Stochastic quantization of Einstein gravity

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We determine a one-parameter family of covariant Langevin equations for the metric tensor of general relativity corresponding to DeWitt's one-parameter family of supermetrics. The stochastic source term in these equations can be expressed in terms of a Gaussian white noise upon the introduction of a stochastic tetrad field. The only physically acceptable resolution of a mathematical ambiguity in the ansatz for the source term is the adoption of Ito's calculus. By taking the formal equilibrium limit of the stochastic metric a one-parameter family of covariant path-integral measures for general relativity is obtained. There is a unique parameter value, distinguished by any one of the following three properties: (i) the metric is harmonic with respect to the supermetric, (ii) the path-integral measure is that of DeWitt, (iii) the supermetric governs the linearized Einstein dynamics. Moreover the Feynman propagator corresponding to this parameter is causal. Finally we show that a consistent stochastic perturbation theory gives rise to a new type of diagram containing "stochastic vertices."

I. INTRODUCTION

The stochastic quantization method of Parisi and Wu[1] provides an interesting alternative to the standard quantization of gauge theories, as no gauge fixing and associated Faddeev-Popov ghosts are required. It appears worthwhile to apply this scheme also to the gravitational field, although, pessimistically, one might expect that all the well-known difficulties of the standard (i.e., Hamiltonian and path integral) approaches to quantum gravity will show up again, though, maybe, in a different guise. From the optimist's point of view it is precisely this transformation of an old problem into a new guise that might offer some new insight, however. We feel that stochastic quantization does indeed live up to this expectation in the case of the gravitational field: We hope that our main result, contained in Sec. V, is of interest also to those who have reservations about the stochastic approach.

The main obstacle to applying the Parisi-Wu method in its original form to the metric tensor field of general relativity is the fact that the Euclidean Einstein-Hilbert action is not bounded from below. This necessitates a modification of the standard Euclidean path integral[2] and hence also of the stochastic formalism in which this path integral is expected to define the equilibrium distribution of a stochastic relaxation process. Several *ad hoc* modifications of stochastic quantization have been proposed to overcome this difficulty.[3-5] In our opinion the physically most promising one is to generalize stochastic quantization so as to make it applicable to fields in physical space-time with Lorentzian metric signature.[6,7] The linearized gravitational field has already been treated in this manner.[8] In this paper we propose a nonperturbative stochastic quantization scheme for the metric tensor in the full nonlinear Einstein theory. This scheme differs from another one proposed some time ago.[9] The perturbation theory implied by our scheme is found to be different from that employed in Ref. 5.

In Sec. II we show that the principle of general covariance with respect to field redefinitions determines a one-parameter family of Langevin equations for the metric tensor field of general relativity. The parameter is the same as in DeWitt's metric on the space of four-metrics. In Sec. III the stochastic source term of the Langevin equation is expressed explicitly in terms of a stochastic vielbein functional and a Gaussian white noise. A so-called Ito-Stratonovich ambiguity in this expression is resolved by requiring that the stochastic metric be independent of the choice of the vielbein functional, while maintaining the covariance of the Langevin equation with respect to general coordinate transformations. In Sec. IV we associate an "analytically continued" Fokker-Planck equation with every complex Langevin equation and derive from its equilibrium limit a covariant path-integral measure, again depending on one parameter, for general relativity. In Sec. V the existence of a preferred parameter is pointed out, distinguished by (i) harmonicity of the metric with respect to the DeWitt metric, (ii) coincidence of the measure with the DeWitt measure, (iii) the special role it plays in the linearized Einstein dynamics and causality of the corresponding propagator. The linearization corresponding to this parameter is not the standard linearization, however, since it has been shown[8] that in the latter the propagator of stochastic quantization is noncausal. In Sec. VI we sketch the perturbation theory implied by our covariant quantization scheme and show that it gives rise to a new type of diagram containing "stochastic vertices." A final section is reserved for concluding remarks.

II. COVARIANT LANGEVIN EQUATIONS

Our main guideline for the formulation of a Langevin equation for the metric tensor field $g_{\alpha\beta}(x)$ will be the notion of general covariance with respect to field redefinitions in field configuration space. The usefulness of this concept in quantum field theory has been stressed recently.[10,11] For conciseness we shall adopt the notation of DeWitt (see, e.g., Ref. 12) and represent a general stochastic field by $\Phi^A(s)$ where s is the fictitious evolution parameter and $A=(\alpha,\beta,\ldots;x)$ comprises component indices as well as the space-time coordinates. By covariance, then, the general form of the Lorentzian generalization[6] of the Parisi-Wu ansatz[1] for the Langevin equation for Φ^A must be

$$\dot{\Phi}^A \equiv \frac{\partial \Phi^A}{\partial s} = iG^{AA'}\frac{\delta S[\Phi]}{\delta \Phi^{A'}(s)} + \xi^A(s) . \tag{2.1}$$

Here $G^{AA'}$ is the inverse of a metric tensor functional $G_{AA'}[\Phi]$ in field configuration space, and the summation over A' includes a space-time integration. The stochastic source term $\xi^A(s)$ will be defined here only implicitly by requiring that all its correlations vanish with the exception of

$$\langle \xi^A(s)\xi^{A'}(s')\rangle = 2\langle G^{AA'}[\Phi(s)]\rangle \delta(s-s') . \tag{2.2}$$

A more explicit definition of ξ^A will be given in the next section.

Note that (2.2) implies

$$\langle \xi^A(s)\xi^{A'}(s')\rangle = 2G^{AA'}\delta(s-s') \tag{2.3}$$

only if $G_{AA'}$ is independent of Φ (and hence of ξ). For nongravitational fields there exists a field coordinate system (defining the "natural field variables") for which $G_{AA'}$ is indeed independent of Φ. However, for the gravitational field

$$\Phi^A \equiv g_{\alpha\beta}(x) , \tag{2.4}$$

$G_{AA'}$ must depend on Φ for reasons of ordinary general covariance (except we adopt a bimetric theory). [The index positions in (2.4) should cause no confusion, as usually the covariant metric is taken as the gravitational field variable, which, when considered as a field coordinate, must carry an upper index.] Indeed, ordinary covariance requires that the actions of general coordinate transformations on $g_{\alpha\beta}(x)$ be isometries with respect to the field metric $G_{AA'}$. The most general *local* metric $G_{AA'}$ that has this property is known[12] to be

$$\begin{aligned} G_{AA'} &= G^{\alpha\beta,\alpha'\beta'}(x,x') \\ &= \frac{C}{2}\,|g|^{1/2}(g^{\alpha\alpha'}g^{\beta\beta'} + g^{\alpha\beta'}g^{\beta\alpha'} + \lambda g^{\alpha\beta}g^{\alpha'\beta'}) \\ &\qquad \times \delta^{(4)}(x-x') , \end{aligned} \tag{2.5}$$

$$g = \det(g_{\alpha\beta}) , \tag{2.6}$$

$$\lambda \neq -\tfrac{1}{2} . \tag{2.7}$$

The constant C in (2.5) must be different from zero, but otherwise its choice is irrelevant: It can be seen from (2.1) and (2.2) that a positive rescaling of C, $C \to \gamma C$, $\gamma > 0$, corresponds merely to a rescaling of the parameter s, $s \to \gamma s$. Even a change of the sign of C does not affect the equilibrium limit as will be seen in Sec. IV. We are thus left with a one-parameter (λ) family of field metrics. For later reference we note the determinant and the inverse of the metric (2.5):

$$G \equiv \det(G_{AA'}) = \prod_x C^{10}(-1-2\lambda) , \tag{2.8}$$

$$\begin{aligned} G^{AA'} &\equiv G_{\alpha\beta,\alpha'\beta'}(x,x') \\ &= (2C\,|g|)^{-1/2}\left[g_{\alpha\alpha'}g_{\beta\beta'} + g_{\alpha\beta'}g_{\beta\alpha'} - \frac{\lambda}{2\lambda+1}g_{\alpha\beta}g_{\alpha'\beta'} \right] \\ &\qquad \times \delta^{(4)}(x-x') . \end{aligned} \tag{2.9}$$

In particular the determinant G is constant in the usual parametrization of geometries, a property that holds only if the number of space-time dimensions is four.[12] Moreover we see from (2.8) that the signature of $G_{AA'}$ depends on λ.

Let us now insert the Einstein-Hilbert action

$$S[g] = \frac{1}{2\kappa}\int d^4x\,|g|^{1/2}R , \tag{2.10}$$

$$\frac{\delta S}{\delta g_{\alpha\beta}} = -\frac{1}{2\kappa}|g|^{1/2}(R^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}R) , \tag{2.11}$$

into the Langevin equation (2.1). As we want to retain the canonical dimension L^2 of s, we choose

$$C = 4\kappa . \tag{2.12}$$

Then we obtain

$$\dot{g}_{\alpha\beta} = -2i\left[R_{\alpha\beta} - \frac{\lambda+1}{2(2\lambda+1)}g_{\alpha\beta}R \right] + \xi_{\alpha\beta} . \tag{2.13}$$

Note that ξ does not have its usual canonical dimension L^{-3} in Eq. (2.13). By the field rescaling

$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta}/(2\kappa^{1/2}) \tag{2.14}$$

one recovers this canonical dimension, however, and moreover, $\tilde{G}_{AA'}$ becomes dimensionless with

$$|\tilde{G}| = 1 \quad \text{if } \lambda = 0 . \tag{2.15}$$

As is easily seen, the drift term in the Langevin equation (2.13) is proportional to the Einstein tensor only if $\lambda = 0$. Also, only in this case does one obtain the standard Langevin equation for the linearized gravitational field discussed in Ref. 8 [the numerical factor in (2.12) has been chosen such that then also the parameter s is the same]. This appears to give the value $\lambda = 0$ a preferred status. It will be argued in Sec. V, however, that a different value ($\lambda = -1$) has all the prospects of describing the correct physics. For the present we shall leave λ unspecified.

III. RESOLUTION OF AN ITO-STRATONOVICH DILEMMA

Definitions (2.1) and (2.2) of the stochastic processes Φ^A and ξ^A are purely formal and not useful for practical calculations. Our aim is to reduce (2.1) to a more familiar type of stochastic differential equation by giving a more explicit definition of ξ^A. This may be accomplished as follows. We introduce a field-independent "reference metric" $G^{(0)}_{MM'}$ in field configuration space and associate with it a stochastic vielbein functional $E^M{}_A[\phi]$ obeying

$$G^{(0)}_{MM'}E^M{}_A[\phi]E^{M'}{}_{A'}[\phi'] = G_{AA'}[\phi] \;. \tag{3.1}$$

With the help of the vielbein we may define the anholonomic "vector components" $\xi^{(0)M}$ by

$$\xi^{(0)M} = E^M{}_A[\phi]\xi^A \tag{3.2}$$

or

$$\xi^A = E_M{}^A[\phi]\xi^{(0)M} \tag{3.3}$$

with

$$E_M{}^A E^M{}_{A'} = \delta^A{}_{A'} \;. \tag{3.4}$$

It is then natural to take (3.3) as an ansatz to satisfy (2.2) and to define $\xi^{(0)M}$ as a generalized Gaussian white noise with nonvanishing correlation

$$\langle \xi^{(0)M}(s)\xi^{(0)M'}(s')\rangle = 2G^{(0)MM'}\delta(s-s') \;, \tag{3.5}$$

where $G^{(0)MM'}$ is the inverse of $G^{(0)}_{MM'}$.

When (3.3) is substituted into (2.1), the Langevin equation is recognized to be of the type studied by Ito[13] and Stratonovich.[14] It is well known (see, e.g., Ref. 15) that a stochastic differential equation of this type is not well defined because the Wiener-type process $W^M(s)$ whose increment

$$dW^M = \xi^{(0)M}ds \tag{3.6}$$

appears in the associated stochastic integral is of unbounded variation (or, put differently, $\xi^{(0)M}$ is a generalized process, its sample "functions" being distributions). The product $E_N{}^A[\Phi]dW^N(s)$ may be interpreted in infinitely many different ways:

$$\begin{aligned} E_N{}^A[\phi]dW^N(s) &= \lim_{\Delta s\to 0^+}\{(1-\alpha)E_N{}^A[\Phi(s)]+\alpha E_N{}^A[\Phi(s+\Delta s)]\} \\ &\quad \times[W^N(s+\Delta s)-W^N(s)] \;, \end{aligned} \tag{3.7}$$

$$0\le\alpha\le 1 \;. \tag{3.8}$$

From the mathematical point of view there are two natural choices of α: $\alpha=0$, defining Ito's calculus, and $\alpha=\frac{1}{2}$, defining Stratonovich's calculus. In the following, we show that $\alpha=0$ is dictated by physical requirements. We shall make extensive use of results on covariant stochastic calculus obtained by Graham.[16]

Let us consider the Stratonovich interpretation first. Its attractive feature is that it implies that each term in (2.1) transforms as a "contravariant vector" under arbitrary coordinate transformations in field configuration space. On the other hand, for $\alpha>0$ (3.7) and (2.1) imply that $\Phi^A(s)$ depends statistically on $W^N(s+ds)-W^N(s)$, and hence in general

$$\langle F[\Phi(s)]\xi^A(s)\rangle \neq 0 \;, \tag{3.9}$$

where F is an arbitrary functional of $\Phi(s)$. Consequently there appear spurious terms in

$$\begin{aligned} \langle \xi^A(s)\xi^{A'}(s')\rangle &= \langle E_M{}^A(s)E_{M'}{}^{A'}(s')\rangle \\ &\quad \times\langle \xi^{(0)M}(s)\xi^{(0)M'}(s')\rangle + \cdots \end{aligned} \tag{3.10}$$

that violate (2.2). Equation (2.1) with the Stratonovich interpretation for (3.3) is also not form invariant under a Φ-dependent change of the vielbein, in contrast with naive expectations. Indeed, if we perform a Φ-dependent, locally $G^{(0)}$-isometric transformation of the vielbein functional,

$$E_M{}^A[\Phi]\to\Lambda_M{}^N[\Phi]E_N{}^A[\Phi] \;, \tag{3.11}$$

$$\xi^{(0)M}\to\Lambda^M{}_N[\Phi]\xi^{(0)N} \;, \tag{3.12}$$

$$G^{(0)MN}\Lambda_M{}^P\Lambda_N{}^Q = G^{(0)PQ} \;, \tag{3.13}$$

$$\Lambda_M{}^P\Lambda^N{}_P = \delta_M{}^N \;, \tag{3.14}$$

then a spurious term

$$-\frac{\delta\Lambda^M{}_P}{\delta\Phi^B}\Lambda_N{}^P E_M{}^A E^{NB} \;, \tag{3.15}$$

$$E^{NB} := G^{BC}E^N{}_C \;, \tag{3.16}$$

adds to the "drift vector" in (2.1). (It is even possible to transform the drift vector to zero by an appropriate Φ-dependent "rotation."[17]) Therefore a different choice of vielbein defines a different process Φ^A, which is clearly not desirable.

The physical consequences of the vielbein dependence can be read off from the Fokker-Planck (FP) equation implied by the Stratonovich interpretation of (3.3) in (2.1). Since Φ^A is a complex process, this FP equation involves partial derivatives with respect to the real and imaginary parts of Φ^A. Because we are interested in the limit $s\to\infty$ which is expected to exist only in the distributional sense,[6] we shall not write down this correct FP equation, but a modified one, obtained from the FP equation for the Euclidean analog of Φ^A by the analytic continuation

$$S_{\mathrm{Eucl}}\to -iS \;, \tag{3.17}$$

where S_{Eucl} is the Euclidean action. It has been conjectured by Parisi[18] that the *real* process (with *complex* probabilities) described by this modified FP equation has the same expectation values as the original *complex* process with *real* probabilities).

The modified FP equation for the Stratonovich interpretation of (3.3) and (2.1) reads

$$\begin{aligned} \frac{\partial P[\Phi,x]}{\partial s} &= \left\{-\frac{\delta}{\delta\Phi^A}K^A[\Phi]\right. \\ &\quad \left.+\frac{\delta^2}{\delta\Phi^A\delta\Phi^B}G^{AB}[\Phi]\right\}P[\Phi,s] \;, \end{aligned} \tag{3.18}$$

$$K^A = iG^{AB}\frac{\delta S}{\delta\Phi^B}+\frac{\delta E_M{}^A}{\delta\Phi^B}E_N{}^B G^{(0)MN} \;, \tag{3.19}$$

$$\langle K^A\rangle = \langle\dot\Phi^A\rangle \;. \tag{3.20}$$

Equation (3.19) shows the explicit dependence of the probability distribution $P[\Phi,s]$ on the choice of the vielbein even in the equilibrium case $\partial P/\partial s=0$. The Stratonovich interpretation is therefore not viable in our case. The same conclusion may be reached for all other values of α in (3.7) different from zero.

If the Langevin equation (2.1) with the ansatz (3.3) and (3.5) is interpreted in the sense of Ito, the unphysical dependence on the choice of the vielbein disappears. In particular (3.10) reduces to (2.2) because of the "nonanticipating" nature of $\Phi(s)$, and hence of $E_M{}^A[\Phi(s)]$, with respect to $dW(s)$. On the other hand, in Ito's calculus $\dot{\Phi}^A$ is not a contravariant "vector," but transforms inhomogeneously according to

$$\dot{\Phi}'^{A}=\frac{\delta\Phi'^{A}}{\delta\Phi^{B}}\dot{\Phi}^{B}+G^{BC}\frac{\delta^2\Phi'^{A}}{\delta\Phi^{B}\delta\Phi^{C}} \tag{3.21}$$

(the same rule holds for K^A in Stratonovich's calculus). Therefore the Langevin equation (2.1) is *not* manifestly covariant with respect to field redefinitions, but assumes its particular form only in a special class of "coordinate" systems.

A generally covariant stochastic quantization scheme may be based on the covariantized Langevin equation

$$\dot{\Phi}^{A}-\Delta_G\Phi^A=iG^{AB}\frac{\delta S[\Phi]}{\delta\Phi^B}+\xi^A\ , \tag{3.22}$$

where Δ_G denotes the Laplace-Beltrami operator of the field metric G_{AB}. The left-hand side of (3.22) is indeed a "vector," because

$$\Delta_G\Phi^A=|G|^{-1/2}\frac{\delta}{\delta\Phi^B}(|G|^{1/2}G^{AB}) \tag{3.23}$$

transforms as

$$\Delta_{G'}\Phi'^{A}=\Delta_G\Phi^A+G^{BC}\frac{\delta^2\Phi'^{A}}{\delta\Phi^B\delta\Phi^C}\ . \tag{3.24}$$

In the present paper, however, we shall stick mainly to the "naive" Langevin equation (2.1) with the field variable (2.4), because only this version allows us to obtain also the noncovariant (in field configuration space) path-integral measures that have been proposed in the literature. Note that (2.1) is still covariant with respect to diffeomorphisms, because the latter act linearly on Φ^A and hence $\delta^2\Phi'^{A}/\delta\Phi^B\delta\Phi^C$ vanishes.

Finally we mention a further advantage of Ito's calculus. As indicated above, the process $\Phi^A(s)$ is "causal," $\Phi^A(s)$ and $dW^N(s')$ being uncorrelated for $s\le s'$. Therefore a discretized version of equation (2.1) can in principle be used to simulate sample "paths" of $\Phi^A(s)$ on a computer by repetitive calls of a Gaussian random number generator for the Wiener increments.[19]

IV. FORMAL EQUILIBRIUM LIMIT AND PATH-INTEGRAL MEASURE

The modified FP equation implied by the Ito version of the Langevin equation (2.1) is

$$\frac{\partial P[\Phi,s]}{\partial s}=\left[-\frac{\delta}{\delta\Phi^A}F^A[\Phi]+\frac{\delta^2}{\delta\Phi^A\delta\Phi^B}G^{AB}[\Phi]\right]P[\Phi,s]\ , \tag{4.1}$$

$$F^A=iG^{AB}\frac{\delta S}{\delta\Phi^B}\ , \tag{4.2}$$

$$\langle F^A\rangle=\langle\dot{\Phi}^A\rangle\ . \tag{4.3}$$

The probability distribution $P[\Phi,s]$ being a scalar density in field configuration space, it is natural to introduce the corresponding "scalar field" Q defined by

$$Q[\Phi,s]=|G|^{-1/2}P[\Phi,s]\ . \tag{4.4}$$

Equation (4.1) implies

$$\frac{\partial Q}{\partial s}=-|G|^{-1/2}\frac{\delta}{\delta\Phi^A}|G|^{1/2}\times\left[F^A-\Delta_G\Phi^A-G^{AB}\frac{\delta}{\delta\Phi^B}\right]Q\ . \tag{4.5}$$

This equation is not covariant, because the term in parentheses is not a "vector." This is not surprising, as we have seen in the last section that (2.1) is not covariant in the Ito interpretation. If we adopted (3.22) instead of (2.1), then the spurious term $-\Delta_G\Phi^A$ would disappear in the parentheses in (4.5) and we would obtain

$$\frac{\partial Q}{\partial s}=-\nabla^B\left[\frac{\delta S}{\delta\Phi^B}Q-\frac{\delta Q}{\delta\Phi^B}\right] \tag{4.6}$$

which is manifestly covariant. Here $\nabla^B=G^{BC}\nabla_C$ with ∇_C the covariant derivative with respect to Φ^C implied by the Levi-Cività connection $\Gamma^A{}_{BC}$ of the field metric G_{AB}:

$$\Gamma^A{}_{BC}=\frac{1}{2}G^{AJ}\left[\frac{\delta G_{JB}}{\delta\Phi^C}+\frac{\delta G_{JC}}{\delta\Phi^B}-\frac{\delta G_{BC}}{\delta\Phi^J}\right]\ . \tag{4.7}$$

With regard to the question of the equivalence of stochastic quantization with path-integral quantization it is interesting to check whether the modified FP equation admits an equilibrium solution $Q_{\rm eq}$ proportional to e^{iS}. The stochastic equilibrium averages will then coincide with the Schwinger averages of quantum field theory. A sufficient condition for the stationarity of $Q_{\rm eq}$ in (4.5) is

$$\frac{\delta Q_{\rm eq}}{\delta\Phi^A}=\left[i\frac{\delta S}{\delta\Phi^A}-G_{AB}\Delta_G\Phi^B\right]Q_{\rm eq}\ . \tag{4.8}$$

Evaluation of the second term on the right-hand side (RHS) of this equation gives

$$\begin{aligned}-G_{AB}\Delta_G\Phi^B&\equiv-\int d^4y\,G^{\alpha\beta,\gamma\delta}(x,y)\frac{\delta G_{\gamma\delta,\alpha'\beta'}(y,x')}{\delta g_{\alpha'\beta'}(x')}\\&=-\tfrac{9}{2}(1+\lambda)g^{\alpha\beta}(x)\delta^{(4)}(0)\ .\end{aligned} \tag{4.9}$$

Now

$$\delta^{(4)}(0)g^{\alpha\beta}(x)=\frac{\delta}{\delta g_{\alpha\beta}(x)}\left[\sum_y\ln|g(y)|\right]\ . \tag{4.10}$$

Therefore

$$Q_{\rm eq}[g]\propto\prod_x|g(x)|^{\gamma}e^{iS[g]}\ , \tag{4.11}$$

$$\gamma=-\tfrac{9}{2}(1+\lambda)\ . \tag{4.12}$$

Thus $Q_{\rm eq}$ contains in addition to e^{iS} also a nontrivial path-integral measure, and we obtain the following formal expression for the equilibrium expectation value of an arbitrary functional $F[g]$:

$$\langle F[g]\rangle \propto \int \prod_{x} \prod_{\alpha\leq\beta} |g(x)|^{\gamma} dg_{\alpha\beta}(x) F[g] e^{iS[g]} . \quad (4.13)$$

This expression shows that the Langevin equation (2.1) with $\Phi^A = g_{\alpha\beta}$ can reproduce any covariant (with respect to diffeomorphisms) measure based on the integration variable $g_{\alpha\beta}$ with the exception of $\gamma=-\frac{9}{4}$. The values $\gamma=-\frac{5}{2}$ (Refs. 20–22) and $\gamma=0$ (Refs. 23 and 24) have been considered most frequently. A unique noncovariant measure with weight $|g|^{-5/2}|{}^{(3)}g|$, resulting from a lattice discretization[25] and also from a canonical formulation[26], has also been proposed. Note that DeWitt's choice $\gamma=0$ is obtained only for $\lambda=-1$. Some unique implications of this parameter value will be pointed out in the next section.

As is well known, the path integral (4.13) is not normalizable even in a formal sense because of the invariance of S under diffeomorphisms. By analogy with the situation in gauge theories[27,28] one is led to conjecture, however, that stochastic gauge fixing[29] will yield an equilibrium distribution that is formally identical with the Faddeev-Popov distribution of standard quantization in an appropriate gauge. An investigation in this direction has already been carried out by Sakamoto,[9] who uses an approach different from ours.

V. HARMONICITY, DeWITT MEASURE AND LINEARIZED EINSTEIN DYNAMICS

The DeWitt path-integral measure [$\gamma=0$ in (4.13)] appears attractive because of its invariance under general coordinate transformations. It is obtained from (2.1) only if $\lambda=-1$. The reason is that only in this case

$$\Delta_G \Phi^A \equiv \nabla_C \nabla^C \Phi^A = 0 ; \quad (5.1)$$

i.e., $\lambda=-1$ is the unique parameter value for which $g_{\alpha\beta}(x)$ are harmonic "coordinates."

Of course the covariantized Langevin equation (3.22) implies via (4.6) the generally covariant equilibrium distribution

$$P_{\rm eq}[\Phi] \propto |G[\Phi]|^{1/2} e^{iS[\Phi]} \quad (5.2)$$

which is independent of λ because of (2.8) and coincides with (4.11) in the case $\lambda=-1$.

We note in passing that besides $g_{\alpha\beta}$ also $|g|^{5/2} g_{\alpha\beta}$ is harmonic for $\lambda=-1$. In general there are two real solutions of the type $|g|^r g_{\alpha\beta}$ to the harmonic coordinate condition (5.1) except in the interval $\lambda>\frac{7}{18}$ where r becomes complex. There is a unique parameter λ for a given exponent r such that $|g|^r g_{\alpha\beta}$ is harmonic, with the exception of $r=-\frac{1}{4}$ and $-\frac{9}{4}$, where no solution exists (note that for $r=-\frac{1}{4}$, $|g|^r g_{\alpha\beta}$ are not valid coordinates because they comprise only the conformally invariant part of the metric $g_{\alpha\beta}$). Another remarkable feature of $\lambda=-1$ is that only in this case $G_{AB} d\Phi^B$ is a differential, namely,

$$\delta(-|g|^{1/2} g^{\alpha\beta}) = \int d^4x' G^{\alpha\beta,\alpha'\beta'} \delta g_{\alpha'\beta'} \quad \text{if } \lambda=-1 .$$

Therefore, and because G is constant, the integration variable $g_{\alpha\beta}$ in (4.13) may be replaced by $|g|^{1/2} g^{\alpha\beta}$, if $\lambda=-1$. The variable $g_{\alpha\beta}$ is the only one for which a relationship of this type and harmonicity hold at the same value of λ.

Up to this point in the paper our considerations have been completely independent of the special choice of action $S[\Phi]$. We now want to examine the stochastic dynamics generated by the Einstein action in the linear limit defined by

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 2\kappa^{1/2} \psi_{\alpha\beta} , \quad (5.3)$$

$$S[g] = S^{(0)}[\psi] + S^{(\rm int)}[\psi] , \quad (5.4)$$

$$S^{(0)}[\psi] = \frac{1}{2} \int d^4x\, \psi_{\alpha\beta} V^{\alpha\beta\gamma\delta} \psi_{\gamma\delta} . \quad (5.5)$$

Here η is the Minkowski metric, and all indices will be transvected with this metric. For the parameter value $\lambda=0$ the Langevin equation for $\psi_{\alpha\beta}(x,s)$ implied by the action $S^{(0)}$ has already been studied in an earlier paper,[8] to which we refer the reader for more details on the technical points of the discussion that follows. Our aim is to derive the Feynman propagator for a general value of λ. It is convenient to work in momentum space where the linear operator V of (5.5) is given by

$$V = k^2 (P^2 - 2P^{0'}) . \quad (5.6)$$

Here P^2 and $P^{0'}$ are members of a complete orthogonal set $\{P^2, P^1, P^0, P^{0'}\}$ of projection operators on the space of symmetric tensor fields, the superscript indicating their (massive) spin value.[30]

In the linear limit the Langevin equation (2.15) becomes

$$\dot\psi_{\alpha\beta} = i W_{\alpha\beta}{}^{\gamma\delta} \psi_{\gamma\delta} + \xi_{\alpha\beta} , \quad (5.7)$$

$$W_{\alpha\beta}{}^{\gamma\delta} \equiv G^{-1}{}_{\alpha\beta\mu\nu} V^{\mu\nu\gamma\delta} , \quad (5.8)$$

where the inverse supermetric is simply

$$G^{-1}{}_{\alpha\beta\gamma\delta} = \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} + \mu \eta_{\alpha\beta}\eta_{\gamma\delta}) , \quad (5.9)$$

$$\mu \equiv -\lambda/(2\lambda+1) , \quad (5.10)$$

and ξ is Gaussian with correlation

$$\langle \xi_{\alpha\beta}(k,s) \xi_{\alpha'\beta'}(k',s')\rangle = 2(2\pi)^4 G^{-1}{}_{\alpha\beta\alpha'\beta'} \delta^{(4)}(k+k') \delta(s-s') . \quad (5.11)$$

If $\psi(x,s)$ is subjected to the initial condition $\psi(x,0)=0$ as usual (for a discussion of different initial conditions see Ref. 8), then the solution of (5.7) is given by

$$\psi(s) = \int_0^s H(s-s') \xi(s') ds' , \quad (5.12)$$

where H is the "Schrödinger kernel"

$$H(s) = e^{iWs} \quad (5.13)$$

corresponding to the "Hamiltonian" W. Equation (5.8) implies

$$W_{\alpha\beta}{}^{\gamma\delta} = V_{\alpha\beta}{}^{\gamma\delta} - \mu k^2 \eta_{\alpha\beta} T^{\gamma\delta} , \quad (5.14)$$

$$T^{\gamma\delta} := \eta^{\gamma\delta} - L^{\gamma\delta} , \quad (5.15)$$

$$L^{\gamma\delta} := k^\gamma k^\delta/(k^2+i0) \; . \tag{5.16}$$

Since W is not a linear combination of orthogonal projections, H is most practically calculated using the relation

$$G(k,s)=\theta(s)H(k,s) \; , \tag{5.17}$$

where G is the retarded Green's function of the deterministic part of Eq. (5.7) obeying

$$\left[\frac{\partial}{\partial s}-iW\right]G(k,s)=\delta(s)\mathbb{1} \; . \tag{5.18}$$

Here $\mathbb{1}$ is the unit operator on symmetric tensor fields,

$$\mathbb{1}_{\alpha\beta}{}^{\gamma\delta}=\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} \; . \tag{5.19}$$

The Green's function G can be computed by Fourier transforming (5.18) with respect to s:

$$G(k,s)=\frac{i}{2\pi}\int_{-\infty}^{\infty} d(m^2)K(k,m^2-i0)e^{im^2 s} \; , \tag{5.20}$$

$$(W-m^2\mathbb{1})K(m^2)=\mathbb{1} \; . \tag{5.21}$$

Equation (5.21) defines the "propagator" corresponding to the "field equation"

$$(W-m^2\mathbb{1})\phi=0 \; , \tag{5.22}$$

$$K=\frac{P^2}{k^2-m^2}-\frac{P^1}{m^2}-\frac{P^0}{m^2}-\frac{P^{0'}}{2k^2+m^2}+\frac{\lambda k^2\eta T}{[(3\lambda+2)k^2+m^2](2k^2+m^2)}+\frac{2\lambda k^4 LT}{m^2[(3\lambda+2)k^2+m^2](2k^2+m^2)} \; , \tag{5.23}$$

where $(\eta T)_{\alpha\beta}{}^{\gamma\delta}=\eta_{\alpha\beta}T^{\gamma\delta}$, etc. Using complex integration in (5.20) we obtain via (5.7)

$$\begin{aligned} H&=e^{ik^2 s}P^2+P^1+P^0+e^{-2ik^2 s}P^{0'}+\tfrac{1}{3}(e^{-i(3\lambda+2)k^2 s}-e^{-2ik^2 s})\eta T-\left[\frac{\lambda}{3\lambda+2}+\frac{2}{3(3\lambda+2)}e^{-i(3\lambda+2)k^2 s}-\tfrac{1}{3}e^{-2ik^2 s}\right]LT \\ &=e^{ik^2 s}P^2+P^1+P^0+\tfrac{1}{3}e^{-i(3\mu+2)k^2 s}\eta T-\left[\frac{\mu}{3\mu+2}+\frac{2}{3(3\mu+2)}e^{-i(3\mu+2)k^2 s}\right]LT \; . \end{aligned} \tag{5.24}$$

From (5.12), (5.24), and (5.11) we may calculate the "stochastic propagator"

$$D_{\alpha\beta\alpha'\beta'}(k,s;k',s') := \langle \psi_{\alpha\beta}(k,s)\psi_{\alpha'\beta'}(k',s')\rangle \tag{5.25}$$

$$=2(2\pi)^4\delta^{(4)}(k+k')\int_0^{\min(s,s')} d\sigma\, H_{\alpha\beta}{}^{\gamma\delta}(k,s-\sigma)H_{\alpha'\beta'}{}^{\gamma'\delta'}(k',s'-\sigma)G^{-1}{}_{\gamma\delta\gamma'\delta'} \; . \tag{5.26}$$

In this section we are not interested in its explicit form, but only in its equilibrium limit, which defines the Feynman propagator K:

$$\lim_{s\to\infty} D_{\alpha\beta,\alpha'\beta'}(k,s;k',s)=i(2\pi)^4\delta^{(4)}(k+k')K_{\alpha\beta,\alpha'\beta'}(k) \; . \tag{5.27}$$

Since $H(k)=H(-k)$, we have

$$K_{\alpha\beta,\alpha'\beta'}(k)=-2i\int_0^\infty d\tau\left[H_{\alpha\beta}{}^{\gamma\delta}(k,\tau)H_{\alpha'\beta'\gamma\delta}(k,\tau)+\frac{\mu}{2}H_{\alpha\beta}{}^{\gamma}{}_{\gamma}(k,\tau)H_{\alpha'\beta'}{}^{\delta}{}_{\delta}(k,\tau)\right] \; . \tag{5.28}$$

The explicit result is

$$K=K^{(0)}+R \; , \tag{5.29}$$

$$K^{(0)}=\frac{P^2}{k^2+i0}-\frac{1}{2}\frac{P^{0'}}{k^2-i0}-i\infty^2(P^1+P^0) \; , \tag{5.30}$$

$$\begin{aligned} R=&\frac{1}{6}\left[\frac{1}{k^2-i0}-\frac{1}{k^2-iq0}\right]\eta\eta+\frac{1}{3}\left[\frac{1}{(3\mu+2)k^2-i0}-\frac{1}{2}\frac{1}{k^2-i0}\right](\eta L+L\eta) \\ &+\left[\frac{1}{6k^2-i0}-\frac{2}{3(3\mu+2)}\frac{1}{3(\mu+2)k^2-i0}-i\frac{\mu}{3\mu+2}\infty^2\right]LL \; , \end{aligned} \tag{5.31}$$

$$q := \mathrm{sgn}(3\mu+2) \; , \tag{5.32}$$

$$\infty^{2n}=2\int_0^\infty ds\, s^{n-1} \; . \tag{5.33}$$

One recognizes in $K^{(0)}$ the propagator of the standard linearization[8] ($\lambda=\mu=0$) and checks easily that R vanishes for $\mu=0$. Recalling that

$$P^{0'}=\tfrac{1}{3}TT \; , \tag{5.34}$$

$$P^0=LL \; , \tag{5.35}$$

we conclude from (5.29)—(5.32) that

$$K=\frac{P^2}{k^2+i0}-\frac{1}{6}\frac{\eta\eta}{k^2-iq0}-i\infty^2P^1-\left[\frac{2}{3(3\mu+2)}\frac{1}{(3\mu+2)k^2-i0}+\frac{4\mu+2}{3\mu+2}i\infty^2\right]P^0+\frac{1}{3}\frac{1}{(3\mu+2)k^2-i0}(\eta L+L\eta)\ . \tag{5.36}$$

Thus the whole propagator is causal if

$$\mu<-\tfrac{2}{3}\Longleftrightarrow -2<\lambda<-\tfrac{1}{2}\ . \tag{5.37}$$

The quadratic divergences in K are harmless as they do not contribute to gauge-invariant quantities. In fact the only part of K contributing to gauge-invariant expectation values is

$$K^{(\mathrm{inv})}_{\alpha\beta,\alpha'\beta'}(k)=\tfrac{1}{2}(\eta_{\alpha\alpha'}\eta_{\beta\beta'}+\eta_{\alpha\beta'}\eta_{\beta\alpha'}-\tfrac{2}{3}\eta_{\alpha\beta}\eta_{\alpha'\beta'})\frac{1}{k^2+i0}-\frac{1}{6}\eta_{\alpha\beta}\eta_{\alpha'\beta'}\frac{1}{k^2-iq0} \tag{5.38}$$

$$=\frac{G^{-1}{}_{\alpha\beta,\alpha'\beta'}(\lambda=-1)}{k^2+i0}\quad \text{if } \mu<-\tfrac{2}{3}\ . \tag{5.39}$$

We find it remarkable that the field metric with parameter value $\lambda=-1$ appears in the propagator. Another remarkable property is that the whole of K assumes an especially simple form if $\lambda=-1$. Making use of

$$P^2{}_{\alpha\beta\gamma\delta}=\tfrac{1}{2}(T_{\alpha\gamma}T_{\beta\delta}+T_{\alpha\delta}T_{\beta\gamma})-\tfrac{1}{3}T_{\alpha\beta}T_{\gamma\delta} \tag{5.40}$$

we obtain

$$K^{(-1)}_{\alpha\beta,\alpha'\beta'}=\tfrac{1}{2}(\eta_{\alpha\alpha'}\eta_{\beta\beta'}+\eta_{\alpha\beta'}\eta_{\beta\alpha'}-\eta_{\alpha\beta}\eta_{\alpha'\beta'}+\eta_{\alpha\alpha'}k_\beta k_{\beta'}+\eta_{\alpha\beta'}k_\beta k_{\alpha'}+\eta_{\beta\alpha'}k_\alpha k_{\beta'}+\eta_{\beta\beta'}k_\alpha k_{\alpha'})\frac{1}{k^2+i0}-i\infty^2P^1-2i\infty^2P^0\ . \tag{5.41}$$

The case $\mu=-\frac{2}{3}$ must be treated separately. Since in this case there is a double pole in the last term of (5.23), the last term of (5.24) becomes

$$(-1+\tfrac{2}{3}ik^2s)LT\ . \tag{5.42}$$

Consequently

$$R=\frac{\eta\eta}{6k^2-i0}+\left[\frac{i}{3}\infty^2-\frac{1}{6k^2-i0}\right](\eta L+L\eta)+\left[\frac{1}{6k^2-i0}-\frac{i}{3}\infty^2-\frac{2}{3}k^2\infty^4\right]LL\ ,\qquad (\mu=-\tfrac{2}{3})\ . \tag{5.43}$$

Note that the first term on the RHS of (5.43) differs from what would be obtained if the limit $\mu\to-\frac{2}{3}$ were taken after the limit $s\to\infty$, and in a symmetrical manner, namely,

$$\frac{1}{6}\left[\frac{1}{k^2-i0}-P\frac{1}{k^2}\right]\eta\eta\ , \tag{5.44}$$

P denoting the principal value. This discrepancy will be seen shortly to be related to the so-called van Dam—Veltman mass discontinuity.[31]

Having evaluated the Feynman propagator implied by stochastic quantization it is natural to seek an inverse to it. This is a nontrivial problem because of the divergences present in K. In standard quantization the Feynman propagator is obtained by inverting the linear differential operator defined by the free part of the action. If a gauge invariance is present, as in the case of the linearized gravitational field, the propagator does not exist. The usual way around this difficulty is to break the gauge invariance of the action by adding a gauge-fixing term to the Lagrangian. Adding a mass term to the Lagrangian may be considered as a special case of gauge fixing. In the following we show that K may be identified with the inverse of a certain massive extension of linearized gravity in the (singular) limit as the mass goes to zero.

It is natural to consider the massive extension

$$V^{\alpha\beta\gamma\delta}\to V^{\alpha\beta\gamma\delta}-M^2G^{\alpha\beta\gamma\delta} \tag{5.45}$$

with G the inverse of the field metric (5.9). The extension (5.45) implies

$$W_{\alpha\beta}{}^{\gamma\delta}\to W_{\alpha\beta}{}^{\gamma\delta}-M^2 1_{\alpha\beta}{}^{\gamma\delta}\ , \tag{5.46}$$

$$H\to e^{-iM^2s}H\ . \tag{5.47}$$

The propagator corresponding to the massive extension is

$$K_{M^2}=\frac{P^2}{k^2-M^2+i0}-\frac{1}{6}\frac{3\mu+2}{(3\mu+2)k^2+M^2-i0}\eta\eta-\frac{P^1}{M^2-i0}-\left[\frac{2}{3(3\mu+2)}\frac{1}{(3\mu+2)k^2+M^2-i0}+\frac{4\mu+2}{3\mu+2}\frac{1}{M^2-i0}\right]P^0+\frac{1}{3}\frac{1}{(3\mu+2)k^2+M^2-i0}(\eta L+L\eta)\quad(\mu\neq-\tfrac{2}{3})\ . \tag{5.48}$$

If $\mu \neq -\frac{2}{3}$, the P^0 term in (5.48) has to be replaced by

$$\left[\frac{2k^2}{3(M^2-i0)^2} - \frac{4}{3}\frac{1}{M^2-i0} \right] P^0 . \tag{5.49}$$

Comparison of (5.48) and (5.49) with (5.31) and (5.43), respectively, shows that indeed

$$K_{M^2} \underset{M^2 \to 0}{\to} K , \tag{5.50}$$

where also the correct singularity structure of K is implied: If $i0$ is replaced by $i\epsilon$ in K_{M^2}, then (5.33) is consistent with

$$\infty^{2n} = \lim_{\epsilon \to 0+} \epsilon^{-n} . \tag{5.51}$$

This is a confirmation of the fact that K can be obtained also from the Langevin equation with real damping,

$$\dot{\psi}_{\alpha\beta} = -\epsilon\psi_{\alpha\beta} + iW_{\alpha\beta}{}^{\gamma\delta}\psi_{\gamma\delta} + \xi_{\alpha\beta} \tag{5.52}$$

if the limit $\epsilon \to 0+$ is taken after the calculation has been performed.

It is easy to evaluate $K^{-1}{}_{M^2}$ once it is noted that the operator W is Hermitian with respect to the scalar product

$$\langle \psi, \psi \rangle = \int d^4x\, \psi_{\alpha\beta} G^{\alpha\beta\gamma\delta} \psi_{\gamma\delta} \equiv \int d^4x\, \psi^T G \psi . \tag{5.53}$$

Denoting the Hermitian adjoint of W by $\widetilde{W}$, we have

$$\widetilde{W} = G^{-1} W^T G , \tag{5.54}$$

where T denotes the ordinary matrix transposition. Since $W = G^{-1}V$ [cf. (5.8)] and $G = G^T, V = V^T$, it follows that

$$\widetilde{W} = W . \tag{5.55}$$

Therefore the stochastic propagators (5.25) and (5.26) can also be written as

$$\begin{aligned} D(k,s;k',s') &= 2(2\pi)^4\delta^{(4)}(k+k') \\ &\quad \times \int_0^{\min(s,s')} d\sigma\, e^{iW(s-\sigma)} G^{-1} (e^{iW(s'-\sigma)})^T \\ &= 2(2\pi)^4\delta^{(4)}(k+k') \\ &\quad \times \int_0^{\min(s,s')} d\sigma\, e^{iW(s+s'-2\sigma)} G^{-1} \end{aligned} \tag{5.56}$$

and

$$K = -2i \int_0^\infty d\tau\, e^{2iW\tau} G^{-1} . \tag{5.57}$$

If $W \to W - M^2 \mathbf{1}$, we obtain

$$K_{M^2} = -2i \int_0^\infty d\tau\, e^{2i(W-M^2)\tau} G^{-1} = (W - M^2\mathbf{1})^{-1} G^{-1} \tag{5.58}$$

$$= (V - M^2 G)^{-1} . \tag{5.59}$$

Thus all the propagators K are "inverses" of the singular differential operator V in a certain sense specified by (5.59) and (5.50). With the exception of $\lambda = -2$ ($\mu = -\frac{2}{3}$) the principal value of the gauge-independent part is the same for all propagators. The "critical" value $\lambda = -2$ corresponds to the so-called Fierz-Pauli mass term[32] appearing in the Lagrangian for a massive pure spin-2 particle,

$$L_M^{(2)} = \tfrac{1}{2}\psi^T V \psi - \frac{M^2}{2}(\psi_{\alpha\beta}\psi^{\alpha\beta} - \psi^\gamma{}_\gamma \psi^\delta{}_\delta) . \tag{5.60}$$

The "anomalous" behavior of the corresponding propagator for $M^2 \to 0$ is known as the van Dam–Veltman mass discontinuity.[31]

The Fierz-Pauli extension is one of the two massive extensions characterized by the property that only one physical mass value is present [in the other cases there is a second pole of $K_{M^2}(k)$ at $k^2 = -M^2/(3\mu+2)$]. The other extension is that of $\lambda = -1$. In this case we have

$$\begin{aligned} K_{M^2}^{(-1)} &= \frac{1}{k^2 - M^2 + i0}[P^2 - \tfrac{1}{6}(\eta + 2L)(\eta + 2L)] \\ &\quad - \frac{P^1}{M^2 - i0} - \frac{2P^0}{M^2 - i0} . \end{aligned} \tag{5.61}$$

Summarizing the results of this section, we have seen that the parameter value $\lambda = -1$ is preferred both kinematically and dynamically. It is distinguished dynamically because (i) it governs the linearized Einstein equations [cf. (5.39)], (ii) the corresponding Feynman propagator is causal (though this is the case also for other parameter values), and (iii) considerable simplifications occur in K [cf. (5.41)] and K_{M^2} [cf. (5.61)]. Another manifestation of property (i) is the fact that by adding the gauge-fixing term

$$S_{\rm gf} = \int d^4x\, (\partial_\gamma \psi_\alpha{}^\gamma - \tfrac{1}{2}\partial_\alpha \psi_\gamma{}^\gamma) \tag{5.62}$$

to the action $S^{(0)}$ one obtains

$$S^{(0)} + S_{\rm gf} = \int d^4x\, \psi_{\alpha\beta} G^{\alpha\beta\gamma\delta}(\lambda = -1)(-\Box)\psi_{\gamma\delta} . \tag{5.63}$$

It is not possible to change λ in (5.63) by a different gauge fixing. It is amusing to observe that the usual harmonic coordinate condition involved in (5.62) serves to make the harmonicity of the metric itself, as implied by the Einstein action, manifest.

VI. PERTURBATION THEORY

The appearance of the stochastic source (3.3) in the Langevin equation (2.1) implies that in the case of the gravitational field a new type of diagram will contribute in the perturbative calculation of stochastic averages. In the following we sketch this perturbation theory briefly. It can be formulated for the stochastic perturbation $\psi_{\alpha\beta}(x,s)$ of an arbitrary deterministic and s-independent background metric $g^{\rm cl}_{\alpha\beta}(x)$. The full metric is split according to

$$g_{\alpha\beta}(x,s) = g^{\rm cl}_{\alpha\beta}(x) + 2\kappa^{1/2}\psi_{\alpha\beta}(x,s) . \tag{6.1}$$

A convenient choice of vielbein functional $E_M{}^A$ is the following. We introduce a deterministic, nonsingular "reference metric" $g^{(0)}_{ab}(x,s)$ and an associated stochastic tetrad field (in the ordinary sense) $e^a{}_\alpha(x,s)$ with

$$g^{(0)}_{ab}e^a{}_\alpha e^b{}_\beta = g_{\alpha\beta} \;. \tag{6.2}$$

Since $g_{\alpha\beta}$ is a complex process, we admit also complex $e^a{}_\alpha$. It is therefore not necessary to impose a signature restriction on $g^{(0)}_{ab}$. Then

$$E_M{}^A = |g|^{-1/4} e^a{}_\alpha e^b{}_\beta \tag{6.3}$$

yields (3.1) via (3.4) with

$$G^{(0)}_{MM'} = \frac{C}{2}|g^{(0)}|^{1/2}[g^{(0)aa'}(x)g^{(0)bb'}(x') + g^{(0)ab'}(x)g^{(0)ba'}(x') + \lambda g^{(0)ab}(x)g^{(0)a'b'}(x')]\delta^{(4)}(x-x') \;. \tag{6.4}$$

For simplicity we shall choose in the following for the reference metric

$$g^{(0)}_{ab}(x,s) = \delta_{ab} \tag{6.5}$$

and for the background metric

$$g^{\rm cl}_{\alpha\beta}(x) = \eta_{\alpha\beta} \;. \tag{6.6}$$

Moreover we make a concrete choice of orthonormal tetrad fields:

$$e^a{}_\alpha(x,s) = (g_+^{1/2})_{a\alpha} \;. \tag{6.7}$$

Here $g_+^{1/2}$ is defined by

$$\begin{aligned} g_+^{1/2} &= \eta_+^{1/2}(1+2\kappa^{1/2}\eta\psi)^{1/2} \\ &= \eta_+^{1/2}\left[1+\kappa^{1/2}\eta\psi - \frac{\kappa}{2}(\eta\psi)^2 + \cdots\right] , \end{aligned} \tag{6.8}$$

where $\eta_+^{1/2}$ is the square root of η with one positive imaginary eigenvalue.

We may now expand the drift term and the noise term in the Langevin equation for the rescaled metric $\tilde{g}_{\alpha\beta}$ defined by (2.14) in powers of $\kappa^{1/2}$:

$$|g|^{-1/2}\left[g_{\alpha\gamma}g_{\beta\delta} + \frac{\mu}{2}g_{\alpha\beta}g_{\gamma\delta}\right]\frac{\delta S}{\delta\psi_{\gamma\delta}} = W_{\alpha\beta}{}^{\gamma\delta}\psi_{\gamma\delta} + I_{\alpha\beta}(\psi,\partial\psi) \;, \tag{6.9}$$

$$|g|^{-1/4}e^a{}_\alpha e^b{}_\beta \xi^{(0)}_{ab} = 2\kappa^{1/2}[\tilde{\xi}^{(0)}_{\alpha\beta} + J_{\alpha\beta}{}^{\gamma\delta}(\psi)\tilde{\xi}^{(0)}_{\gamma\delta}] \;, \tag{6.10}$$

$$I = \sum_{n=2}^{\infty}\kappa^{(n-1)/2}I^{(n)} \;, \tag{6.11}$$

$$J = \sum_{n=1}^{\infty}\kappa^{n/2}J^{(n)} \;, \tag{6.12}$$

$$\begin{aligned} &\langle \tilde{\xi}^{(0)}_{\alpha\beta}(x,s)\tilde{\xi}^{(0)}_{\alpha'\beta'}(x',s')\rangle \\ &\quad = (\eta_{\alpha\alpha'}\eta_{\beta\beta'} + \eta_{\alpha\beta'}\eta_{\beta\alpha'} + \mu\eta_{\alpha\beta}\eta_{\alpha'\beta'})\delta^{(4)}(x-x')\delta(s-s') \;. \end{aligned} \tag{6.13}$$

The last equation is the Fourier transform of (5.11) with respect to k and k'. The Langevin equation for ψ implied by (2.13) is

$$\dot{\psi} - iW\psi = iI(\psi,\partial\psi) + J(\psi)\tilde{\xi}^{(0)} + \tilde{\xi}^{(0)} \;. \tag{6.14}$$

This may be solved iteratively using the Schrödinger kernel H introduced in (5.13). With the initial condition $\psi(x,0)=0$ Eq. (6.14) implies

$$\psi(s) = \int_0^\infty d\sigma\, H(s-\sigma)[\,iI(\psi(\sigma),\partial\psi(\sigma)) + J(\psi(\sigma))\tilde{\xi}^{(0)}(\sigma) + \tilde{\xi}^{(0)}(\sigma)] \;. \tag{6.15}$$

The iterative solution of (6.15) yields a series of tree diagrams whose first terms are depicted in Fig. 1. The main new feature in these gravitational tree diagrams is the appearance of "stochastic vertices" corresponding to the terms $J^{(n)}$, with $n+1$ prongs, $n=1,2,\ldots$. These vertices are represented by an encircled cross in Fig. 1.

The tree series expansion implies a perturbative expansion of the stochastic average of products of ψ fields. The corresponding "stochastic diagrams" are obtained by joining all possible pairs of crosses (including the encircled ones) in the tree diagrams. (Because of the Gaussian character of $\tilde{\xi}^{(0)}$ no joinings of higher order need be considered.) A joined pair of crosses is usually denoted by just one cross, because diagrams containing single crosses do not contribute to expectation values, again because of the Gaussian character of $\tilde{\xi}^{(0)}$. We note that because we have adopted Ito's interpretation of the stochastic integral (6.15), all diagrams containing bubbles attached to stochastic vertices vanish (Fig. 2). This brings about a great simplification as compared with the rules that would follow from Stratonovich's calculus. Figure 3 shows the lowest-order nonvanishing contributions to the 1-, 2-, and 3-point functions. The crossed line denotes the stochastic propagator D introduced in (5.25).

Because of the divergences in the Feynman propagator (5.36) all stochastic diagrams will actually diverge in the limit $s\to\infty$, although the divergences will not contribute to gauge-invariant expectation values. For practical calculations it is more convenient to use the method of stochastic gauge fixing introduced by Zwanziger.[29] A complete discussion of all covariant linear stochastic gauges in linearized gravity for the case $\lambda=0$ can be found in Ref.

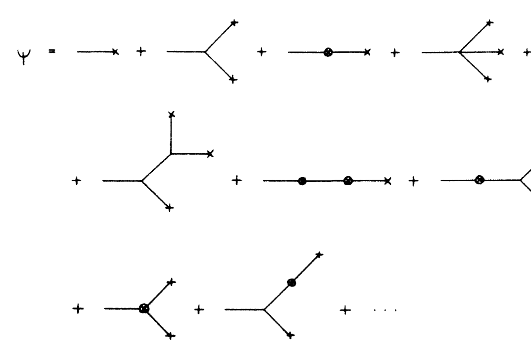

FIG. 1. Diagrammatical representation of the first terms in the tree series expansion for $\psi(s)$. The line denotes the Schrödinger kernel H, the cross denotes the stochastic source $\tilde{\xi}^{(0)}$, and the encircled cross with $n+1$ prongs denotes $\kappa^{n/2}J^{(n)}\tilde{\xi}^{(0)}$.

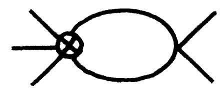

FIG. 2. Example of a subdiagram that implies the vanishing of a stochastic diagram.

8. In higher-order perturbation theory the method was first applied to the gravitational field by Fukai and Okano.[5] It appears, however, that their discussion is incomplete as they did not take into account the J term in (6.14) and the associated stochastic vertices.

VII. CONCLUSION

There are two different contexts in which the results of this paper may be interpreted. The point of view that was adopted throughout the paper was to accept the Langevin equation (2.1), though it turned out to be not covariant with respect to general field transformations, but only diffeomorphisms. Only in this way was it possible to obtain a whole family of nontrivial equilibrium path-integral measures. From this point of view the following interpretation of our results is possible: A characteristic feature of stochastic quantization is that it is based directly on a classical field equation, and no classical action or Hamiltonian is required. Classically equivalent field equations may give rise to inequivalent quantizations. This is what happens in the case of the vacuum Einstein field equations. As can be seen from (2.13), the one-parameter family of Langevin equations corresponds to the classical field equations

$$G_{\alpha\beta}-\frac{\mu}{2}g_{\alpha\beta}R=0\ , \tag{7.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor. Although $\mu=0$ looks like a natural choice, it implies the rather odd exponent $\gamma=-\frac{9}{2}$ in the equilibrium path-integral measure contained in (4.14), and the Feynman propagator of the corresponding linearization turns out to be noncausal. We have argued in this paper that the natural choice is $\mu=-1$, in which case (7.1) becomes

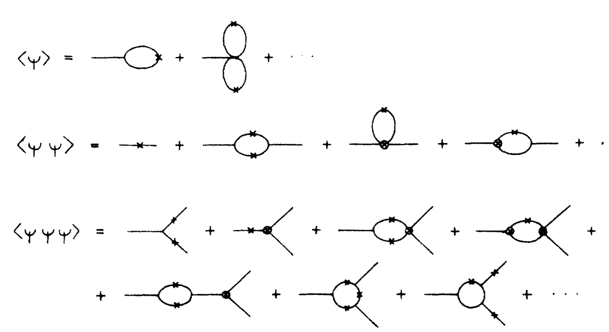

FIG. 3. Nonvanishing stochastic diagrams contributing to the 1-, 2-, and 3-point function up to third order in $\kappa^{1/2}$ (without permutations and rotations).

$$R_{\alpha\beta}=0 \tag{7.2}$$

with $R_{\alpha\beta}$ the Ricci tensor. Note the relation

$$R_{\alpha\beta}(x)=\int d^4x' G_{\alpha\beta,\alpha'\beta'}(x,x')G^{\alpha'\beta'}(x') \quad \text{for } \mu=-1 \tag{7.3}$$

in which the Ricci and Einstein tensors appear with their natural index positions.

We found that $\mu=-1$ implies $\gamma=0$ and that the corresponding field metric appears in the gauge-invariant part of the Feynman propagator. The propagator corresponding to $\mu=-1$ is causal and assumes a particularly simple form in its gauge-dependent part. Causality alone, however, allows a whole range of parameter values μ. The corresponding range of exponents γ is

$$-\tfrac{9}{4}<\gamma<9 \tag{7.4}$$

which excludes Misner's choice $\gamma=-\frac{5}{2}$.

Our results may, however, also be seen in a different context: Stochastic quantization should be generally covariant also with respect to changes of the field variable. Therefore the covariantized Langevin equation (3.22) appears to be more fundamental than (2.1). This observation does not make our calculations obsolete. Since the spurious term $-\Delta_G\Phi^A$ in (3.22) is divergent unless it vanishes, e.g.,

$$\Delta_G g_{\alpha\beta}(x)=\tfrac{9}{2}(1+\mu)\delta^{(4)}(0)\,|\,g\,|^{-1/2}g_{\alpha\beta}(x) \tag{7.5}$$

the Langevin equation can be solved only if the field variable is harmonic. In particular the linearization is meaningful only in harmonic "coordinates." The propagator in these coordinates may be easily inferred from the result of Sec. V. Since it is related to the latter by a bilinear transformation, the causality properties are the same. The conclusion then is that one and the same equilibrium distribution, viz., (5.2), gives rise to different stochastic perturbation theories based on different variables. In general these perturbation theories are inequivalent; in particular the causality properties of the corresponding propagators may be different.

It appears worthwhile to study the perturbative aspects of the stochastic quantizations of the gravitational field. It cannot be excluded that there is a parameter value for which the theory is finite. If nontrivial divergences are present, however, a subtraction scheme at finite s will have to be devised before meaningful calculations can be made.

Finally we remark that in the causal range $\mu<-\frac{2}{3}$ the stochastic perturbation theory may even be performed in Euclidean space-time, because then the original Parisi-Wu ansatz implies only decreasing exponentials in (5.24).

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