Evolution of cosmic strings

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(Received 16 September 1985)

The evolution of a system of cosmic strings is studied using an extended version of an analytic formalism introduced by Kibble. It is shown that, in a radiation-dominated universe, the fate of the string system depends sensitively on the fate of the closed loops that are produced by the interactions of very long strings. The strings can be prevented from dominating the energy density of the Universe only if there is a large probability ($\gtrsim 50\%$) that a closed loop will intersect itself and break up into smaller loops. A comparison with the numerical simulations of Albrecht and Turok indicates that the probability of self-intersection is indeed large enough to allow the energy density in strings to stabilize at a small fraction of the radiation density, but there is a potential problem with the gravitational radiation that is produced by the strings. If the string tension μ is too large, then the gravitational radiation will be so copious that it interferes with primordial nucleosynthesis. By assuming that the probability of self-intersection is less than 85%, as the comparison with the results of Albrecht and Turok indicates, an upper bound on the string tension is obtained: $G\mu < 10^{-6}$. (G is Newton's constant.) This is slightly smaller than the values ($\gtrsim 2\times 10^{-6}$) predicted for the cosmic-string theory of galaxy formation. This bound would become significantly lower if the probability of intersection is less than 85%.

I. INTRODUCTION

Topologically stable cosmic strings appear naturally as a consequence of spontaneous symmetry breaking in many grand unified theories,[1,2] as well as in the low-energy sector of superstring theories.[3] There has recently been some interest in the idea that these cosmic strings, formed at a phase transition in the early Universe could have served as seeds for the primordial density fluctuations that are responsible for the large-scale structure of the Universe. This idea, originally due to Zel'dovich[4] and Vilenkin,[5] is based on the assumption that, in a radiation-dominated universe, the strings will evolve in a scale-invariant manner, so that at any given time, the length scale and the energy density of the string system will depend only on the horizon size. In this picture, the energy density of the string system would evolve as a small, constant fraction of the energy density in radiation. Many authors[6-11] have explored the details of Vilenkin's scenario in which individual loops of string serve as the "seeds" for the collapse of individual galaxies, and loops of a much larger size may be responsible for the formation of clusters and superclusters.

Very recently, many papers have appeared arguing that the cosmic-string model for galaxy formation has many advantages not shared by other galaxy-formation scenarios. For instance, Turok and Brandenberger have found that the string theory induces density fluctuations that do not conflict with the observed microwave-background-radiation isotropy.[12] They have also claimed[13,14] that the cosmic-string model accurately predicts the correlation function of Abell's rich clusters of galaxies, and is the only theory of galaxy formation to do so. Similarly, Albrecht, Brandenberger, and Turok[15] have argued that voids and superclusters, which seem to present a severe problem for other theories of galaxy formation, can be explained by the cosmic-string theory of galaxy formation. Finally, Schramm[16] has argued that the large-scale structure of the Universe can be described as scale-free, and that such a scale-free spectrum might be best described by the string scenario.

Despite these successes, the basic premise that the string system evolves in a scale-invariant manner has not been proven to be correct, although substantial progress in this direction has been made by Albrecht and Turok.[17] In order to see why energy loss presents a difficult problem, let us consider a very simple string system: a radiation-dominated universe filled with long straight strings. In this case, as the Universe expands, the length of string in a comoving volume grows as R while the volume increases as R^3, where R is the scale factor. Thus, the string density falls as $1/R^2$, much slower than the $1/R^4$ falloff required for scale invariance. In fact, in this case, the strings will quickly come to dominate the Universe, a cosmological disaster. Now, assume that instead of being straight, the strings have a Brownian configuration with some persistence length L, which is much smaller than the horizon. In this case, the stretching of the strings due to the expansion is negligible compared to their total length. This is because for a Brownian string, the length of string between two points separated by a distance $d \gg L$ is $\sim d^2/L$. So, for Brownian strings, the density falls off as $\sim 1/R^3$, which would be safe in the matter-dominated epoch but disastrous in a Universe dominated by radiation. For a consistent cosmic-string scenario, it is necessary that there exist some energy-loss mechanism for the system of strings.

The production and decay of closed loops is the only such mechanism. A closed loop can be formed when two segments of string intersect. If the segments intercommute (change partners) when they intersect, then loops can be formed if the colliding segments are part of the same string or upon two intercommutings of different strings. The probability of intercommuting is not known *a priori*, but the preliminary indications from a calculation by Shellard[18] are that the probability is of order one. If all closed-loop trajectories intersect themselves at some point during their period (the trajectories are all periodic inside the horizon), then the loops can rapidly decay into smaller loops by successive intercommutings. Because the period of the loops is proportional to their size, it will only take about one expansion time until the loops are small enough to radiate massive particles such as Higgs bosons or gauge bosons. For gauge strings (those formed as a result of a spontaneously broken gauge symmetry), this mechanism can be ruled out because of an important result obtained by Kibble and Turok.[19] They have shown that there is a large class of loop trajectories that never self-intersect, so that the fragmentation of the loops will be truncated when the average loop size is not much smaller than the horizon, much too large to decay into heavy particles.

The only option then is for the strings to radiate massless particles. For global strings (those formed as a result of the spontaneous breaking of a global symmetry), loop decay is very fast ($\sim$ a few expansion times) through the radiation of Goldstone bosons.[20] (Global strings may also decay by successive intercommutings.) But since gauge strings couple to no massless particles except for gravitons,[21] they can only radiate gravitationally. Since the strings couple to gravity only very weakly, they will live for many expansion times before they decay. It is essential to the galaxy formation scenario that this occurs because, in order to give rise to a significant density fluctuation, a loop must live for a long time.

It is not clear that this energy-loss mechanism will be efficient enough so that the energy density of strings will scale as $1/R^4$ as is necessary for the consistency of the string scenario. Since a loop lives for a long time, there is a large probability that it will collide with and be absorbed by a long string before it can radiate away much of its energy. (The probability for a loop to be absorbed by a long string decreases considerably if there is a large probability that a loop will self-intersect and split up into smaller loops. This is one of the reasons that the fate of a system of cosmic strings depends sensitively on the probability of self-intersection.)

Once a loop has been absorbed by a long string, any energy lost to gravitational radiation will be more than compensated for by an energy gain due to the stretching of the string. The numerical calculation of Albrecht and Turok is an important step toward confirming previous speculations that the string scenario is indeed consistent, but because of the limitations of their calculation some doubts still remain. Their main constraint is the finite number of points on their lattice. (They use an 80^3 cubic lattice.) Because of this limitation, they can only run their simulations for a factor of ~ 10 in time before the horizon grows as large as their whole lattice. As a consequence of this they can never achieve a solution that is scale invariant, at least on the scale of small loops. Of course, small loops tend to decouple, and Albrecht and Turok have checked and found that their results are not strongly dependent on the loop size they use as a lower cutoff. But, it is conceivable that their results are an artifact of their initial conditions which had a much smaller proportion of closed loops than their "steady state" (or scale-invariant) solution. The relative lack of closed loops in the initial state means that the long or "infinite" strings can lose energy through the production of loops, but that they will not gain as much energy by the absorption of loops as they would if more loops were present. Although it seems likely that they have evolved their simulations long enough to avoid this problem, it is an important question to check. Another possibility is that the evolution of the string system might differ only slightly from the $1/R^4$ falloff of a scale-invariant system, changing too slowly to be seen in the numerical simulation, but fast enough to cause a conflict with known cosmology.

An alternative approach for calculating the evolution of a network of cosmic strings has been attempted by Kibble.[22] He set up equations for the formation and absorption of closed loops by long strings in order to study this problem analytically. In this paper, an improved version of Kibble's formalism is developed and used to study the evolution of a system of cosmic strings and test the consistency of the string theory of galaxy formation. With this approach, we have no limit to the size of the loops that we can consider or to the length of time that the system can evolve. It is also easy to see the effect of varying the intercommuting probability. The drawback is that we must absorb many of the details of the string evolution into unknown parameters. Although we can obtain bounds on these parameters analytically, we can only determine their actual values through comparison with a calculation like that of Albrecht and Turok. On the other hand, these parameters are actually useful if we wish to understand which physical processes have a significant influence on the evolution of the string system because we are able to vary them. For instance, we find that, if a scale-invariant solution is to exist, then there must be a high probability ($\gtrsim 0.5$) that an arbitrary loop produced by the intercommutation of long strings with self-intersect and break up into smaller loops.

The final advantage to this approach is that small loops can be treated almost exactly, and since the energy density of the string system is dominated by small loops, it is important to what the density of small loops really is. For instance, in the cosmic-string theory of galaxy formation, the small loops are responsible for density fluctuations, so if we want to know the magnitude of the initial fluctuations, we must know the density in small loops. Also, because the density of gravitational radiation is proportional to the total energy density in strings, we find that we can obtain a limit on the string tension from the requirement that the density of gravitational radiation be small enough to satisfy primordial nucleosynthesis constraints. This constraint is much stronger, at present, than the constraint on the density of gravitational radiation obtained from the timing of the millisecond pulsar.[23,24] With an estimate of the probability of self-intersection obtained from the results of Albrecht and Turok's simulation we obtain $G\mu < 10^{-6}$. Such a limit is marginally in conflict with the values, $G\mu \gtrsim 2\times 10^{-6}$, obtained from the galaxy formation calculations of Refs. 2, 7–9, 13, and 14. Thus, primordial nucleosynthesis seems likely to provide an important constraint on the cosmic-string theory of galaxy formation.

This paper is organized as follows. Section II is devoted to a review of Kibble's formalism with a few minor extensions. In the third section, we solve Kibble's string evolution equation and show that no scale-invariant solution exists if we neglect the self-intersection of loops. In Sec. IV we modify Kibble's formalism to include several processes that Kibble neglected including the self-intersection of loops and gravitational radiation. We then present the solution of these equations. In Sec. V we compare our results with those of Albrecht and Turok, and finally, in Sec. VI, we discuss our conclusions.

II. STRING EVOLUTION EQUATIONS

We will now review the formalism for the evolution of a system of strings in an expanding universe as developed by Kibble.[22] In Sec. II A we will introduce the equations of motion for a noninteracting system of strings. Interactions between loops and long strings are added in Sec. II B.

A. String equations of motion

In the cosmological setting that concerns us, we may neglect the thickness of the string and treat it as a truly one-dimensional object. With the exception of gravitational radiation (which will be added in a later section) the gravitational interactions of the strings will also be neglected. The justification for this is that the strings couple to gravity with strength $G\mu \sim 10^{-6}$ so gravity should have very little influence on the string motion.[25] We will work in a Robertson-Walker space-time with the metric

$$ds^2 = R^2(\tau)(d\tau^2 - d\mathbf{x}^2) . \tag{2.1}$$

If we take σ to be the parameter denoting position along the string we can write the expression for the total string energy inside a comoving volume V as[26]

$$E = R \int_V d\sigma\, \epsilon , \tag{2.2}$$

where

$$\epsilon = \left\{ \left[\frac{\partial \mathbf{x}}{\partial \tau} \right]^2 \Big/ \left[1 - \left[\frac{\partial \mathbf{x}}{\partial \sigma} \right]^2 \right] \right\}^{1/2} , \tag{2.3}$$

and the integration is over all strings in the volume V. Implicit in these formulas is the assumption that

$$\frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \sigma} = 0 , \tag{2.4}$$

which can be satisfied by a proper choice of the parameter σ provided we are in the center-of-mass reference frame.

In these coordinates the equation of motion for the string is[26]

$$\frac{\partial}{\partial \tau} \left[\epsilon \frac{\partial \mathbf{x}}{\partial \tau} \right] + \frac{2}{R} \frac{\partial R}{\partial \tau} \epsilon \frac{\partial \mathbf{x}}{\partial \tau} = \frac{\partial}{\partial \sigma} \left[\frac{1}{\epsilon} \frac{\partial \mathbf{x}}{\partial \sigma} \right] . \tag{2.5}$$

From (2.5) we can immediately derive expressions for the time derivatives of ϵ and E,

$$\frac{\partial \epsilon}{\partial \tau} = -\frac{2}{R} \frac{\partial R}{\partial \tau} \epsilon \left[\frac{\partial \mathbf{x}}{\partial \tau} \right]^2 , \tag{2.6}$$

$$\dot{E} = E \frac{\dot{R}}{R} (1 - 2\langle v^2 \rangle) , \tag{2.7}$$

where

$$\langle v^2 \rangle_\sigma \equiv \frac{\int d\sigma\, \epsilon \left[\frac{\partial \mathbf{x}}{\partial \tau} \right]^2}{\int d\sigma\, \epsilon} .$$

The dots in (2.7) denote derivatives with respect to ordinary time, $dt = R\, d\tau$.

For very straight strings, we can neglect the term on the right-hand side of (2.5) and obtain a very small terminal velocity. Thus, in this limit (2.7) implies that the energy in a comoving volume grows as R, so the string density would fall as $1/R^2$ as noted above. For very small loops or Brownian strings with a persistence length much smaller than the horizon, we can neglect the damping term on the left-hand side of (2.5). It is then simple to derive the result that $\langle v^2 \rangle = \frac{1}{2}$ ($\langle\ \rangle$ denotes an average over t as well as σ). In this case (2.5) implies that the energy in a comoving volume is constant, as we might expect. Hence, in the absence of intercommuting the density of strings drops no faster than $1/R^3$.

B. Interactions between loops and long strings

In order to test the standard string evolution scenario, we will now set up rate equations for the interactions between infinite or long strings and loops, following Kibble. We will classify as long strings all the infinite strings as well as all the "large" loops. (A precise definition of large loops will be given below.) The scale of the long strings, L, will be defined by

$$E = \frac{\mu V}{L^2} , \tag{2.8}$$

where E is the energy in long strings contained in the volume V. If we define $\omega(v)dv$ as the probability that a given string segment has a velocity in the range v to $v+dv$, then the number of string segments between v and $v+dv$ intersecting a surface of area A is given by

$$A \frac{(1-v^2)^{1/2}}{2L^2} \omega(v) dv . \tag{2.9}$$

The factor of $(1-v^2)^{1/2}$ in (2.9) comes from the fact that a string segment of energy $\mathscr{E}$ has a length equal to $(1-v^2)^{1/2}\mathscr{E}/\mu$, and the factor of $\frac{1}{2}$ is just an average of $\cos\theta$.

If we ignore the correlations between different string segments, the probability that, in the time interval δt, a string segment of proper length $2\pi l_1$ (energy $= 2\pi\mu l_1$) and velocity v_1 will collide with a long string is

$$\int |\mathbf{v}_1+\mathbf{v}_2|\,\delta t 2\pi l_1(1-v_1{}^2)^{1/2}\frac{(1-v_2{}^2)^{1/2}}{2L^2}\omega(v_1)dv_1\omega(v_2)dv_1\frac{\sin\theta_{12}d\theta_{12}}{2} = \frac{\pi l_1 \delta t}{L^2}\langle \max(v_1,v_2)(1-v_1{}^2)^{1/2}(1-v_2{}^2)^{1/2}\rangle \ , \quad (2.10)$$

where, to obtain the last expression, we have integrated over θ_{12} (the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$). In order to evaluate

$$\langle \max(v_1,v_2)(1-v_1{}^2)^{1/2}(1-v_2{}^2)^{1/2}\rangle \ ,$$

we would need to know the correct form for $\omega(v)$, which is not easy to obtain. The case in which we are the most interested is when the scale size of the strings is slightly smaller than the horizon size, but we can only obtain a good estimate of

$$\langle \max(v_1,v_2)(1-v_1{}^2)^{1/2}(1-v_2{}^2)^{1/2}\rangle$$

when the string scale is much smaller than the horizon. If we take $\omega(v)$ to be the same as it is for a small circular loop ($v=\sin t/\pi l$), then we obtain

$$\frac{8}{3\pi^2}=\langle \max(v_1,v_2)(1-v_1{}^2)^{1/2}(1-v_2{}^2)^{1/2}\rangle \ . \quad (2.11)$$

Fortunately, the results of our calculation will not be very sensitive to the exact value of

$$\langle \max(v_1,v_2)(1-v_1{}^2)^{1/2}(1-v_2{}^2)^{1/2}\rangle \ .$$

An essential requirement of the standard energy-loss scenario for the string model is that, once formed, many loops will survive without reconnection for a very long time in order to radiate away. Thus, it is important to know the probability that a loop will survive once it is formed. This depends on the probability p that two string segments will change partners when they collide. Although it is not really known what this probability is, it is usually assumed that $p\approx 1$ is required for the standard scenario to work, and there is apparently some indication that this is true.[18] We will use $p=1$ in all our calculations, and in the concluding section, it will be shown how to generalize these results to other values of p. Before we write down the expression for the probability of loop survival, it is useful to introduce some new notation: let $\gamma=L/t$ be a measure of the ratio between the string scale and the horizon, and let $L\sim t^{\alpha}$. ($\alpha=1$ gives the scale-invariant evolution that is assumed in the standard scenario of string evolution.) From (2.10), the probability for a loop of proper radius l formed at time t to survive until $t=\infty$ is

$$\exp\left[-(2\alpha-1)\frac{pl\pi}{L\gamma}\langle \max(v_1,v_2)(1-v_1{}^2)^{1/2}(1-v_2{}^2)^{1/2}\rangle\right] \sim e^{-p\bar{v}l/\gamma L} \ . \quad (2.12)$$

Thus, if $\gamma\sim 1$, loops of a size much smaller than L will almost certainly survive while large loops will not. On the other hand, if $\gamma \ll 1$, we can only expect extremely small ($l<\gamma L$) loops to survive.

In order to discuss interactions between loops and long strings, it is convenient to define the number density of loops with radii between l and $l+dl$ to be

$$n(l)dl=\frac{1}{2\pi\mu l}\frac{E}{V}f\left[\gamma,\frac{l}{L}\right]\frac{dl}{L} \ , \quad (2.13)$$

so that the energy density in loops of size l to $l+dl$ is

$$\frac{E}{V}f\left[\gamma,\frac{l}{L}\right]\frac{dl}{L} \ .$$

We can use (2.10) to write down the expression for the energy gained by long strings by absorption of closed loops

$$\dot{E}_{\text{from loops}}=E\frac{p\bar{v}}{L}\int f(\gamma,x)x\,dx \ , \quad (2.14)$$

where we have defined

$$\bar{v}\equiv\pi\langle \max(v_1,v_2)(1-v_1{}^2)^{1/2}(1-v_2{}^2)^{1/2}\rangle \ , \quad (2.15)$$

and

$$x\equiv\frac{l}{L} \ .$$

Similarly, the production of closed loops can be described by

$$\dot{E}_{\text{to loops}}=-E\frac{p\bar{v}}{L}\int a(x)x\,dx \ , \quad (2.16)$$

where the loop production function $a(x)$ is defined by this expression. We should note that we have already made an implicit assumption by writing down (2.16). That is, we have assumed that $a(x)$ depends only on x and not on γ, the ratio between L and the horizon size. It is possible that the detailed form of $a(x)$ may depend on γ, but most of the dependence on γ has been factored out in (2.16). Since the fate of the string system depends sensitively on the detailed form of $a(x)$ for only a small range of γ, we are probably safe in neglecting any residual dependence.

Combining (2.16) and (2.14) with (2.7) we obtain an expression for the rate of change of the energy density in long strings:

$$\frac{\dot{E}}{E}=\frac{\dot{R}}{R}(1-2\langle v^2\rangle)+\frac{p\bar{v}}{L}\int x[f(\gamma,x)-a(x)]dx \ . \quad (2.17)$$

We can write down a similar equation for the time derivative of the energy in closed loops:

$$\frac{d}{dt}\left[\frac{E}{L}f\left[\gamma,\frac{l}{L}\right]\right]=E\frac{p\bar{v}}{L^2}[xa(x)-xf(\gamma,x)] \ . \quad (2.18)$$

(Note that we have neglected any stretching of the loops.) Using (2.8) to eliminate E from (2.17) and (2.18), we can obtain two coupled equations which describe the evolution of the system of loops and long strings:

$$3\frac{\dot{R}}{R}-2\frac{\dot{L}}{L}=\frac{\dot{R}}{R}(1-2\langle v^2\rangle)+\frac{p\bar{v}}{L}\int x[f(\gamma,x)-a(x)]dx\ , \tag{2.19}$$

$$3\left[\frac{\dot{R}}{R}-\frac{\dot{L}}{L}\right]f(\gamma,x)-\frac{\dot{L}}{L}xf'(\gamma,x)+\dot{\gamma}\frac{\partial}{\partial\gamma}f(\gamma,x)=\frac{p\bar{v}}{L}[xa(x)-xf(\gamma,x)]\ . \tag{2.20}$$

(If we drop the $\dot{\gamma}$ term and set $\bar{v}=v$, these are just the evolution equations derived by Kibble.)

We have taken two different approaches to solve these equations. The first approach, discussed in Secs. III and IV, is motivated by the scaling solution discussed by Kibble. We assume specific forms for $R(t)$, $L(t)$, $\gamma(t)$, and $a(x)$, which allow an exact solution of (2.19) and (2.20). This exact solution can then be used to calculate the deviations of $R(t)$, $L(t)$, and $\gamma(t)$ from their assumed forms. We use this approach for our main analysis, and it has the advantage that we can evolve the system for an arbitrarily long time. The second approach will be to integrate (2.18) and (2.20) numerically. Here, we are limited because the solution eventually develops numerical instabilities, but we are able to include arbitrary initial conditions. This approach is useful when trying to check the results of Albrecht and Turok.[17]

III. SOLUTION TO KIBBLE'S EVOLUTION EQUATIONS

In this section we solve Kibble's evolution equations with a simple choice for the loop production function $a(x)$ and we show that, unless the self-intersection of loops or string correlations play an important role, a scale-invariant (or scaling) solution cannot exist. Before we can show this, however, we must obtain some limits on $a(x)$.

A. Limits on the loop production function

Before we attempt to solve (2.19) and (2.20), we need to discuss the unknown loop production function $a(x)$. An estimate of $\int a(x)dx$ can be obtained by considering the number of collisions between segments of long strings in a volume V. The probability of a collision can be obtained by multiplying the probability that a string segment of proper length $2\pi l_1$ will collide with a long string (2.10), by the number of segments in the volume V, $V/(2\pi l_1 L^2)$. Multiplying the collision probability by p, we obtain the intercommuting probability

$$\frac{V}{L^2}\frac{\delta t}{2\pi L^2}p\bar{v}\ . \tag{3.1}$$

From (2.16), we can see that the energy converted to loops of size between l and $l+dl$ in the same time interval is

$$\Delta E_{\text{to loops}}(l)dl=E\delta t\frac{p\bar{v}}{L}\frac{l}{L}a\left[\frac{l}{L}\right]\frac{dl}{L}\ . \tag{3.2}$$

Dividing (3.2) by $2\pi\mu l$ (the energy per loop) and setting it equal to (3.1) we obtain

$$F_l=\int a(x)dx,\quad F_l<1\ , \tag{3.3}$$

where F_l is the fraction of long string intercommutings that produce new loops. $F_l<1$ is strictly true only when the loops are produced by collisions of uncorrelated segments of long strings, but loops are generally produced as a result of collisions of waves on a single long string.[27] This process clearly involves correlations between the different string segments. Nevertheless, it seems quite unlikely that loop production will occur at a higher rate than the crossing of random segments of long strings.

An upper bound on $\int a(x)dx$ has been suggested by Kibble.[22] He noted that, in a nonexpanding universe (neglecting gravitational radiation), the equilibrium solution of (2.19) and (2.20) is

$$a(x)=f(\gamma,x),\quad \dot{L}=0\ . \tag{3.4}$$

[We have implicitly assumed that $a(x)$ does not depend on the expansion rate of the Universe.] If the equilibrium solution resembles the random configuration in which the strings are formed, then only about 20% of the total length of string would be in the form of loops.[24] This implies that

$$\int a(x)dx=\frac{\text{length in loops}}{\text{length not in loops}}=0.25\ . \tag{3.5}$$

B. Solution to the evolution equations neglecting correlations between string segments

Now, let us see if either of these restrictions (3.3) or (3.5) are consistent with the scaling solution that is usually assumed. We will take $L\sim t^{\alpha}$ and $R\sim t^{1/2}$ (for a radiation-dominated universe) so that a solution with $\alpha=1$ corresponds to the Kibble's scaling solution. Thus, (2.19) and (2.20) become

$$2(\alpha-1)=\langle v^2\rangle-1+\frac{p\bar{v}}{\gamma}\int x[a(x)-f(\gamma,x)]dx\ , \tag{3.6}$$

$$0=\alpha xf'(\gamma,x)-(\alpha-1)\gamma\frac{\partial}{\partial\gamma}f(\gamma,x)+3(\alpha-\tfrac{1}{2})f(\gamma,x)+\frac{p\bar{v}}{\gamma}x[a(x)-f(\gamma,x)]\ . \tag{3.7}$$

Using $\alpha=1$, the solution to (3.7) was found by Kibble to be

$$f(\gamma,x)=\frac{p\bar{v}}{\gamma}\int_x^{\infty}dy\left[\frac{y}{x}\right]^{3/2}e^{p\bar{v}(x-y)/\gamma}a(y)dy\ . \tag{3.8}$$

Substituting this into (3.6) we obtain the condition

$$2(\alpha-1)=\langle v^2\rangle-1+\frac{p\bar{v}}{\gamma}\int_0^\infty dy\, y a(y)\left[1-\frac{p\bar{v}}{\gamma}\int_0^y dx\left[\frac{y}{x}\right]^{1/2} e^{p\bar{v}(x-y)/\gamma}\right]$$

$$=\langle v^2\rangle-1+\int_0^\infty dy\, a(y)\frac{p\bar{v}}{\gamma} y e^{-p\bar{v}y/\gamma} M\left[-\frac{1}{2},\frac{1}{2},\frac{p\bar{v}y}{\gamma}\right], \tag{3.9}$$

where $M(-\frac{1}{2},\frac{1}{2},p\bar{v}y/\gamma)$ is Kummer's function.[28]

In order for a scaling solution to exist the integral on the right-hand side of (3.9) must be large enough so that it can cancel the term $\langle v^2\rangle-1$. It is somewhat surprising, then, that the integrand in (3.9) becomes negative for $p\bar{v}y/\gamma>0.85$. This implies that the production of large loops is not an efficient way for the long strings to lose energy. Because they have a high probability to reconnect, they are very unlikely to survive long enough to radiate away. However, we have not included the process in which a large loop produces smaller ones in our Eqs. (2.19) and (2.20), and, therefore, we are in danger of underestimating the energy loss through large loops. For this reason, we will always cut off the loop production function and include loops larger than the cutoff with the long strings. If some of these large loops survive long enough to shrink below the cutoff size, we can treat them as if they are formed when they are at the cutoff size.

As a first approximation, let us ignore loop fragmentation and take

$$a(x)=F_l\delta(x-x_0)\ ,$$

where x_0 is an unknown constant presumably of order one. Then, (3.9) becomes

$$2(\alpha-1)=\langle v^2\rangle-1+F_l I\left[\frac{p\bar{v}x_0}{\gamma}\right], \tag{3.10}$$

where $I(z)\equiv ze^{-z}M(-\frac{1}{2},\frac{1}{2},z)$. A graph of $I(z)$ vs z is given in Fig. 1. Setting $\langle v^2\rangle=\frac{1}{2}$ and $I(p\bar{v}x_0/\gamma)\simeq 0.15$ (their maximum values), and using F_l, 1, we obtain an upper limit on α,

$$\alpha=\tfrac{3}{4}+\tfrac{1}{2}F_l(0.15)\le 0.825\ , \tag{3.11}$$

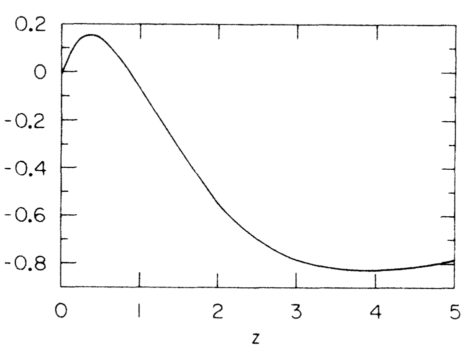

FIG. 1. $I(z)=ze^{-z}M(-\frac{1}{2},\frac{1}{2},z)$ vs z.

which is close to its minimum value: $\alpha=0.75$. Note that we have chosen the function $a(x)$ such that the integral in (3.9) is maximized, subject to the constraint (3.3), so no other choice for $a(x)$ will give as large a value for α. [If we have used Kibble's constraint (3.5), we would obtain a more stringent limit: $\alpha\le 0.769$, and if we had used the loop production function that Kibble suggested, $a(x)=0.4/x^3$, we would get an even smaller result.] Thus, the scaling solution, $\alpha=1$, cannot be satisfied. This means that the density in strings will decrease more slowly than that in radiation, and the strings will soon come to dominate the Universe. Therefore, an additional energy-loss mechanism is required for the string theory of galaxy formation to be consistent.

IV. CORRECTIONS AND SOLUTIONS TO THE EVOLUTION EQUATIONS

The purpose of this section is to extend Kibble's formalism to include some of the effects of correlations between different string segments, and to solve the resulting evolution equations.

A. The self-intersection of loops

The mechanism that we have ignored is the self-intersection and fragmentation of loops. In showing the failure of the scaling solution we considered the production and reabsorption of loops by long strings but we ignored all loop-loop interactions. In principle, we should include both the process in which two loops intersect and form a larger loop and the inverse process in which a loop self-intersects and splits up into a number of smaller loops. This second process is particularly important because self-intersection is likely to be quite common, and because the smaller loops are less likely to recombine with the long strings. Kibble and Turok[19,8] have shown that there is a large class of loops that never self-intersect which are, in fact, essential for the galaxy-formation scenario. However, it is unknown what fraction of the loops formed by the long strings will self-intersect. It seems likely, however, from the work of Turok and Kibble, that this fraction is neither very close to zero nor to one. We will account for the self-intersection process by an appropriate modification of the loop production function $a(x)$. The other process in which two loops combine to form a larger loop is likely to have a much smaller influence on the overall evolution of the string system, and we can include it as a minor correction to the self-intersection probability.

We can model this behavior very simply if we assume that there is a probability p_{SI} that a loop will break up into two equal-sized pieces. Then the ratio of the energy in loops of proper radius between 0 and $l/2$ to the energy in loops of size between $l/2$ and l is $p_{SI}/(1-p_{SI})$. We can then define an effective loop production function $a_{eff}(x)$ to have the form

$$a_{eff}(x)=A_n x^n \theta(\xi-x), \quad \xi\sim 1, \quad p_{SI}=(\tfrac{1}{2})^{n+2} \tag{4.1}$$

which has the correct behavior for all x less than ξ. [Recall that $xa(x)$ is proportional to the energy produced in loops of size x.] We have implicitly assumed that the process of successive self-intersections actually takes place rather rapidly and this is indeed the case. Because the loop oscillations are periodic with a period proportional to the proper length of the loop, daughter loops will self-intersect in roughly half the time it took their parent to split up, so the loop fragmentation process should complete itself in roughly one oscillation time. If $p<1$, the fragmentation process should take roughly $\sim 1/p$ oscillation times.

The correct value for ξ is given by our assumption that the scale of curvature on a string is roughly the same as the average distance between strings. (An argument justifying this assumption is given in the final section.) If we take $\xi=1.5$, as we have done in most of our calculations, then the average parent loop has a radius of $\approx 0.8L$ (assuming its average velocity is $v_{rms}=1/\sqrt{2}$), while from (2.9), the average distance between neighboring segments of long strings is $\approx 1.7L$. Thus, $\xi=1.5$ implies that the diameter of the average parent loop is roughly the same as the average distance between the long strings.

Another way to check that we have picked an appropriate value for ξ is to recall that our justification for cutting off the loop production function at $x=\xi$ was that loops of size larger than ξ would not survive long before being readsorbed by the network of long strings. When p_{SI} is large, it is particularly important to check whether a large parent loop will survive long enough to intersect itself. If $\xi>2/\gamma$, a loop of proper radius ξ will be stretched by the expansion and will not self-intersect until the horizon has grown much larger, but in the meantime, it will probably be reconnected to a long string. Thus, $\xi<2/\gamma$ should be an upper limit on ξ. For $\gamma<1$ (which holds in all the cases of interest), this constraint is easily satisfied for $\xi=1.5$.

We must also check that loops of size $\gtrsim\xi$ have only a small probability (p_{sur}) to survive long enough to self-intersect before they intercommute with a long string. By integrating (2.10), we can obtain the probability that, in a time interval, $\delta t=\pi\xi L/2$, a loop of proper radius ξL will not collide with a long string. (This value of δt is just the time it takes for a loop initially at rest to self-intersect.) Using (2.11), we obtain

$$p_{sur}=\exp\left[-\frac{4\xi^2}{3(1+\pi\xi\gamma/2)}\right]. \tag{4.2}$$

For $\xi=1.5$ and $\gamma=0.6$, this probability is 29%. ($\gamma=0.6$ is the value that seems to match Albrecht and Turok's results.) So, perhaps a larger value for ξ is appropriate, but since this would make my final limit on $G\mu$ more strict, I will take $\xi=1.5$ to be a conservative choice.

In order to relate $a_{eff}(x)$ to F_l, we will take

$$a_p(x)=\frac{F_l}{\ln 2}\frac{1}{x}\theta(x-\xi/2)\theta(\xi-x), \quad \xi\sim 1 \tag{4.3}$$

to be the "primordial" loop production function which is reduced to $a_{eff}(x)$ by the self-intersection process. [Equation (4.3) is normalized according to (3.3).] Since energy is conserved during these self-intersections, we require that

$$\int_0^\infty xa_p(x)dx=\int_0^\infty xa_{eff}(x)dx ,$$

or

$$A_n=\frac{n+2}{2\ln 2}\xi^{-n-1}F_l . \tag{4.4}$$

We should note that there are two effects that may tend to interfere with the relation between the probability of self-intersection p_{SI} and n. First, we have assumed that the loop fragmentation process takes place instantaneously and neglected the possibility that a loop may reattach itself to a long string before it can break up into smaller loops. A loop of radius l that is formed almost at rest will first intersect itself (if it is of the self-intersecting variety) at time $\sim\pi l/2$, so the probability that the loop will split into two before it can recombine is $\sim e^{-pvx^2}$, where $x=l/L$. Unless $x\ll 1$, the probability that the loop will recombine before it splits up is not negligible, so our assumption that the loops instantaneously fragment may be faulty. On the other hand, although one part of the loop may recombine with a long string, the rest of the loop will not "know" that it has recombined until enough time has elapsed for a wave to travel half way around the loop. In the meantime, the rest of the loop will continue to fragment; hence, the final spectrum of daughter loops may be almost the same as if the parent loop finished fragmenting before the collision with the long string. Our other assumption in making the identification (4.1) was that the probability of self-intersection is independent of loop size. In fact, this is unlikely to be the case. The periodic non-self-intersecting solutions of Turok and Kibble only exist when the loops are much smaller than the horizon; larger loops are not periodic and are therefore more likely to self-intersect. Since the larger loops are also the most likely to be absorbed by the long strings, the errors from these two assumptions tend to cancel, so our treatment of the self-intersection process is probably reasonable.

B. Other corrections to the string evolution equations

Before we attempt to solve (2.19) and (2.20), we must introduce a further correction factor to account for the fact that we have overestimated the probability of loop reconnection for small loops. In our original treatment, we treated all the string segments as if they were uncorrelated; this assumption is particularly bad in the case of small loops. If $p=1$, a long string can only intercommute when it strikes the near side of the loop. Thus, a small loop has an effective length equal to $\sim\frac{1}{2}$ of its true length. For a loop size $x\sim 1$, however, the effective length is much closer to its true length. Similarly, when a loop intersects a long string and is absorbed, the inertia of the original loop may cause the trailing edge of the loop to intersect the long string again, producing a new loop. We can take both processes into account by adding a correction factor $\delta<1$ in front of the $(p\bar{v}/L)xf(\gamma,x)$ terms in (2.19) and (2.20), implying that the energy returned to the long strings by the reconnection of loops is only a fraction δ of the energy that would be returned from a network of straight segments of equal total length. Presumably, these processes together should contribute a factor of ~ 0.5 to δ. In our calculations, we will take $\delta \approx 0.5$ to be the most likely value and $\delta \geq 0.3$ as a lower limit on δ.

Another effect that we can incorporate into the correction factor δ is the red-shifting of loops. So far we have implicitly assumed that loops are formed at rest. Since the rms velocity of the strings is $\sim 1/\sqrt{2}$, we might expect that the loops formed will have a non-negligible velocity. If so, the loops could lose a large fraction of their energy by red-shifting as the Universe expands. It seems likely, however, that the rms velocity of the loops produced will be considerably smaller than $1/\sqrt{2}$. This means that the loops can lose some fraction $<30\%$ of their energy by red-shifting if they survive for a time of the order of one expansion time. The simulations of Albrecht and Turok indicate that the kinetic energy of the loops is indeed much smaller than this;[13] the average velocity of the daughter loops in their simulation is only ~ 0.1. Thus, we can expect that the loops will lose $<3\%$ of their energy by red-shifting. Loops formed as a result of very many self-intersections may gain kinetic energy at each self-intersection, and so they may have a higher velocity than this. (These loops would be smaller than the cutoff in Albrecht and Turok's simulations, so they would not see them.) Fortunately, only a small portion of the total loop energy is likely to be carried by these very small loops so we will need only a small correction to account for them.

The final correction to the string-evolution equations that we must include is the loop shrinkage and energy loss due to gravitational radiation. The total power radiated by an oscillating string is proportional to $G\mu^2$, so it is convenient to write it as $P=\lambda G\mu^2$ where λ is a dimensionless constant. This implies that $\dot{l}=-(\lambda/2\pi)G\mu$, independent of the size of the loop. Because our ultimate limit on the string tension will be proportional to λ, it is important that we use an accurate value for λ.

Vachaspati and Vilenkin[29] have calculated λ for several different loop trajectories, and they have found that λ is usually about 50, but some of their loop trajectories had λ values greater than 100. The appropriate choice for the purposes of our calculation would be an average over the non-self-intersecting loops that would be produced by the long strings by the processes that we have described. The λ values for the few non-self-intersecting loop trajectories calculated by Vachaspati and Vilenkin were all around 50. Furthermore, the loop trajectories that yielded much larger values of λ seemed to be rather degenerate cases that would be unlikely to be produced by long strings or by the fragmentation of parent loops; they either had very large angular momentum, resembling a rapidly rotating double line, or were almost circular with very little angular momentum such that the whole loop passes through a short line during each oscillation. (These are the types that may form black holes.) Hence, a λ value not much larger than 50 is probably appropriate for our calculation. We will use $\lambda=20\pi$, so that

$$\dot{l}=-10G\mu \ . \tag{4.5}$$

Thus, (2.18) becomes

$$\frac{d}{dt}\left[\frac{E}{L}f\left[\gamma,\frac{l}{L}\right]\right]=E\frac{p\bar{v}}{L^2}[xa(x)-x\delta f(\gamma,x)]-2\pi\mu\,\dot{l}n(l), \tag{4.6}$$

and our final string evolution equations, the analogs to (2.19) and (2.20), are

$$3\frac{\dot{R}}{R}-2\frac{\dot{L}}{L}=\frac{\dot{R}}{R}(1-2\langle v^2\rangle)+\frac{p\bar{v}}{L}\int x[\delta f(\gamma,x)-a(x)]dx, \tag{4.7}$$

$$3\left[\frac{\dot{R}}{R}-\frac{\dot{L}}{L}\right]f(\gamma,x)+\left[\frac{\dot{l}}{L}-\frac{\dot{L}}{L}x\right]f'(\gamma,x)+\dot{\gamma}\frac{\partial}{\partial\gamma}f(\gamma,x)$$
$$=\frac{p\bar{v}}{L}[xa(x)-x\delta f(\gamma,x)]+\frac{\dot{l}}{xL}f(\gamma,x) \ . \tag{4.8}$$

C. Solution of the string evolution equations

Now, after discussing all our assumptions and parameters, we are ready to solve (4.7) and (4.8). We will write the time dependence of R, L, and γ as

$$R\sim t^N,\quad L\sim t^\alpha,\quad \gamma\sim t^{\alpha-1}\ , \tag{4.9}$$

and we will neglect all terms proportional to $\dot{N}$ or $\dot{\alpha}$. (Our calculations show that this assumption is self-consistent in the cases of interest.) Inserting (4.5) and (4.9) into (4.7) and (4.8), we obtain

$$2\alpha=2N(1+\langle v^2\rangle)+\frac{p\bar{v}\delta}{\gamma}\int x\left[\frac{a(x)}{\delta}-f(\gamma,x)\right]dx\ , \tag{4.10}$$

and

$$\frac{p\bar{v}\delta}{\gamma}x\left[f(\gamma,x)-\frac{a(x)}{\delta}\right]$$
$$=\left[\alpha x+\frac{10G\mu}{\gamma}\right]f'(\gamma,x)+(1-\alpha)\gamma\frac{\partial}{\partial\gamma}f(\gamma,x)$$
$$+3(\alpha-N)f(\gamma,x)-\frac{10G\mu}{\gamma x}f(\gamma,x)\ . \tag{4.11}$$

If we take $a(x)$ to have the form given by (4.1) and (4.4), we can find an exact solution[30] of (4.11):

$$f(\gamma,z)=\frac{1}{\xi}\frac{n+2}{2\ln 2}\frac{F_l}{\delta}B$$
$$\times\int_z^{z_m}du\frac{u^n z}{z+\epsilon}\left[\frac{u+\epsilon}{z+\epsilon}\right]^{q+B\epsilon}e^{B(z-u)}du\ , \tag{4.12}$$

where

$$B=\frac{p\bar{v}}{\gamma}\frac{\delta\xi}{2\alpha-1},\ z=x/\xi,\ \epsilon=\frac{10G\mu}{(2\alpha-1)\gamma\xi},$$
$$q=3\frac{\alpha-N}{2\alpha-1}+1-\alpha,\ (z_m+\epsilon)z_m^{\alpha/(1-\alpha)-1}=z+\epsilon\ .$$

To obtain (4.12) we have used the boundary condition that $f(\gamma,x/\xi)$ vanishes when $x=\xi$. This boundary condition holds for $\alpha\leq 1$, which will be true if we choose a sufficiently large initial value for γ. When we substitute (4.12) back into (4.10) it becomes

$$2\alpha=2N(1+\langle v^2\rangle)+(2\alpha-1)\frac{n+2}{2\ln 2}\frac{F_l}{\delta}B\times\int_0^1 dz\left[z^{n+1}-B\int_{z_0}^{z}du\frac{z^n u^2}{u+\epsilon}\left[\frac{z+\epsilon}{u+\epsilon}\right]^{q+B\epsilon}\times e^{B(u-z)}\right],\quad (4.13)$$

where $z_0=\max(0,(z+\epsilon)z^{\alpha/(1-\alpha)-1}-\epsilon)$.

Before we can evaluate (4.13) to obtain the time evolution of the string system, we need two additional ingredients: an expression for $\langle v^2\rangle$, and an equation for $N(t)$ [i.e., $R(t)$]. As we saw in Sec. III, $\langle v^2\rangle$ is a function of the curvature scale of the long strings, which we have assumed to be γ. (The possibility that the scale of curvature of the strings is different from the mean distance between the strings will be discussed in Sec. VI.) In Sec. II, we saw that when $\gamma\ll 1$, $\langle v^2\rangle=\frac{1}{2}$, and when $\gamma\gg 1$, $\langle v^2\rangle\approx 0$. Numerical simulations by Turok and Bhattacharjee[26] seem to indicate that the transition between these two regions is rather abrupt. In our calculations, we used

$$\langle v^2\rangle=\frac{1}{2}\left[\frac{1}{1+k\gamma^4}\right]^2,\quad (4.14)$$

with k being varied from 0 to $\frac{1}{2}$. It should be noted that, although we have merely guessed a form for $\langle v^2\rangle$, an error here will not have a great influence on our results because scaling solutions tend to occur only for fairly small values of γ. (Unless we take $F_l>1$.) In fact, our main conclusions will be valid even if $\langle v^2\rangle=\frac{1}{2}$ for all values of γ.

The last ingredient we need before we can solve for the time evolution of the string system is Einstein's equation to determine how N evolves. It is

$$\frac{\dot{R}}{R}=\frac{N}{t}=\left[\frac{8\pi G}{3}\right]^{1/2}(\rho_r+\rho_s+\rho_{gr})^{1/2},\quad (4.15)$$

where ρ_r, ρ_s, and ρ_{gr} are the densities of ordinary radiation (or relativistic matter), strings, and gravitational radiation, respectively. ρ_r is calculated from

$$\rho_r=\frac{3}{32\pi G t_0{}^2}\left[\frac{R_0}{R}\right]^4.\quad (4.16)$$

The expression for ρ_s can be found from (2.8) and (2.13) to be

$$\rho_s=\frac{\mu}{L^2}\left[1+\int dx\, f(x)\right],\quad (4.17)$$

and the energy transferred from loops to gravitational radiation is given by the last term of (4.6). With (4.5), this gives

$$\frac{d\rho_{gr}(x)}{dt}dx=\frac{E}{V}\left[\frac{10G\mu}{Lx}\right]f(x)dx\quad (4.18)$$

for the rate that the density of gravitational radiation increases. Initially, we set $\rho_{gr}=0$, and then we use (4.18) to calculate the density of gravitational radiation produced by the strings in a given time interval. After it is produced the density of the gravitational radiation scales as $1/R^4$.

Now we can use (4.13)–(4.18) to calculate evolution of the string system in the early Universe. We start at time[31]

$$t_0=\frac{m_p{}^3}{\mu^2}=7\times 10^{-32}\ \sec\frac{(10^{16}\ \text{GeV})^4}{\mu^2},\quad (4.19)$$

when the string damping becomes negligible. Our initial condition for γ is to set $\gamma(t_0)=1$, and it is implicitly assumed that at $t=t_0$ we already have an "equilibrium" configuration of small loops so that Eqs. (4.7) and (4.8) can be used to describe the subsequent evolution of the system. Presumably, it would probably take some time (until $t\sim t_0/10G\mu$) before the very small loops could be formed, but thereafter, we should expect that (4.7) and (4.8) will describe the evolution of the string system accurately. We start our calculation by choosing a value for α and evaluating (4.13) several times until a self-consistent solution for α is found. Once we find α, we evolve

$$\gamma\sim t^{\alpha-1},\quad \rho_r,\rho_{gr}\sim t^{-4N}\quad (4.20)$$

from t_0 until t_1 (usually $t_n=10t_{n-1}$). Then we calculate α from (4.13), ρ_s from (4.17), and

$$\frac{\Delta\rho_{gr}}{\rho_0}=\frac{32\pi G\mu}{3\gamma^2}\frac{10G\mu}{\gamma}\ln\left[\frac{t_n}{t_{n-1}}\right]\int\frac{f(x)dx}{x}\quad (4.21)$$

for the change in $\Delta\rho_{gr}/\rho_0$ in the interval t_{n-1} to t_n ($\rho_0=3/32\pi G t^2$ is the energy density of a radiation-dominated universe). We can continue this procedure until $t\approx 2\times 10^{12}$ sec$\approx 10^{43}t_0$ when baryons ordinarily become important.

D. Cosmological constraints on ρ_s and ρ_{gr}

In order for the string scenario to be consistent with known cosmology, we must require that the strings do not come to dominate before the Universe becomes matter dominated at $t\approx 2\times 10^{12}$ sec. If the strings should come to dominate much before this time, then the microwave background radiation will reach 2.7 K long before the age of the Universe is 4×10^9 yr (the age of Earth). Actually, the situation is not quite so simple. It is possible that, if the strings do not dominate over the radiation by a large amount, the Universe could pass from the string-dominated phase to a baryon-dominated phase at roughly the usual time, so that the strong-dominated phase would have little influence on the current relationship between the age of the Universe and the microwave background temperature.

A more stringent constraint on the string scenario can be obtained by considering primordial helium synthesis. (This constraint was found independently by Davis.[32]) Our constraint is essentially the same as the familiar bound on the number of light neutrinos, except that our bound will be on the maximum density allowed in strings and gravitational radiation. (We can treat gravitational radiation and strings, like neutrinos, because like neutrinos they have decoupled from ordinary matter.) This constraint arises from the time delay between the time ($T \simeq 1$ MeV, $t \approx 1$ sec) when neutrinos and protons drop out of equilibrium and the time ($T \approx 0.1$ MeV, $t \approx 100$ sec) when the protons are too cold to photodissociate deuterons and nucleosynthesis commences. If the density of the Universe is too large when $T_\gamma \sim 1$ MeV, then the Universe will expand too fast and too few neutrons will decay before they can be combined into ^{4}He nuclei, resulting in a ^{4}He abundance that is larger than what is observed. The constraint given by Boesgaard and Steigman[33] is

$$\frac{\rho_s + \rho_{gr}}{\rho_r} < 0.08 \ . \tag{4.22}$$

It turns out that this constraint is difficult for the string scenario to satisfy. This can be seen quite easily by considering a system of strings satisfying a scaling solution with the density of strings of some fraction $\eta \equiv \rho_x / \rho_r$ of the radiation density. If it were not for gravitational radiation, the string density would scale as $1/R^3 \sim t^{1/2}\rho_r$. Thus, without gravitational radiation the string density would grow to

$$\frac{\rho_s}{\rho_r} = \eta \left[\frac{t+dt}{t} \right]^{1/2} ,$$

or

$$\frac{d\rho_s}{\rho_r} = \eta \frac{1}{2} \frac{dt}{t} \tag{4.23}$$

in a time interval dt. In order for the scaling solution to hold, most of this energy must be lost through gravitational radiation. This implies that

$$\frac{d\rho_{gr}}{\rho_r} = \eta \frac{1}{2} \frac{dt}{t} (1-r) , \tag{4.24}$$

where we have defined r as the fraction of loop energy in the form of kinetic energy of small loops. Since this energy can be red-shifted away as the Universe expands, it need not be radiated gravitationally. As noted previously, the simulations of Albrecht and Turok indicate that r is very small ($r < 0.03$).

Since nucleosynthesis occurs at $t \sim 10^{31} t_0$, we should integrate (4.24) for a factor of $\sim 10^{31} \times 10 G\mu \approx 10^{26}$ in time, where the factor $10 G\mu$ is just the time it takes for the smallest loops (those formed at t_0) to decay by gravitational radiation. Then, the constraint (4.22) becomes

$$\eta (1-r) \tfrac{1}{2} \ln 10^{26} < 0.08$$

or

$$\eta (1-r) < 0.0027 \ . \tag{4.25}$$

So, if r is as large as 0.1, then we must require that $\eta < 0.0030$ in order to satisfy the nucleosynthesis constraint. This constraint (4.25) is modified somewhat in our numerical calculations by several factors that change the expansion rate of the Universe. Between 10^{-31} sec and 1 sec, many particle species become nonrelativistic and annihilate, depositing their energy in the particles that are still relativistic and in thermal equilibrium.[34] This makes the Universe expand slightly faster, red-shifting the gravitational radiation slightly. The presence of the cosmic strings also makes the Universe expand slightly faster than it would otherwise. (For the choice of parameters that seem to be in the best agreement with Albrecht and Turok, we obtain $N \approx 0.5007$.) This effect red-shifts the ordinary matter more than the gravitational radiation, so it tends to cancel the previous correction. Finally, during the time it takes for the first loops formed to decay by gravitational radiation, $\sim 10 G\mu t_0$, some gravitational radiation is produced, but we have neglected it.

Our calculations show that these corrections combine to reduce the density in gravitational radiation by a factor of about 1.4, implying a factor of 2 correction to the bound on $G\mu$ that can be obtained from (4.25). These corrections have been included in the calculations to obtain the bounds on $G\mu$ given in Figs. 3 and 4.

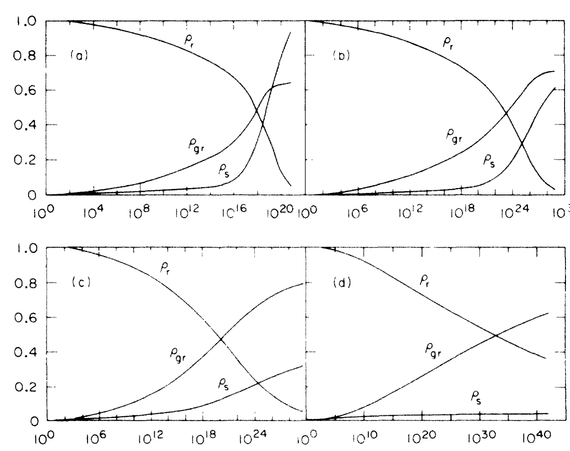

FIG. 2. The evolution of ρ_r, ρ_s, and ρ_{gr} as a function of time for $G\mu = 10^{-6}$, $k = \frac{1}{16}$, and $r = 0.1$: in (a), $p_{SI} = 0.25$, $F_l = 0.6$, and $\delta = 0.3$, in (b), $p_{SI} = 0.5$, $F_l = 0.8$, and $\delta = 0.5$, in (c), $p_{SI} = 0.62$, $F_l = 0.55$, and $\delta = 0.5$, and in (d), $p_{SI} = 0.76$, $F_l = 0.35$, and $\delta = 0.5$.

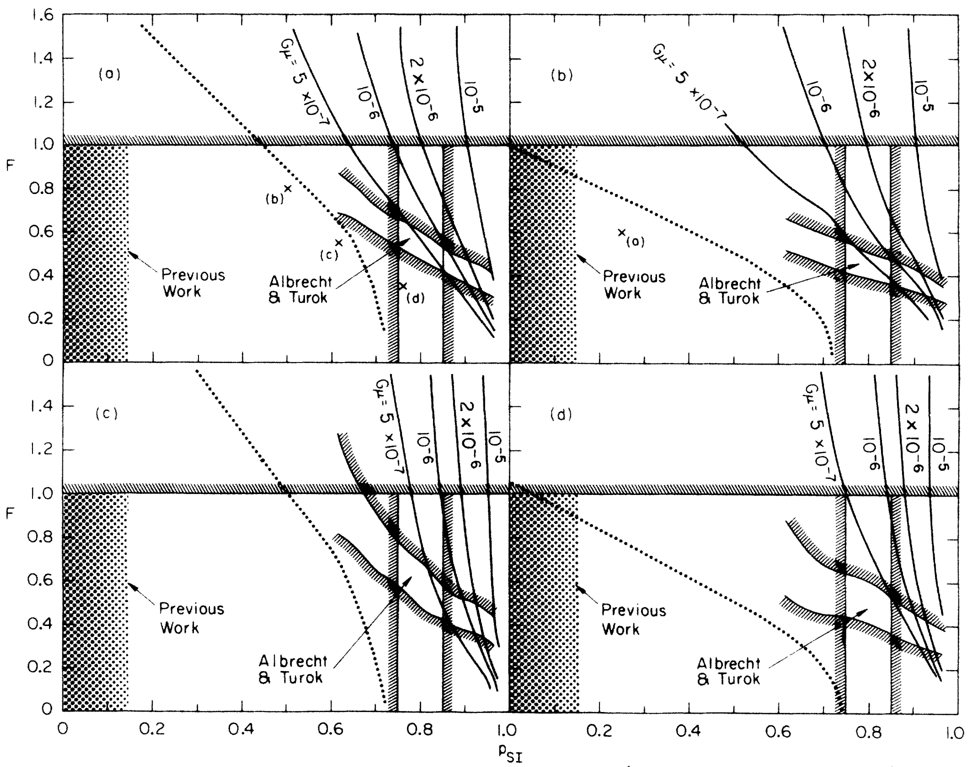

FIG. 3. Constraints on $G\mu$ for various values of our parameters: in (a), $k=\frac{1}{16}$ and $\delta=0.5$, in (b), $k=\frac{1}{16}$ and $\delta=0.3$, in (c), $k=\frac{1}{2}$ and $\delta=0.5$, and in (d), $k=\frac{1}{2}$ and $\delta=0.3$.

The results of our numerical calculations are summarized in Fig. 2 and Fig. 3. In Fig. 2, ρ_s, ρ_r, and ρ_{gr} are plotted as a function of time for selected values of the parameters, and Fig. 3 consists of four constraint diagrams indicating the behavior of the string system as a function of F_l and p_{SI} with k [from (4.14)] and δ taking the values $k=\frac{1}{16}$, $\frac{1}{2}$, and $\delta=0.3$, 0.5. For both of these figures, we have set $\xi=1.5$, $p=1$, and $r=0.1$.

Our constraint diagrams (Fig. 3) require some explanation. They display the constraints on $G\mu$ as a function of four parameters. The axes of Fig. 3 are the probability of self-intersection p_{SI} and the loop production amplitude F_l, while the different graphs correspond to different values of the correction factor δ and the velocity parameter k. The solid lines labeled with different values of $G\mu$ indicate the constraint on $G\mu$ as a function of the parameters: p_{SI}, F_l, δ, and k. The region above and to the right of each curve is the region of parameter space for which the given value of $G\mu$ is consistent with standard nucleosynthesis. The regions of parameter space below the dotted lines are the regions for which no scaling solutions exist. The shaded lines in Fig. 3 indicate the constraints on p_{SI} and F_l that we have obtained from a comparison with the numerical results of Albrecht and Turok (as well as the requirement that $F_l<1$). The shaded line at the top of each diagram implies that we expect that $F_l<1$, while the other shaded lines indicate the region of parameter space suggested by a fit with the numerical simulations of Albrecht and Turok. (This fit will be discussed in the next section.) Finally, the crosses in Fig. 3 represent the points in parameter space corresponding to the graphs in Fig. 2.

The shaded regions, labeled "previous work," on the left side of each diagram indicate that previous authors[2,22,32] have neglected the self-intersection of loops, corresponding to small values of p_{SI}. Note that these regions fall almost entirely underneath the dotted lines. This means that, if the probability of self-intersection was negligible as was assumed in all the previous analytic work, then a scaling solution would almost certainly not exist. In fact, if $\delta\geq0.5$, then it is necessary that $p_{SI}\gtrsim0.5$ for a scaling solution to exist. Hence, the self-intersection process is critically important for the consistency of the string model.

We also find that the question as to whether the strings come to dominate depends only on F_l/δ and p_{SI}, and is essentially independent of p, the probability of intercommuting (assuming $p\not\approx0$). This can be understood if we examine (4.13). For some value of $B\propto p/\gamma$ the integral on the right-hand side of (4.13) has its maximum value, but if this maximum value is not large enough to obtain $\alpha=1$, then there is no scaling solution and the strings will come to dominate. Therefore, our results will only be sensitive to p in those cases when a scaling solution exists. In that case p determines the scale of the string system (γ) at the scaling solution.

The main conclusion to be drawn from Fig. 3 is that a string tension of $G\mu\gtrsim2\times10^{-6}$ can be consistent with helium synthesis constraints only for large values of F_l and very large values of p_{SI}. Of course, it is possible that the correct values of these parameters are really quite large, so we must make some attempt to determine what these parameters really are.

V. COMPARISON WITH ALBRECHT AND TUROK

A. Constraints on $p_{\rm SI}$, F_l, and ρ_s

Although we have obtained stringent restrictions on the parameters describing the string scenario, we can only make definite predictions if we have some idea what the correct values of the parameters might be. The best way to obtain these parameters is to compare our results with those of Albrecht and Turok. It is of particular importance to determine the probability of self-intersection $p_{\rm SI}$ both because our results are the most sensitive to this parameter, and because there seems to be no other way to obtain an estimate of it.

Unfortunately, it is not completely straightforward to determine $p_{\rm SI}$ from the simulations of Albrecht and Turok. According to Turok,[13] their simulations indicate that the "average" loop splits up into about 10 daughter loops. The question follows: How do we relate this to $p_{\rm SI}$? Clearly, if we calculate the average number of loops produced by parent loops with $p_{\rm SI} \geq \frac{1}{2}$, we get a divergent result. If we take into account the fact that the simulations use a finite lattice size, we can introduce an upper cutoff for the number of self-intersections to make this number finite. If we allow the parent loops to self-intersect a maximum number of times m then we obtain

$$\overline{N}_{\rm loops} = 1 + \sum_{k=1}^{m} 2^{k-1} p^k . \tag{5.1}$$

Evaluating this for $m=4$, 5, and 6, gives

$$\begin{aligned} \overline{N}_{\rm loops}(m=4) &= 10 \quad \text{for } p_{\rm SI}=0.85 , \\ \overline{N}_{\rm loops}(m=5) &= 10 \quad \text{for } p_{\rm SI}=0.74 , \\ \overline{N}_{\rm loops}(m=6) &= 10 \quad \text{for } p_{\rm SI}=0.66 . \end{aligned} \tag{5.2}$$

Perhaps a better way to estimate $p_{\rm SI}$ would be to take the median daughter loop to be one-tenth the size of the parent loop. This would imply that

$$n+2 = \log_{10} 2 ,$$

or

$$p_{\rm SI} = 2^{-\log_{10} 2} = 0.81 . \tag{5.3}$$

A third estimate of $p_{\rm SI}$ can be made if we assume that the average parent loop undergoes $\log_2 10 = 3.32$ self-intersections. This identification yields

$$\frac{p_{\rm SI}}{1-p_{\rm SI}} = \log_2 10 \quad \text{and} \quad p_{\rm SI} = 0.77 , \tag{5.4}$$

in rough agreement with (5.2) and (5.3).

Combining these estimates, we see that $p_{\rm SI} = 0.80 \pm 0.05$ should result in the production of roughly 10 daughter loops from each parent loop that Albrecht and Turok see in their simulations. From Fig. 3, we can see that this choice for $p_{\rm SI}$ almost certainly implies that a scaling solution exists, but that the energy density in strings depends on the string production amplitude F_l. If, as we expect, $F_l < 1$, it will be difficult to satisfy the nucleosynthesis constraint for $G\mu \gtrsim 10^{-6}$.

We can obtain a more definite result if we fit our calculations with another result of the numerical simulations of Albrecht and Turok. In order to demonstrate that they have reached a scaling solution, they plot the evolution of the energy density in strings as a function of time, but because their initial state has no small loops they cannot hope to obtain as many small loops as a real scaling solution would have. Since small loops tend to decouple this omission will probably not have a great influence on the evolution of larger loops, but it does mean that they will find that the energy density in small loops will always be increasing. Therefore, in order to test to see whether a scaling solution has been reached, they consider only the energy density (ρ_c) in loops of radius larger than a cutoff, $r_c = 0.2t$. (It should be mentioned that r_c refers to the real radius of the loop, not the proper radius, l that we have used in most of our discussions. The relationship between l_c and r_c is $l_c = \sqrt{2} r_c$.) From Fig. 2 of Ref. 17 we can estimate the value of ρ_c at the scaling solution to be $\rho_c t^2/\mu = 3.0^{+1.0}_{-0.8}$.

We can use this number to fix our parameter F_l and the results of this fit are given in Fig. 3. Thus, if we are given $p_{\rm SI}$, we can use our calculation of $\rho_c t^2/\mu = 3.0^{+1.0}_{-0.8}$ to bound F_l, which according to (4.13) determines the equilibrium density of the string network, ρ_s. This procedure gives the shaded curves in Fig. 3. It turns out, however, that the ratio of ρ_c to ρ_s depends mainly on $p_{\rm SI}$ and only very weakly on F_l and δ, so by fitting our results for ρ_c to those of Albrecht and Turok, we can obtain ρ_s as a function of only $p_{\rm SI}$, as shown in Fig. 4. Figures 4(a) and 4(b) correspond to different values of the $\langle v^2 \rangle$ parameter k. The error bars come from matching our value of $\rho_c t^2/\mu$ to the value, $3.0^{+1.0}_{-0.8}$, obtained from Albrecht and Turok's results. Since $k = \frac{1}{16}$ probably gives an underestimate of $\langle v^2 \rangle$, we can use Fig. 4(a) to obtain a bound on $G\mu$ as a function of $p_{\rm SI}$. (Actually, the difference between $k = \frac{1}{16}$ and $k=0$ is negligible when $\gamma \approx 0.6$ as is necessary to obtain the correct value for ρ_c.)

The constraints on $G\mu$ implied by matching our value of ρ_c with that of Albrecht and Turok are given in Table I. If $p_{\rm SI} < 0.85$ as the simulations of Albrecht and Turok seem to indicate, we must require $G\mu < 10^{-6}$ for the string model to be consistent with primordial nucleosynthesis.

TABLE I. Constraints on $G\mu$ as a function of $p_{\rm SI}$.

$p_{\rm SI}$	$G\mu$
<0.66	$<3\times10^{-7}$
<0.77	$<5\times10^{-6}$
<0.87	$<10^{-6}$
<0.91	$<2\times10^{-6}$
<0.93	$<4\times10^{-6}$
<0.97	$<10^{-5}$

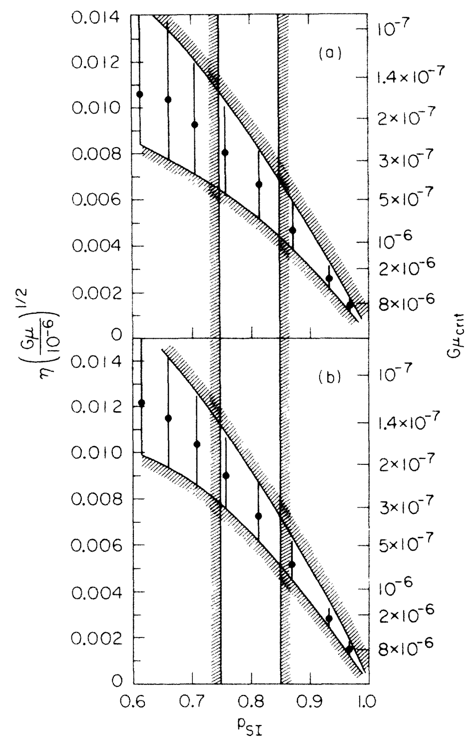

FIG. 4. Constraints on $\eta(G\mu/10^{-6})^{1/2}$ and $G\mu_{\rm crit} = 10^{-6}(0.003/\rho_s)^2$ from comparison to the simulations of Albrecht and Turok. $k=\frac{1}{16}$ in (a) and $k=\frac{1}{2}$ in (b).

B. Checking the numerical simulation

In addition to using the work of Albrecht and Turok to fix some of our parameters, we can also use our results to check those of Albrecht and Turok. As we have noted in the Introduction, their most serious limitation is the limited amount of time that they can run their simulation. In view of this limitation and the fact that their initial state has fewer small loops than are present in the final scale-invariant configuration, it is worthwhile to check to see if we can find any transient behavior that might influence their results. Already, we have found some support for their conclusions because, if $p_{\rm SI} \approx 0.8$ as their work seems to indicate, then our calculations show that a scaling solution does exist. But, if the probability of self-intersection should be slightly smaller, $p_{\rm SI} \lesssim 0.7$, then it is possible that a scale-invariant solution may not exist.

In order to test whether this might be a problem, we have solved (4.6) and (4.8) by direct numerical integration to allow the use of arbitrary initial conditions. Unfortunately, with this approach, we were unable to use the form for $a_{\rm eff}$ given in (4.1) because it is not continuous, and it gives rise to large instabilities in the numerical integration. Instead, we used

$$a_{\rm eff} = A_n x^n e^{-x^2}, \tag{5.5}$$

which gave results quite similar to those obtained by solving (4.10) and (4.11) using (4.1). The initial condition used was simply $f(x)=0$. Since the initial state used by Albrecht and Turok had about 20% of the string density in loops, we would expect that any transient that appeared in Albrecht and Turok's simulation would be slightly exaggerated in our analysis.

In general, for most choices of $p_{\rm SI}$, F_l, and the initial energy density in strings, we found that the cutoff string density ρ_c fell initially because of the lack of loops in the initial state, but this transient behavior usually disappeared by the time the horizon doubled in size. Because Albrecht and Turok can easily run their simulations for a longer time than this, it seems unlikely that they have been fooled by a transient. On the other hand, it is conceivable that their apparent observation of a scaling solution is merely an artifact of their initial conditions if $p_{\rm SI}$ and F_l happen to lie close to a boundary separating regions where scaling solutions do and do not exist. Of course, this would require that $p_{\rm SI}$ be far from the value that we have estimated above. An example of such behavior is plotted in Fig. 5(a), which is a plot of the evolution of $\rho_c t^2/\mu$ as a function of $t^{1/2}$. The different curves in Fig. 5(a) correspond to different initial string energy densities.

The relative lack of small loops in Albrecht and Turok's initial data is more likely to cause problems if we wish to determine the total energy density of strings once the scaling solution has been reached. In addition to the fact that they cannot run their simulations long enough to get an accurate picture of the density of small loops, it is also possible that their value for ρ_c may be an underestimate. An example of such behavior is plotted in Fig. 5(b) for three different initial string energy densities with $p_{\rm SI}=0.81$ and $F_l \approx 0.5$. An initial condition with few

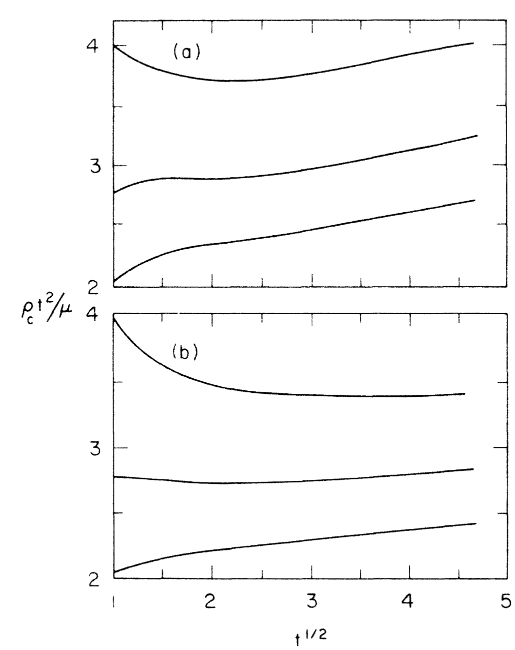

FIG. 5. $\rho_c t^2/\mu$ vs $t^{1/2}$ with $\delta=0.5$ for (a) $p_{\rm SI}=0.62$, $F_l \sim 0.5$, and (b) $p_{\rm SI}=0.81$, $F_l \sim 0.4$.

loops ensures that ρ_c will tend to drop below its scaling solution value initially. Once enough small loops have been created so that they are in equilibrium with the long strings, ρ_c will begin to rise, but only very slowly. As we can see from Fig. 5(b), for each of the three initial states the cutoff string density seems to approach some equilibrium state for a short while, but then the curves seem to flatten out. What may not be apparent from Fig. 5(b), but is apparent from our calculations, is that, in all three cases, ρ_c is slowly increasing at the end of each curve. ($\rho_c \sim t^{0.065}$, $t^{0.040}$, and $t^{0.006}$, respectively.) This means that the equilibrium value of ρ_c is greater than the values at all three end points. Thus, if we saw this graph with the resolution of the numerical simulations, we would probably conclude that the correct value of ρ_c at the steady-state configuration is $\rho_c t^2/\mu = 2.8$, when the actual value is $\rho_c t^2/\mu > 3.4$. Thus, it is possible that our fit with the results of Albrecht and Turok gives an underestimate of the energy density in strings at the scaling solution. If so, the upper bound on $G\mu$ could be as much as a factor of 2 lower, corresponding to the upper curves in Fig. 5 rather than the lower ones.

Despite these possible discrepancies, our calculations generally confirm the main conclusion of Albrecht and Turok: that cosmic strings do indeed evolve in a scale-invariant fashion. If we assume that the correct value for p_{SI} is in the range suggested by their simulations, then we find that a scaling solution does exist. For a reasonable choice of our parameters, we can obtain the same energy density in large loops and long strings.

The situation is quite different, however, when we consider their conclusions regarding the energy density in small loops. This result is important because the galaxy-formation calculations,[2,14] the microwave-background calculations,[11,12] and the nucleosynthesis constraint all depend on the density in small loops. Using the range for p_{SI} that seems to correspond to the numerical simulations, we find that the density in small loops is greater than that obtained from the numerical simulations by a factor of 4.5 ± 1.5. In order to obtain the same value as Albrecht and Turok, we must set $p_{SI} \approx 0.97$. In Albrecht and Turok's simulations such a high probability for self-intersection would probably be indistinguishable from $p_{SI} = 1$, which they have ruled out.

It is not surprising that Albrecht and Turok's result for the density in small loops may be too low because they have obtained it after running their simulation for roughly a factor of 3 in time[35] starting from an initial condition that had very few small loops. Actually, for $0.75 < p_{SI} < 0.85$, at the loop size corresponding to the smallest loops in Albrecht and Turok's simulation, we obtain a loop density that is not much different from that of Albrecht and Turok, but the difference occurs when we extrapolate to smaller loops. With p_{SI} so large, we find that many daughter loops are still being produced at a loop size smaller than this, so an extrapolation to smaller loops assuming negligible self-intersection would lead to an underestimate of the energy density in small loops. It would seem that, while the parameters for our calculation must be fixed by comparison to the numerical simulation, a more accurate determination of the density in small loops can be obtained from our calculation than from the simulation of Albrecht and Turok.

VI. DISCUSSION AND CONCLUSIONS

As we have shown in the previous two sections, our analysis seems to lead to some important constraints on the properties of cosmic strings. We should, therefore, understand how our results depend on the assumptions we have made and how our conclusions might change if we vary some of our parameters.

A. Discussion of our assumptions and approximations

One of our most fundamental assumptions is that the network of long strings can be described by one scale L. As originally defined in (2.8), L defined the distance scale of the separations between long strings, but we subsequently used it in several slightly different contexts. In the expression (2.16) for the energy loss from long strings to loops, we assumed that the loop production function $a(x)$ depends only on $x = l/L$, so that the distance scale between the strings determines the average size of the loops that are produced. Similarly, the expression we used for estimating $\langle v^2 \rangle$ (4.14) depends only on the ratio γ between L and the horizon. Essentially, we have assumed that the scale of curvature of a string is the same as the scale of distance between the strings. This seems to be a natural assumption when the intercommuting probability is of order one, but what if it is wrong?

Suppose, for instance, that the average radius of curvature of the string is smaller than L. Then, the stretching of strings would be less important, and $\langle v^2 \rangle$ would take a larger value, but our bound on $G\mu$ was obtained in the case where string stretching was already negligible, so the absence of stretching would not change our results. A very bumpy string would also produce loops and lose energy at a higher rate than the ones we have considered. (Recall that loops are generally produced as a result of collisions of waves on a single string.) Since the separation between loops is large, they would not be likely to be absorbed by another string, so highly curved strings would lose energy faster than the strings we have considered. Thus, the scale of their curvature would increase faster than the separation between strings, so apparently, a system of highly curved strings will evolve into the type of string system that we have considered where the curvature has the same scale as the separation between the strings.

In the opposite case of very straight strings, we expect that curvature on the scale of L would be restored by intercommuting between different strings. If this is not the case, then the stretching of strings would be more important than we have assumed while loop production would be very much suppressed. Thus, in this case, the energy density in strings would grow faster than our calculations show, and a more stringent bound on $G\mu$ would be appropriate.

It is also of interest to see how our results vary under changes of ξ, p, or corrections to our expressions for $\bar{v}$, (2.15), and (2.11). If we set $\langle v^2 \rangle = \frac{1}{2}$ and take $\alpha = 1$ (since we are concerned with scaling solutions), we can derive a simple scaling law for these variations. When $\langle v^2 \rangle = \frac{1}{2}$, the evolution equation (4.13) depends on p, $\bar{v}$, ξ, and γ only through the combinations $B = p\bar{v}\delta\xi/\gamma$ and $\epsilon = 10G\mu/\gamma\xi$, but since the evolution equation is almost independent of ϵ, we can neglect it. (This just means that the smallest loops have decoupled, so they cannot influence the evolution of the long strings.) Thus, if we know a scaling solution exists for one set of the variables, p, $\bar{v}$, ξ, and γ, then we can find a solution for different values of p, $\bar{v}$, and ξ by choosing γ such that B remains unchanged. The new value of ρ_s is then obtained from (4.12) and (4.17). When we do the integral (4.17) we see that the explicit factor of ξ in (4.12) drops out, but that we can no longer ignore ϵ because the dominant contribution to ρ_s is from the smallest loops. In fact, we find that

$$\rho_s \sim \frac{1}{\gamma^2}\epsilon^{-1/2} \sim \frac{\xi^{1/2}}{\gamma^{3/2}} , \tag{6.1}$$

when B and F_l are held fixed. Equation (6.1) implies that, if we change the product $p\bar{v}$ keeping all other variables fixed, then γ will scale as $p\bar{v}$, and $\rho_s \sim (p\bar{v})^{3/2}$. Similarly, if we vary only ξ, we obtain $\gamma \sim \xi$ and $\rho_s \sim 1/\xi$. We can apply these formulas to see how our constraint diagrams in Fig. 3 change under variation of p, $\bar{v}$, or ξ.

We should note that the scaling law given above, (6.1), does not apply to the constraints (given in Table I and Fig. 5) that we derived from comparison with Albrecht and Turok, so if it turns out that there is an error in our estimate of $\bar{v}$, it would not be correct to use (6.1) to calculate the corrections to ρ_s and Table I. This is because the comparison with Albrecht and Turok's results involved matching our value of the cutoff string density ρ_c with their value. For large values of p_{SI} ($p_{\text{SI}} \gtrsim 0.75$), more than 85% of the contribution to ρ_c comes from the long strings, and this contribution is just $1/\gamma^2$. Thus, when we constrain ρ_c to be the value given by Albrecht and Turok, we are essentially fixing γ, not F_l and B, so (6.1) does not apply. Instead, we find that in most cases, we can vary p, $\bar{v}$, or δ by a factor of 2, and our constraints on $G\mu$, as shown in Fig. 4, will change by less than 5%. This rough rule only breaks down when $p_{\text{SI}} < 0.7$, and our constraint on $G\mu$ becomes very stringent.

Our constraint is not quite so insensitive to the parameter ξ, which is a measure of the initial size of the parent loops. This can be understood by noticing that a change in ξ has the same effect as changing p and ϵ by the same factor. By the arguments given above, the change in p will have no effect on our constraint. The change in ϵ affects $\rho_s \sim \xi^{1/2}$ but not $1/\gamma^2$, which is the main contribution to ρ_c. ξ also determines the fraction of the loops that will contribute to ρ_c. These two effects tend to cancel and the result is, very roughly, that $\rho_s \sim \xi^{0.3}$, for fixed ρ_c. Thus, since our constraint on $G\mu$ will have some sensitivity to ξ ($G\mu \sim \xi^{-0.6}$), it is important to know the uncertainty in our value for ξ. From the arguments given in Sec. IV, it seems likely that $\xi = 1.5$ is a reasonable choice, but it may be wise to allow for some uncertainty. Actually, if we set $\xi = 0.75$ as would be necessary in order to weaken our constraint on $G\mu$ by a factor of ~ 1.5, then, from (4.2), we find that a loop with a slightly larger radius than ξ has a probability of 64% to survive long enough to fragment into smaller loops. So, it seems that even decreasing ξ by a factor of 2 is unreasonable, but we should recall that, although well motivated, our expression for the loop production function is actually a fairly crude approximation. Thus, the factor of 2 or so might more properly reflect our uncertainty about a_{eff} rather than ξ.

As we have seen in the previous section, our constraint on $G\mu$ is independent of almost all the parameters that we have had to introduce to describe the string system. This is because by fixing our parameters so that our value of ρ_c corresponds to that obtained from the numerical simulations of Albrecht and Turok, we are in effect just calculating the ratio of ρ_s to ρ_c, and the majority of unknown parameters simply cancel out of this ratio. This leads to the somewhat remarkable result that if we ignore the possible weak dependence on ξ, our bound on $G\mu$ depends only on the probability of self-intersection p_{SI}. Apparently then, it is quite important to obtain a better determination of this parameter. Presumably, this can be accomplished by a closer comparison between our analytic approach and the numerical simulations of Albrecht and Turok.

B. Implications for galaxy formation

We should make some comments about the implications of our constraint ($G\mu < 10^{-6}$) for the string theory of galaxy formation. Since the disagreement between our constraint and the lowest estimate of $G\mu$ from galaxy- and cluster-formation considerations is only a factor of 2, our constraint is certainly not very severe by cosmological standards. However, there is some reason to believe that this disagreement will become somewhat more severe as the physics of cosmic strings becomes better understood. First, it should be emphasized that we have consistently chosen our parameters so as to underestimate the number density of small loops (n_l) and hence the total energy density in strings. Thus, it seems likely that our bound would become more stringent if we had a better idea of what values these parameters should really take. For instance, if we take $\lambda = 50$ and $r = 0.03$ (which seem likely to be the correct values) instead of the more conservative values that we have used, then we obtain a bound on $G\mu$ that is 40% stronger. Similarly, we can gain additional factors if it turns out that the stretching of strings provides an important contribution to the evolution of the string system, or if the appropriate value of p_{SI} is smaller than 85%. These corrections are, of course, in addition to the factor that we may gain if Albrecht and Turok did indeed underestimate ρ_c as we suggested in Sec. V.

On the other hand, the determination of $G\mu$ from galaxy- and cluster-formation scenarios is also sensitive to the exact value of the energy density in small loops which we have calculated here. For instance, in Ref. 14, the value of $G\mu$ is calculated using a value for the density of small loops that is obtained by an extrapolation from the numerical simulation of Albrecht and Turok. However, as we have mentioned in Sec. V, their simulation is not particularly well suited for this task, and in fact, our results imply a substantially larger value. So if our results are used instead of those of Albrecht and Turok, then the estimate of $G\mu$ obtained in Ref. 14 would decrease by a factor of 2 to $G\mu=10^{-6}$ which is consistent with our bound. However, if we are able to increase our estimate of the number density of small loops, then the discrepancy between the nucleosynthesis bound and Turok and Brandenberger's value for $G\mu$ will increase because the upper bound on $G\mu$ from nucleosynthesis scales as $\sim n_l^{-2}$ while Turok and Brandenberger's result scales as $G\mu \sim n_l^{-2/3}$. If, for example, we let n_l take the maximum value that seems to be consistent with our results (rather than the minimum value that we have used for our constraint) the discrepancy between the nucleosynthesis bound and the galaxy-formation bound will become a factor of 5.

C. Conclusions

We have calculated the evolution of cosmic strings analytically, and our results confirm the main conclusions reached by Albrecht and Turok with their numerical simulations. The analytic approach is particularly helpful when trying to understand the specific physical processes that determine the behavior of the string system. It is also useful for calculating the energy density in small loops which is important for the galaxy-formation scenario, but which is difficult to do numerically. We have found that the fate of a system of cosmic strings depends sensitively on the probability that the closed loops produced from long strings self-intersect and break up into smaller loops. If this probability is low than many of the closed loops intersect and recombine with the long strings before they can radiate away a significant portion of their energy. This means that the energy density of the string system will fall slower than that of radiation, and the Universe will quickly become string dominated. However, if the probability of self-intersection is large, then the string system evolves in a scale-invariant manner so that the energy density in strings is always proportional to that of radiation. Since a large value for the probability of self-intersection ($p_{\rm SI}=0.80\pm0.05$) is indicated by the numerical simulation of Albrecht and Turok, we conclude that the scale-invariant evolution is probably correct.

However, if this estimate of $p_{\rm SI}$ is correct, then we can obtain a bound on the string tension ($G\mu<10^{-6}$) from the requirement that the gravitational radiation produced by the strings not be so copious as to interfere with primordial nucleosynthesis. This constraint is independent of most of our assumptions and approximations (except our estimate for $p_{\rm SI}$), so it seems to be reasonably firm. The constraint is marginally consistent with the value that is required for the galaxy-formation scenarios, but since this conflict may become more serious when the evolution of the string system becomes better understood, it may eventually force a serious revision in the string theory of galaxy formation. This constraint does, however, rely on the correctness of the standard nucleosynthesis calculations, and it may be that nucleosynthesis is not as well understood as we expect.[36]

ACKNOWLEDGMENTS

I would like to thank J. R. Bond, A. Vilenkin, H. Quinn, and A. Albrecht for helpful discussions. I am grateful to S. H. Rhie for the solution to Eq. (4.11), and I would also like to thank H. Quinn for a critical reading of the manuscript. This work was supported by the Department of Energy, under Contract No. DE-AC03-76SF00515.

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the lowest Fourier modes to dominate, it seems quite reasonable that the probability for black hole formation is $\ll 10^{-20}$.

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[32]R. L. Davis, Phys. Lett. **161B**, 285 (1985).

[33]A. M. Boesgaard and G. Steigman, Ann. Rev. Astron. Astr. (to be published); J. Yang, M. S. Turner, G. Steigman, D. N. Schramm, and K. A. Olive, Astrophys. J **281**, 493 (1984).

[34]This correction was pointed out to me by David Seckel.

[35]A. Albrecht (private communication).

[36]Recently Applegate and Hogan [J. H. Applegate and C. J. Hogan, Phys. Rev. D **31**, 3037 (1985)] have argued that details of the QCD phase transition may play an important role in nucleosynthesis, and that perhaps the cosmological bound on the density of extra particles, (4.22), at the time of nucleosynthesis is not correct. If this is correct, then our constraint on the string tension could become weaker, but since Applegate and Hogan have not actually calculated any nuclear abundances with their QCD corrections, we cannot say how our constraint would change. On the other hand, the success of the standard nucleosynthesis calculations at predicting the abundances of several different elements seems to indicate that the standard calculations are not too far wrong.