procedure for the construction of the relevant gauge-covariant derivatives (in terms of which the field equations are expressed) provided an appropriate gauge-connection field is available, is widely known and familiar (see, e.g., Choquet-Bruhat, Morette-DeWitt, and Bleck-Dillard[1]).

The main purpose of the present work is to describe how the standard machinery for gauge-covariant differentiation can be generalized so as to be applicable to fields that are *intrinsically* nonlinear, in the sense of being sections of *nonaffinely* fibered bundles. Such nonaffine fields (as exemplified by nonlinear σ models) have attracted an increasing amount of interest in recent years.

The usual procedure for ordinary vector bundles needs the provision only of a *gauge connection* A, in addition to the requirement that the *base space* should either be flat or at least provided with an *ordinary linear connection* Γ. The natural generalization to be described here requires also that the (curved) *fiber-space should be provided with its own linear connection* $\hat{\Gamma}$.

In the following article we shall describe the application of the general-purpose formalism set up below to the particular case where the fibers have a Riemannian connection induced automatically by the Lagrangian for the natural minimally gauge-coupled generalization of the class of harmonic mappings that was discussed by Misner.[2] These gauged-harmonic mappings will include as a special case with the gauge-coupled generalization of the nonlinear σ model with fully homogeneous-symmetric fibers that was recently described by the present author.[3]

II. THE CONCEPTS OF BITENSORIAL DIFFERENTIATION AND CONNECTOR FIELDS

One of the essential guidelines whose observance qualifies a theoretical treatment for description as *geometric* is the requirement that one should work as far as possible in terms of entities that are *invariant* in the sense of being independent of any arbitrarily chosen system of reference that one might wish to introduce for the sake of explicitness at some intermediate stage in the treatment. However, the strictest observance of this precept risks giving a treatment that either needs to be unduly abstract as the price of being elegant or else that needs unwieldy mathematical machinery as the price of being concrete. For this reason most theoretical physicists do not insist on the exclusive use of entities that are strictly invariant, but as a compromise prefer nevertheless to work as far as possible in terms of entities that are at least *covariant* in the sense of being subject to simply prescribed rules of variation when the relevant reference system is altered. One of the simplest and most convenient examples is that of

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quantities represented in terms of sets of components whose rules of variation are of *tensorial* type in the sense of being expressible in terms of appropriate contractions with relevant coordinate transformation matrices. In the specific context of general field theories we shall be particularly concerned with entities whose covariance is of *bitensorial* type insomuch as they involve two independent matrices expressing independent coordinate changes on the base and fiber spaces, respectively.

As a basic starting point let us consider the case of a field V of simple vectorial type, meaning that its components V^A undergo a change of the form

$$V^A \mapsto G^A{}_B V^B \tag{2.1}$$

under the effect of a fiber-coordinate transformation characterized by the matrix $G^A{}_B$. Suppose that we simultaneously carry out a coordinate transformation

$$x^\mu \mapsto y^\mu(x) \tag{2.2}$$

on the base space $\mathscr{M}$ over which the field V is defined, thereby determining a corresponding base-space transformation matrix given by

$$Q^\mu{}_\nu = \partial_\nu y^\mu \ , \tag{2.3}$$

where we have introduced the abbreviation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

for partial coordinate differentiation of a field over the base space. Then the components

$$D_\mu V^A = \partial_\mu V^A + \omega_\mu{}^A{}_B V^B \tag{2.4}$$

will qualify for description as those of a *covariant* or more explicitly *bitensorial* derivative if they transform according to the corresponding matrix contraction rule as expressed by

$$D_\mu V^A \mapsto Q^{-1\nu}{}_\mu G^A{}_B D_\nu V^B \ . \tag{2.5}$$

It is evident that the bitensorial covariance property (2.5) will hold if and only if the components $\omega_\mu{}^A{}_B$ have a covariance property of a rather more complicated nature, namely,

$$\omega_\mu{}^A{}_B \mapsto Q^{-1\nu}{}_\mu G^{-1C}{}_B (G^A{}_D \omega_\nu{}^D{}_C - \partial_\nu G^A{}_C) \ . \tag{2.6}$$

This will be of bitensorial form only if the base gradient $\partial_\mu G^A{}_B$ of the fiber-coordinate transformation matrix $G^A{}_B$ happens to vanish (which will not, in general, be the case for the examples we wish to consider).

We shall use the term *connector* to denote any field ω having components $\omega_\mu{}^A{}_B$ specified by one (covariant) base-coordinate index and two (mixed) fiber-coordinate indices and transforming according to the rule (2.6). A connector can be considered as a particular kind of biaffinitor, using the term *affinitor* as an abbreviation for affine tensor to denote quantities whose components transform according to a rule that generalizes the ordinary kind of tensorial transformation law by allowing for the presence of an inhomogeneous additive term [having the form $Q^{-1\nu}{}_\mu G^{-1C}{}_B \partial_\nu G^A{}_C$ in the example (2.6)] over and above the usual homogeneous-linear multiplicative term [having the form $Q^{-1\nu}{}_\mu G^{-1C}{}_B G^A{}_D \omega_\nu{}^P{}_C$ in the example (2.6)].

Insomuch as it is subject to the bi-aftensorial transformation rule (2.6), a connector ω can be interpreted as a *genuine field* in the sense that it is a *section* in an appropriately constructed fiber bundle $\mathscr{C}'$ over the base space $\mathscr{M}$, the bundle being of *affine* (rather than ordinary vectorial) kind in the sense that (as well as being subject to the usual group of homogeneous base-coordinate transformations specified by the matrices $Q^\nu{}_\mu$) the fibers of the bundle are subject to an action of the associated *inhomogeneous adjoint* group $\mathscr{G}'^\dagger$ of linear transformations generated by uniform translations and by the adjoint action of the matrices $G^A{}_B$.

We use the term connector (as distinct from connection) for the purpose of emphasizing this interpretation of ω as a genuine (biaffinitorial) field in the sense of being a section in the relevant (affine) fiber bundle, $\mathscr{C}'$, as characterized by an action of the corresponding inhomogeneous adjoint group $\mathscr{G}'^\dagger$. Of course, such an ω can also be given a more traditional mathematical interpretation as a *connection,* meaning an algebra-valued form on an appropriate *principal* fiber bundle $\mathscr{P}'$ (see, e.g., Choquet-Bruhat *et al.*[1] or Carter[4]) associated with the corresponding vector bundle $\mathscr{V}'$ say containing V, as characterized by the left action on itself of the subgroup $\mathscr{G}'$ of $\mathscr{G}'^\dagger$ generated directly by the multiplicative action of the allowed transformation matrices $G^A{}_B$.

The need for rather more care than usual in the interpretation of ω—either as a connector *in* $\mathscr{C}'$ or as a connection *on* $\mathscr{P}'$—arises in situations where our primary purpose is to deal with differentiation of a primary field Φ having values in a *nonaffinely* fibered bundle $\mathscr{B}$ subject to the provision of an ordinary gauge field A with respect to the bundle group $\mathscr{G}$ of $\mathscr{B}$. Such a gauge field A will be interpretable in the traditional way as a connection on the directly associated principle bundle $\mathscr{P}$ of $\mathscr{B}$ (with nonlinear fibers having the form of $\mathscr{G}$ itself, subject to its own left action) and it will also be interpretable as a connector field in an appropriate affine bundle $\mathscr{C}$ subject to the action of the inhomogeneous adjoint group $\mathscr{G}^\dagger$ associated with $\mathscr{G}$ (as well as base coordinate transformations) on the fibers. This *primary* principal bundle $\mathscr{P}$, and the *indirectly* associated *primary* connector bundle $\mathscr{C}$, containing the gauge section A will in the general case (for a nonlinearly fibered primary bundle $\mathscr{B}$) be distinct from what we shall refer to as the *derived* principal bundle $\mathscr{P}'$ and the *derived* connector bundle $\mathscr{C}'$ containing what we shall refer to as the *derived* connector ω (for which the corresponding groups $\mathscr{G}'$ and $\mathscr{G}'^\dagger$ may be larger than $\mathscr{G}$ and $\mathscr{G}^\dagger$). These derived bundles and the connector ω are not (in the nonlinear case) determined in advance by the corresponding primary bundles and the gauge field A, but are specified as functions of a choice of the section Φ in $\mathscr{B}$. Any such section immediately determines a corresponding bundle $\mathscr{V}'$ of ordinary vectorial type (over the same base $\mathscr{M}$) whose elements V are just the tangent vectors to the fibers of $\mathscr{B}$ at the section Φ. This section-dependent vector bundle $\mathscr{V}'$ is the basic building block from which, in conjunction with the ordinary cotangent bundle over $\mathscr{M}$, one can proceed to construct the corresponding tensorially associated vector bundles that are needed to contain bitensorial derivatives of various orders. The derived bundles $\mathscr{P}'$ or $\mathscr{C}'$ that are needed for the definition—as, respectively, a connection or a section—of the connector ω that will be required (for the explicit specification of such bitensorial covariant derivatives) will be, respectively, the directly associated principal bundle $\mathscr{P}'$ of $\mathscr{V}'$ or the corresponding affine bundle $\mathscr{C}'$ as characterized by the bundle group $\mathscr{G}'$ of $\mathscr{V}'$ and of its (inhomogeneous adjoint) extension $\mathscr{G}'^{\dagger}$ acting on $\mathscr{C}'$.

The possibility that the derived bundle group $\mathscr{G}'$ may be considerably larger than the primary bundle group $\mathscr{G}$ results from the fact that it arises from (in general, base-position-dependent) fiber coordinate transformations

$$X^A(X,x) \mapsto G^A(X,x) \tag{2.7}$$

for $X \in \mathscr{X}$, $x \in \mathscr{M}$, where $\mathscr{M}$ is the base space and $\mathscr{X}$ the fiber space of $\mathscr{B}$, that arise *not only* from the action of the primary gauge group $\mathscr{G}$ but *also* from the group of nonlinear transformations between coordinates of the different patches that may be needed to cover the fiber space $\mathscr{X}$ when it has itself a nonlinear manifold structure. In terms of the original fiber coordinates X^A, the elements of $\mathscr{G}'$ will be represented by matrices of the form

$$G^A{}_B(X,x) = G^A{}_{,B} \tag{2.8}$$

as evaluated on the chosen section

$$X = \Phi(x) \ , \tag{2.9}$$

where a comma denotes partial differentiation so that, in particular, the total space gradient components (with respect to the local coordinates x^μ and X^A) by that appear in the connector transformation formula (2.6) will be given explicitly by

$$\partial_\mu G^A{}_B = G^A{}_{,B,\mu} + G^A{}_{,B,C}\Phi^C{}_{,\mu} \ , \tag{2.10}$$

where

$$\Phi^C(x) = X^C(\Phi(x)) \ .$$

In the following sections we shall describe the natural procedure for explicitly constructing a well-defined section-dependent connector field ω obeying the rule (2.7), in terms of a previously given primary gauge field A and of ordinary linear connections Γ and $\hat{\Gamma}$ on the base and fiber spaces $\mathscr{M}$ and $\mathscr{X}$, respectively. Before doing so we remark that because such a section-dependent connector can be interpreted as an ordinary connection on the artificially constructed (section-dependent) vector bundle $\mathscr{V}'$, it follows that ω will automatically have the usual properties that are familiar from the standard theory of fixed (section-independent) connections. In particular, the connector ω will determine a corresponding well-defined (but section-dependent) *bitensorial* curvature field Ω (over the section) according to a formula of the familiar form

$$\Omega_{\mu\nu}{}^A{}_B = 2\partial_{[\mu}\omega_{\nu]}{}^A{}_B + 2\omega_{[\mu}{}^A{}_{|C|}\omega_{\nu]}{}^C{}_B \tag{2.11}$$

(where square brackets denote antisymmetrization) and this field will satisfy a Bianchi identity of the familiar form

$$\partial_{[\mu}\Omega_{\nu\rho]}{}^A{}_B = \Omega_{[\mu\nu}{}^A{}_{|C|}\omega_{\rho]}{}^C{}_B - \omega_{[\mu}{}^A{}_{|C|}\Omega_{\nu\rho]}{}^C{}_B \ . \tag{2.12}$$

III. BITENSORIAL DIFFERENTIATION IN THE ABSENCE OF A GAUGE TRANSFORMATION

Before dealing with the general situation (where there is a nontrivial gauge group $\mathscr{G}$) let us start by dealing with the comparatively simple case for which the fundamental bundle $\mathscr{B}$ under consideration is endowed with a trivial *direct product* structure $\mathscr{X} \times \mathscr{M}$ where $\mathscr{M}$ is the base space, with local coordinates x^μ, and $\mathscr{X}$ is the fiber space, with local coordinates X^A. The imposition of such a direct product structure is equivalent to the specification of an *integrable connection* on the bundle. Its presence enables us to restrict our attention for the time being to fiber-coordinate transformations

$$X^A(X) \mapsto Y^A(X) \tag{3.1}$$

that are *independent* of base-space position, i.e., such that

$$Y^A{}_{,\mu} = 0 \tag{3.2}$$

unlike the more general transformations of the form (2.7) that were mentioned in the introduction and to which we shall return in the next section.

In such integrable cases the procedure described by Misner[2] for the Riemannian case can be taken over directly provided that the base $\mathscr{M}$ and the fiber $\mathscr{X}$ each has its own linear connection. An ordinary linear connection on $\mathscr{M}$ will be specified by a corresponding purely *affinitorial* (as opposed to the more general biaffinitorial) connector field Γ with mixed components $\Gamma_\mu{}^\nu{}_\rho$, which can be used, e.g., for a simple tangent vector v with components v^μ, to specify the covariant variation dv with components $(dv)^\mu$ associated with an infinitesimal component variation $d(v^\mu)$ in conjunction with a base displacement dx^μ by the formula

$$(dv)^\mu = d(v^\mu) + \Gamma_\nu{}^\mu \rho v^\rho dx^\nu \tag{3.3}$$

so that *if* v is defined as a field over $\mathscr{M}$ there will be a corresponding *tensorial* covariant differentiation operation ∇ whose effect is given by

$$\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma_\nu{}^\mu{}_\rho v^\rho \ . \tag{3.4}$$

In an exactly analogous manner, the connection on the fiber space $\mathscr{X}$ will be specified by another such connector field $\hat{\Gamma}$ with components $\hat{\Gamma}_A{}^B{}_C$ whose use can be illustrated as before by the case of a simple fiber-tangent vector V say, with components V^A, whose covariant variation dV will be given in terms of corresponding component variations $d(V^A)$ and fiber displacement components dX^A by

$$(dV)^A = d(V^A) + \Gamma_B{}^A{}_C V^C dX^B \tag{3.5}$$

so that if we were concerned with a field defined over the fiber space we would have a corresponding fiber-covariant differentiation operation whose effect would be given by

$$\hat{\nabla}_B V^A = V^A{}_{,B} + \hat{\Gamma}_B{}^A{}_C V^C \ . \tag{3.6}$$

What we are actually most interested in is situations where the entities such as V under consideration are specified as fields not over the fiber space $\mathscr{X}$ but over the base space $\mathscr{M}$ or to be more explicit where they are specified as fields on some section $\Phi(x)$ of the bundle $\mathscr{B}$ with fibers

$\mathscr{X}$ over $\mathscr{M}$. In such a situation we shall be concerned with variations for which the fiber displacement dX^A appearing in (3.5) will be determined (via the section Φ) by a base-space displacement dx^μ in the form

$$dX^A = (\nabla_\mu \Phi^A) dx^\mu , \tag{3.7}$$

where the *bitensorial gradient components* are defined by

$$\nabla_\mu \Phi^A = \partial_\mu X^A(\Phi(x)) . \tag{3.8}$$

There will thus be a corresponding *bitensorial* generalization of the covariant differentiation operation ∇, whose effect on a fiber-tangent field V at the section Φ over $\mathscr{M}$ will be given by

$$\nabla_\mu V^A = \partial_\mu V^A + \Gamma_\mu{}^A{}_B V^B , \tag{3.9}$$

where the (biaffinitorial) *section-dependent* connector components $\Gamma_\mu{}^A{}_B$ are given by

$$\Gamma_\mu{}^A{}_B = X^C{}_\mu \hat{\Gamma}_C{}^A{}_B , \tag{3.10}$$

where we introduce the abbreviation

$$X^C{}_\mu = \nabla_\mu \Phi^C . \tag{3.11}$$

For the components of the (gradient) projection bitensor defined by the section Φ according to (3.8).

Once the connectors $\Gamma_\mu{}^\rho{}_\nu$ and $\Gamma_\mu{}^A{}_B$ are available one can proceed at once in the usual way to write down the covariant (bitensorial) derivatives of bitensors of arbitrary orders by including a connector term of the appropriate kind for each index. The lowest- (zero) order example is the case of the covariant derivative of the section Φ itself, as given by (3.8), for which no connector term is needed at all.

As one would expect, commuting the order of covariant differentiation operations brings to light torsion and curvature effects resulting from torsion and curvature in $\mathscr{M}$ and $\mathscr{X}$. The ordinary base-space torsion and curvature tensors are defined by the usual expressions

$$\Theta_{\mu\nu}{}^\rho = 2\Gamma_{[\mu}{}^\rho{}_{\nu]} \tag{3.12}$$

and

$$R_{\mu\nu}{}^\rho{}_\sigma = 2\partial_{[\mu}\Gamma_{\nu]}{}^\rho{}_\sigma + 2\Gamma_{[\mu|}{}^\rho{}_\tau \Gamma_{|\nu]}{}^\tau{}_\sigma \tag{3.13}$$

while the analogous fiber torsion and curvature are defined similarly by

$$\hat{\Theta}_{AB}{}^C = 2\hat{\Gamma}_{[A}{}^C{}_{B]} \tag{3.14}$$

and

$$\hat{R}_{AB}{}^C{}_D = 2\hat{\Gamma}_{[B|}{}^C{}_{D,|A]} + 2\hat{\Gamma}_{[A|}{}^C{}_E \hat{\Gamma}_{|B]}{}^E{}_D . \tag{3.15}$$

In terms of these, the effect of commuting two covariant differentiations at the zero-order levels, i.e., when acting on the primary section Φ itself, will be given by

$$2\nabla_{[\mu}\nabla_{\nu]}\Phi = X^C{}_\mu X^D{}_\nu \hat{\Theta}_{CD}{}^A - \Theta_{\mu\nu}{}^\rho X^A{}_\rho . \tag{3.16}$$

At the first-order level, when acting on a base-space vector field we shall obtain an expression of the usual form

$$2\nabla_{[\mu}\nabla_{\nu]} v^\rho = R_{\mu\nu}{}^\rho{}_\sigma v^\sigma - \Theta_{\mu\nu}{}^\sigma \nabla_\sigma v^\rho \tag{3.17}$$

and when acting on a fiber-tangent vector field we shall obtain

$$2\nabla_{[\mu}\nabla_{\nu]} V^A = R_{\mu\nu}{}^A{}_B V^B - \Theta_{\mu\nu}{}^\sigma \nabla_\sigma V^A , \tag{3.18}$$

where the (bitensorial) *section-dependent* base projection of the fiber curvature is given by

$$R_{\mu\nu}{}^A{}_B = X^C{}_\mu X^D{}_\nu \hat{R}_{CD}{}^A{}_B . \tag{3.19}$$

Having seen how the specification of the linear connections Γ and $\hat{\Gamma}$ on the base and fiber spaces $\mathscr{M}$ and $\mathscr{X}$, respectively, will automatically determine a natural bitensorial differentiation operator in the trivial case of a bundle with a direct-product structure (or equivalently with an integrable bundle connection) we now want to consider the generalization of this procedure to the case in which one has a nonintegrable bundle connection A in a bundle whose fibers are subject to a nontrivial action of an automorphism group $\mathscr{G}$. As a preliminary to setting up the actual gauge-covariant differentiation procedure in Sec. V, we shall first describe the appropriate *primary realization* of the gauge algebra in terms of vertical vector fields on the primary bundle $\mathscr{B}$.

IV. THE PRIMARY FIBER-TANGENT VECTOR REALIZATION OF A GAUGE FIELD

Instead of supposing that the primary bundle has a preferred (or indeed any) direct-product structure (as was done in the previous section) we now consider the more general situation in which the bundle fibers are horizontally related only by a nonintegrable connection A subject to a *nonintegrable* action of an automorphism group $\mathscr{G}$ with Lie algebra *a*.

In this more general case, the bundle will still have a simple (albeit no longer uniquely preferred) *local-direct-product* structure $\mathscr{X}\times\mathscr{N}$, i.e., what is traditionally known as a *gauge*, above each (sufficiently small) neighborhood $\mathscr{N}$ in the base space $\mathscr{M}$: in terms of local coordinates X^A on some local fiber-space patch $\mathscr{U}$ and x^μ on the base-space patch $\mathscr{N}$ the bundle points represented in terms of the product structure by the pair (X,x) with $X\in\mathscr{U}$, $x\in\mathscr{N}\subset\mathscr{M}$, will be specified by a corresponding set of local gauge coordinates $\{X^A,x^\mu\}$. However,[1,4] it is now no longer required that any particular such gauge (i.e., direct-product) structure be preserved when the local bundle patches are fitted together. Since a given gauge over $\mathscr{N}$ will specify an isomorphism mapping $J(x)$ of the fiber over each point $x\in\mathscr{N}$ into the abstract fiber space $\mathscr{X}$, and any other gauge over an overlapping neighborhood $\mathscr{N}'$ will specify an analogous isomorphism $J'(x)$ for $x\in\mathscr{N}'$, it follows that there will be a corresponding isomorphism of the form

$$\mathscr{X}\xrightarrow{G}\mathscr{X},\quad X\mapsto GX \tag{4.1}$$

of the fiber space onto itself, determined for any $x\in\mathscr{N}\cap\mathscr{N}'$ by the product mapping $G=J'\circ J^{-1}$. If the second (new) gauge is represented in an overlapping patch by the local gauge-coordinates $\{Y^A,x^\mu\}$ where the Y^A are coordinates on some local path $\mathscr{U}'\subset\mathscr{X}$, then there will be a relation of the general form (2.7) specifying the *new* gauge coordinates $\{G^A, x^\mu\}$ of a point represented by the pair (X,x) with local coordinates $\{X^A(X), x^\mu(x)\}$ in the *original* gauge by a prescription of the form

$$G^A(X,x) = Y^A(G(x)X) . \tag{4.2}$$

If the elements G of the gauge group $\mathcal{G}$ were allowed to consist of arbitrary diffeomorphisms then the form (4.2) would allow quite arbitrary transformation $X^A \to G^A$. Even when $\mathcal{G}$ is restricted by the requirement that the transformations G should be *isomorphisms*, i.e., that they must preserve any intrinsic structure (such as the connection $\hat{\Gamma}$) given on the fiber space $\mathcal{X}$, it will still be possible to choose the transformation $X \mapsto G$ arbitrarily on any single given fiber by adjusting the new coordinates Y^A on $\mathcal{X}$ accordingly, but the *variation* of G^A as a function of the base position x will not be arbitrary but will be restricted by the criteria for G to remain an isomorphism.

As usual, the bundle connection over $\mathcal{M}$ will be determined by the specification of a corresponding connector one-form $\mathbb{A}_\mu$ with (gauge-patch-dependent) valves in the Lie algebra $\mathcal{A}$, and there will be a corresponding (gauge-patch-independent) Lie-algebra-valued two-form

$$\mathbb{F}_{\mu\nu} = 2\partial_{[\mu}\mathbb{A}_{\nu]} + 2\mathbb{A}_{[\mu}\mathbb{A}_{\nu]} \tag{4.3}$$

satisfying a Bianchi identity of the form

$$\partial_{[\mu}\mathbb{F}_{\nu\rho]} + [\mathbb{A}_{[\mu}, \mathbb{F}_{\nu\rho]}] = 0 \tag{4.4}$$

and vanishing if and only if the connection is integrable.

In terms of a representation of the form

$$\mathbb{A}_\mu = A_\mu{}^\alpha \mathbf{a}_\alpha \tag{4.5}$$

in terms of a fixed basis $\mathbf{a}_\alpha \in a$ $(\alpha = 1, \ldots, m)$, of the Lie algebra with structure constants specified by

$$[\mathbf{a}_\alpha, \mathbf{a}_\beta] = \mathscr{C}_{\alpha\beta}{}^\gamma \mathbf{a}_\gamma \tag{4.6}$$

the corresponding curvature two-form components in the corresponding representation

$$\mathbb{F}_{\mu\nu} = F_{\mu\nu}{}^\alpha \mathbf{a}_\alpha \tag{4.7}$$

have the explicit expression

$$F_{\mu\nu}{}^\alpha = 2\partial_{[\mu}A_{\nu]}{}^\alpha + \mathscr{C}_{\beta\gamma}{}^\alpha A_\mu{}^\beta A_\nu{}^\gamma . \tag{4.8}$$

In the simple vector bundles that are most commonly used in physics, the algebra $\mathcal{A}$ can conveniently be represented in terms of matrices, but in the general nonlinear case it is more useful to think of the algebra instead as represented by the *vector fields* that generate the corresponding infinitesimal diffeomorphisms on the primary fiber space $\mathcal{X}$ under consideration. The basic function of the gauge field A is to determine, for any infinitesimal base displacement dx, a corresponding algebra element

$$\mathbf{a} = \mathbb{A}_\mu dx^\mu \tag{4.9}$$

which will be realized by a corresponding fiber vector field $\mathbf{a}$ with components

$$a^A = A_\mu{}^A dx^\mu . \tag{4.10}$$

The role of this vector field $\mathbf{a}$ is to specify the (infinitesimal) deflection between the horizontal projection, as determined by the local-direct-product structure associated with the gauge patch under consideration (in effect the local coordinates $\{X^A, x^\mu\}$), between fibers over base points differing by the infinitesimal base displacement dx, and the corresponding horizontal projection as determined by the connection.

The specification of a connection in this way enables one to define a gauge-covariant vertical displacement $\bar{d}X$ between neighboring points on neighboring fibers, as determined with respect to the horizontality specified by the connection. The components of the covariant vertical displacement may be evaluated as the difference,

$$\bar{d}X^A = dX^A - d_a X^A \tag{4.11}$$

between the vertical deviation dX^A determined by the local coordinates (i.e., by the local-product structure of the gauge path) and the vertical deviation

$$d_a X^A = -a^A \tag{4.12}$$

between horizontality with respect to the connection and horizontality as determined by the local coordinates. Hence if we are considering a section Φ, substitution of the corresponding coordinate displacement formula

$$dX^A = X^A{}_\mu dx^\mu \tag{4.13}$$

into (4.11) gives the expression

$$\bar{d}X^A = (X^A{}_\mu + A^A{}_\mu)dx^\mu \tag{4.14}$$

for the corresponding covariant displacement components where $X^A{}_\mu$ are the tangent projection components associated with the section Φ as given by (3.11). (See Fig. 1.)

It is evident that the quantity $\bar{d}X^A$ constructed in this way will be vectorially covariant under the effect of a *general* (base-position-dependent) fiber-coordinate transformation of the form (2.7), which gives

$$dX^A \mapsto G^A{}_B dX^B + G^A{}_{,\mu} dx^\mu \tag{4.15}$$

provided that the gauge-connection field A undergoes the corresponding transformation, which will be given explicitly for the vector realization $\mathbf{A}$ by

$$A_\mu{}^A \mapsto G^A{}_B A_\mu{}^B - G^A{}_{,\mu} \tag{4.16}$$

since the inhomogeneous terms will cancel so as to give the purely vectorial covariant rule

$$\bar{d}X \mapsto G^A{}_B \bar{d}X^B . \tag{4.17}$$

By a (rather longer) calculation one can also verify that (2.7) and (4.16) also imply an analogous purely vectorial covariance rule

$$F_{\mu\nu}{}^A \mapsto G^A{}_B F_{\mu\nu}{}^B \tag{4.18}$$

for the components of the vector realization $\mathbf{F}$ of the gauge curvature, as defined by

$$F_{\mu\nu}{}^A = F_{\mu\nu}{}^\alpha a_\alpha{}^A , \tag{4.19}$$

where $a_\alpha{}^A$ are the components of the vector realization $\mathbf{a}_\alpha$ of $\mathbf{a}_\alpha$, and the base components $F_{\mu\nu}{}^\alpha$ of the gauge curvature are specified by (4.6).

Since the algebra commutator relations will be realized

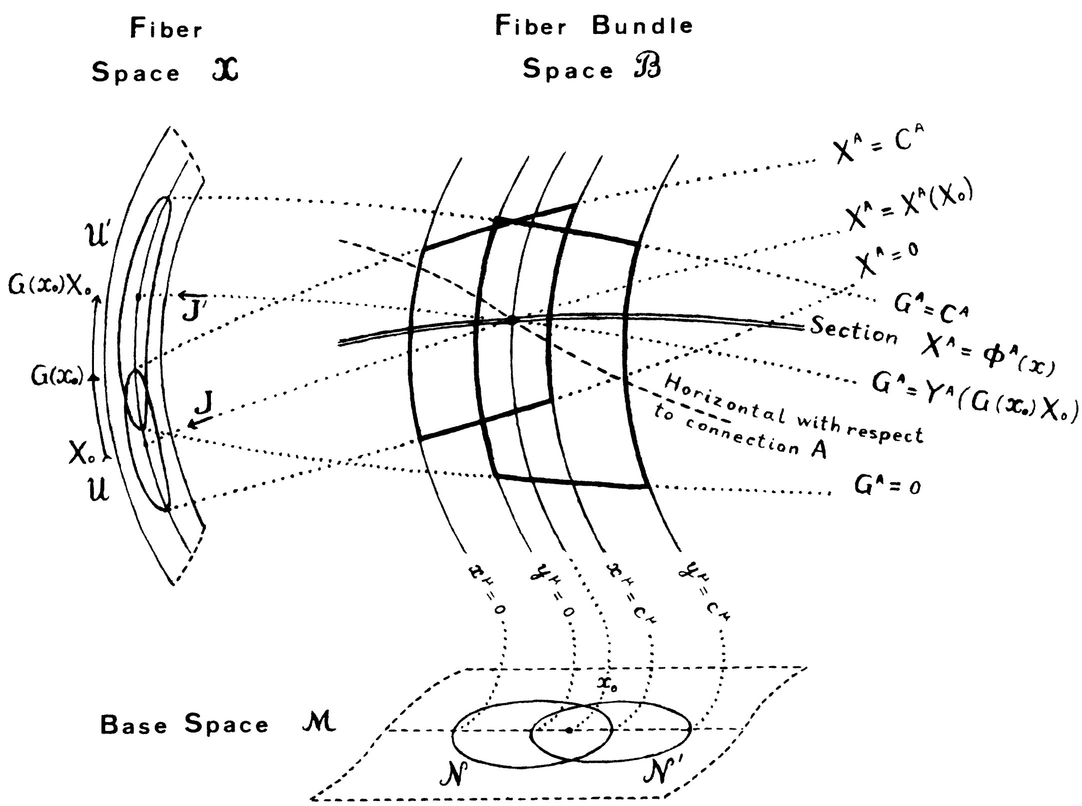

FIG. 1. Schematic representation showing (two-dimensional subspaces of) a (curved) fiber space $\mathscr{X}$, a base space $\mathscr{M}$, and a bundle $\mathscr{B}$ with fiber $\mathscr{X}$ over $\mathscr{M}$, indicating the relationships between the various local coordinate patches mentioned in the text, and showing the distinction between the original gauge projection J determined (for $X \in \mathscr{X}$, $x \in \mathscr{M}$) in the form $J(X,x) = X$ by the local product structure corresponding to some initially given gauge over a neighborhood $\mathscr{N} \subset \mathscr{M}$ and a *new* gauge projection J' over $\mathscr{N}'$ given in terms of the *initial* local product structure over the overlap region $\mathscr{N} \cap \mathscr{N}'$ by $J'(X,x) = G(x)X$. (The positions of the patches $\mathscr{N}, \mathscr{N}'$, in $\mathscr{M}$ and $\mathscr{U}, \mathscr{U}'$ in $\mathscr{X}$ are indicated by pairs of points representing the coordinate origin and some other arbitrary constant values denoted by the letter c.)

by the Lie differentiation commutator of the vector fields on $\mathscr{X}$, the structure relations (4.6) will be realized concretely by

$$2a_{[\alpha|}{}^{A}{}_{,B}\,a^{B}{}_{|\beta]} = \mathscr{C}_{\alpha\beta}{}^{\gamma}a_{\gamma}{}^{A} . \tag{4.20}$$

Hence by substitution in (4.8) we obtain an explicit, Lie-algebra-basis-independent, expression for the components $F_{\mu\nu}{}^{A}$ of the realization **F** of the gauge curvature, namely,

$$F_{\mu\nu}{}^{A} = 2A_{[\nu}{}^{A}{}_{,\mu]} + 2A_{[\nu}{}^{B}A_{\mu]}{}^{A}{}_{,B} . \tag{4.21}$$

It is an essentially straightforward exercise in partial differentiation to verify directly this *fundamental primary bundle realization* of the gauge curvature does indeed undergo a transformation of the vectorial form (4.18) under the effect of a general gauge patch transformation as specified by (2.7) and (4.16). This establishes that the base-space two-form valued vertical (i.e., fiber-space tangent) vector field **F** specified by (4.18) is *globally well defined* over the whole of the primary bundle $\mathscr{B}$, unlike the base space one-form valued vertical vector field **A** which is gauge-patch dependent.

This property of existing as a field over the whole of the primary bundle $\mathscr{B}$ distinguishes the primary gauge curvature realization **F** from the other bitensorial entities introduced in the previous sections, which were defined only over some particular section Φ in $\mathscr{B}$. In detailing with entities such as **F** and **A** which are defined over the whole of the fibers and not just at the section Φ, one must take care to distinguish the partial component derivatives indicated by a comma, from the total base-space gradient components for the field over $\mathscr{M}$ that would be determined by the section Φ. Thus, although we could use the expressions $\partial_\mu A_\nu{}^{\alpha}$ and $A_\nu{}^{\alpha}{}_{,\mu}$ interchangably in (4.8), it is important to notice that $A_\nu{}^{A}{}_{,\mu}$ is not the same as $\partial_\mu A_\nu{}^{A}$ in the algebra-basis-independent expression (4.21), the distinction being specified as a function of the section Φ by

$$\partial_\mu A_\nu{}^{A} = A_\nu{}^{A}{}_{,\mu} + A_\nu{}^{A}{}_{,B}X^{B}{}_{\mu} . \tag{4.22}$$

By paying due attention to this distinction it will be possible to work with an explicit, but Lie-algebra-basis-independent notation scheme throughout the remainder of this work, thereby avoiding any further reference to such cumbersome paraphenalia as the structure constants.

Up to this point we have made no reference to any specific properties required of the gauge group $\mathscr{G}$: the analysis in the present section would be valid for transformations $X^A \mapsto G^A$ resulting from the general action of the *entire (infinite parameter) group of diffeomorphisms* on the fiber space $\mathscr{X}$. However, for the purpose of constructing a gauge-covariant differentiation formalism, as will be done in the section that follows, it will be necessary to restrict ourselves to situations for which $\mathscr{G}$ is included in the at most finite-dimensional diffeomorphism subgroup leaving the chosen fiber-space connection $\hat{\Gamma}$ invariant.

V. GAUGE-COVARIANT BITENSORIAL DIFFERENTIATION

It is immediately evident from the work of the previous section that for any section $\Phi(x)$ in the primary bundle $\mathscr{B}$ the gauge connection A will determine a well-defined covariant derivative field $D\Phi$ over the base-space $\mathscr{M}$, whose components can be read out from the expression

$$\bar{d}X^A = \Phi^A{}_{|\mu} dx^\mu \tag{5.1}$$

for the covariant vertical displacement $\bar{d}X$ resulting from a base-space displacement dx, where we have introduced a heavy *bar* notation convention

$$D_\mu \Phi^A = \Phi^A{}_{|\mu} \tag{5.2}$$

for gauge-covariant differentiation. Recalling our previous abbreviation

$$\partial_\mu \Phi^A = X^A{}_\mu \tag{5.3}$$

we immediately obtain the compact expression

$$\Phi^A{}_{|\mu} = X^A{}_\mu + A_\mu{}^A \tag{5.4}$$

for the bitensorial derivative components $\Phi^A{}_{|\mu}$ by substituting (4.14) in (5.1).

This lowest-order differentiation procedure obviously does not depend on the specification of any intrinsic structures on the fiber $\mathscr{X}$ or base $\mathscr{M}$ of $\mathscr{B}$. However, in order to go on (analogously to the work of Sec. III) to the construction of higher-order bitensorial derivatives, the reintroduction of the fiber connection $\hat{\Gamma}$ on $\mathscr{X}$ and, more routinely, of the base connection Γ on $\mathscr{M}$ will evidently be necessary.

Before continuing, we now make the usual supposition that the gauge group $\mathscr{G}$ acting effectively on the primary bundle $\mathscr{B}$ should be restricted to consist only of fiber isomorphisms, i.e., that it should leave invariant all relevant structure on the fiber space $\mathscr{X}$ in which the primary field is evaluated. As a minimal requirement we must at least demand that the transformation group $\mathscr{G}$ should preserve the only structure that has been introduced so far on $\mathscr{X}$, namely, the indispensible fiber connection $\hat{\Gamma}$; i.e., the gauge transformations specified by (2.7) and (4.2) must be restricted so as not to violate the essential property

$$\hat{\Gamma}_A{}^B{}_{C,\mu} = 0 \tag{5.5}$$

characterizing any allowable local gauge coordinate system $\{X^A, x^\mu\}$. In order to express the corresponding restriction on the gauge algebra, it is convenient, following Yano,[5] to introduce an abbreviation, which we shall indicate by a subscript colon, to indicate a covariant derivative of a vector field that differs from the usual one in that the connection is inserted the *wrong way round.* Thus, for the particular case of the gauge vector one-form $\mathbf{A}$ we introduce a corresponding gauge tensor one-form $\mathbf{A}_:$ defined by

$$A_\mu{}^A{}_{:B} = A_\mu{}^A{}_{,B} + A_\mu{}^C \hat{\Gamma}_C{}^A{}_B \tag{5.6}$$

or equivalently

$$A_\mu{}^A{}_{:B} = \hat{\nabla}_B A_\mu{}^A + A_\mu{}^C \hat{\Theta}_{CB}{}^A , \tag{5.7}$$

where as before $\hat{\nabla}$ denotes the *ordinary* operation of covariant differentiation along the fibers with respect to the fiber connection $\hat{\Gamma}$. In the absence of the torsion $\hat{\Theta}$ the distinction between this Yano covariant derivative and the ordinary covariant derivative disappears. In terms of this notation the essential requirement that the fiber connection $\hat{\Gamma}$ be invariant under the action generated by the primary gauge field A can be obtained (from Yano's formula[5] for the Lie derivative of the connection) in the form

$$\hat{\nabla}_B A_\mu{}^A{}_{:C} = A_\mu{}^D \hat{R}_{BD}{}^A{}_C . \tag{5.8}$$

This basic postulate includes, as a consequence the corresponding decoupled invariance requirement for the torsion tensor, i.e.,

$$A_\mu{}^D \hat{\nabla}_D \hat{\Theta}_{BC}{}^A = A_\mu{}^A{}_{:D} \hat{\Theta}_{BC}{}^D + 2A_\mu{}^D{}_{:[C|} \hat{\Theta}_{D|B]}{}^A . \tag{5.9}$$

For purely base-tensorial entities the question of gauge invariance does not arise. We therefore proceed directly to consider the appropriate gauge-covariant generalization of the definition (3.5) of the absolute variation of the simplest kind of fiber-tensorial quantity, an ordinary vector V between nearby points in nearby fibers separated by a base displacement dx. Evidently the required gauge-covariant variation $\bar{d}V$ should be defined as the covariant variation with respect to the fiber connection $\hat{\Gamma}$ along the covariant vertical displacement $\bar{d}X$ as specified by the projection that is horizontal with respect to the gauge connection. This means that we must take

$$\bar{d}V^A = d(V^A) - d_a V^A + (\bar{d}X^B) \hat{\Gamma}_B{}^A{}_C V^C , \tag{5.10}$$

where $\bar{d}X^A$ are the components of the covariant vertical displacement as specified by (4.11), or more explicitly, (4.14), and $d_a V^A$ are the vector component variations resulting from the fact that horizontality with respect to the local fiber coordinates X^A differs from horizontality with respect to the gauge connection by the effect of infinitesimal Lie displacement induced by the vector field $\mathbf{a}$, specified on the fiber by (4.10), which gives

$$d_a V = -a^A{}_{,B} V^B . \tag{5.11}$$

Thus, explicitly, we shall have

$$\bar{d}V^A = d(V^A) + [A_\mu{}^A{}_{,B} dx^\mu + (A_\mu{}^C dx^\mu + dX^C) \hat{\Gamma}_C{}^A{}_B] V^B . \tag{5.12}$$

In the case where V is a tangent vector defined as a field on a section $\Phi(x)$, there will be a corresponding bitensorial covariant derivative which can be read out from the defining formula

$$dV^A = V^A{}_{|\mu} dx \tag{5.13}$$

using the abbreviated bar suffix notation system

$$D_\mu V^A = V^A{}_{|\mu} \, . \tag{5.14}$$

Thus we obtain the covariant derivative components in the form

$$V^A{}_{|\mu} = \partial_\mu V^A + \omega_\mu{}^A{}_B V^B \, , \tag{5.15}$$

where the section-dependent *connector* ω [as introduced in (2.4)] will be given, using the notation of (5.4), by

$$\omega_\mu{}^A{}_B = \Phi^C{}_{|\mu} \Gamma_C{}^A{}_B + A_\mu{}^A{}_{,B} \tag{5.16}$$

or equivalently, using the notation of (3.11) and (5.6), by

$$\omega_\mu{}^A{}_B = \Gamma_\mu{}^A{}_B + A_\mu{}^A{}_{:B} \, . \tag{5.17}$$

Having thus obtained the required connector ω that is needed for covariant differentiation of a simple fiber vector on the section, one can go on immediately in the usual way to construct the corresponding covariant derivatives of more general fiber-tensorial and bitensorial quantities by adding an appropriate connector term for each index (a term involving $\omega_\mu{}^A{}_B$ with a positive and/or negative sign for each contravariant and/or covariant fiber index, and similarly a term involving $\Gamma_\mu{}^\nu{}_\rho$ for each base-space index).

The resulting generalization of the derivative commutator rule (3.16) for the primary section Φ itself involves the fiber and base torsions and the gauge curvature, taking the form

$$2\Phi^A{}_{|[\nu|\mu]} = \Phi^C{}_{|\mu} \Phi^D{}_{|\nu} \hat{\Theta}_{CD}{}^A - \Theta_{\mu\nu}{}^\rho \Phi^A{}_{|\rho} + F_{\mu\nu}{}^A \, . \tag{5.18}$$

The analogous commutator rule (generalizing (3.18)) for a fiber vector field over the section Φ involves the fiber curvature and the gauge curvature as well as the base torsion, taking the form

$$2V^A{}_{|[\nu|\mu]} = \Omega_{\mu\nu}{}^A{}_B V^B - \Theta_{\mu\nu}{}^\rho V^B{}_{|\rho} \, , \tag{5.19}$$

where the total connector curvature, as defined by (2.12) can be evaluated [using (5.8) and (4.21) as the sum of two *separately* bitensorially covariant terms] in the form

$$\Omega_{\mu\nu}{}^A{}_B = \Phi^C{}_{|\mu} \Phi^D{}_{|\nu} \hat{R}_{CD}{}^A{}_B + F_{\mu\nu}{}^A{}_{:B} \, . \tag{5.20}$$

The first (section-dependent) gauge-covariant term on the right-hand side in (5.20) can evidently be expanded quadratically in the gauge-connection field as

$$\Phi^c{}_{|\mu} \Phi^D{}_{|\nu} \hat{R}_{CD}{}^A{}_B = R_{\mu\nu}{}^A{}_B + 2A_{[\mu}{}^C X^D{}_{\nu]} \hat{R}_{CD}{}^A{}_B + A_\mu{}^C A_\nu{}^D \hat{R}_{CD}{}^A{}_B \, . \tag{5.21}$$

We recapitulate that in the second term on the right-hand side in (5.20) in colon denotes the Yano-type (wrong way round) covariant derivative, i.e.,

$$F_{\mu\nu}{}^A{}_{:B} = \hat{\nabla}_B F_{\mu\nu}{}^A + F_{\mu\nu}{}^C \hat{\Theta}_{CB}{}^A \, . \tag{5.22}$$

Like the undifferentiated gauge curvature field **F** itself, this Yano gauge-curvature gradient $\mathbf{F}_{:}$ is well defined globally over the primary bundle (not just on the section Φ where ω and Ω are defined). Since the gauge curvature belongs, by construction, to the Lie algebra it will automatically satisfy a fiber-connection preservation condition of a form analogous to the fundamental requirement (5.7), namely,

$$\hat{\nabla}_B F_{\mu\nu}{}^A{}_{:C} = F_{\mu\nu}{}^D \hat{R}_{BD}{}^A{}_C \, . \tag{5.23}$$

This relation is useful for the purpose of verifying directly as an exercise that the section-dependent curvature Ω given by (5.20) does indeed satisfy the Bianchi identity (2.12).

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