Scale anomaly and the scalars

H. Gomm, P. Jain, R. Johnson, and J. Schechter
Physics Department, Syracuse University, Syracuse, New York 13210
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We investigate the properties of a possible low-lying scalar glueball as well as the ordinary scalar-quark states using an effective chiral Lagrangian which satisfies the trace anomaly and $U(1)_A$ anomaly of QCD. An interesting mass bound for the lightest particle in the scalar-singlet channel is discussed. Detailed arguments against the existence of a very light (less than 400 MeV) scalar glueball are presented. It is shown that the introduction of a derivative-coupling term in the usual type of linear σ model cures the problem of excessively large widths for the ordinary (nonet) scalar mesons. In fact, chiral symmetry enables one to nicely correlate the widths of the entire scalar nonet. It is noted that the same derivative-coupling term allows a heavy (1–2 GeV) scalar glueball to have sufficiently narrow width to permit its observation as an ordinary resonance, in contrast with a recent claim. Our model can naturally explain the unusually large partial width for the $\eta\eta'$ mode of the glueball candidate $G(1590)$. A symmetrical ansatz for the anomaly terms is suggested which enables one to successfully calculate the η' mass as the ratio of gluon condensate to pion-decay constant. In addition, the theoretically interesting limit where the glueball becomes a true dilaton is formulated in a new and illustrative way.

I. INTRODUCTION AND SUMMARY

The least understood portion of the low-energy particle spectrum is the scalar sector. On the other hand, it is one of the most interesting theoretically. The singlet scalar particles have the same quantum numbers as the vacuum and the trace anomaly and they are expected to include glueballs. Furthermore, the interpretation of the scalars in the quark model is not straightforward and so far calculations in constituent models have failed to give convincing predictions. Also it is difficult to determine experimentally[1] the properties of the scalars since they do not tend to show up as easily identifiable resonances.

Here we would like to explore the properties of a possible low-lying scalar glueball as well as the scalar-quark states using an effective chiral Lagrangian which satisfies the trace anomaly and $U(1)_A$ anomaly of QCD. This approach should enable one to assess the consequences of the symmetry structure of the underlying theory for the observable particle states. In this model physical quantities depend on the vacuum value of the glue-field operator $(F_{\mu\nu}F_{\mu\nu})$ (denoted by H) as well as the vacuum value of the quark-field combination $\bar{q}q$, which is related to the pion-decay constant F_π. The ratio $\langle H\rangle^{1/2}/F_\pi$ is in fact observed to emerge as a characteristic mass scale in the model.

To start things off we review the simple model of pure QCD (no matter fields) dominated by a single scalar glueball. When extended to the case where nonderivative interactions with spin-0 quark matter fields are included (see Sec. II), the mixing between glue and matter scalars is directly calculable. It is noted that the upper bound previously obtained on the lightest particle in the scalar channel, $m_<^2 < 16\langle H\rangle/3d^2F_\pi^2$, where d is the transformation scale dimension of the quarkonium field, also holds when the composite field $F_{\mu\nu}F_{\mu\nu}$ is effectively eliminated from the theory in analogy to the elimination of $\epsilon_{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ in favor of $\eta'(960)$ in the pseudoscalar-singlet channel.

Before giving a detailed discussion of the model we remind the reader (Sec. III) of the peculiar pattern displayed by the experimental candidates for low-lying scalar mesons. We briefly discuss the quark-model treatments which have been proposed for the scalars and also the attempts to make sense of a usual nonet assignment by including unitarity corrections. We point out that, whereas the scalar-meson masses are not simply predictable in a chiral-symmetric framework, the decay widths of the entire postulated scalar nonet are found to be nicely understood on this basis. In particular the ratio $\Gamma(\kappa\rightarrow K\pi)/\Gamma(\delta\rightarrow\eta\pi)$, which is relatively free of the ambiguities afflicting other scalar modes, is correctly predicted by chiral symmetry, in marked contrast with an ordinary SU(3) treatment.

If one chooses the composite field $\bar{q}q$ to have the canonical scaling dimension three, one finds the mass bound of Sec. II to be the rather restrictive value 680 MeV. In this case a scalar glueball would have to be rather light and probably should have already been detected. The characteristics of such a light glueball are described by the model of Sec. IV. In particular, the width into $\pi\pi$ is shown to be only slightly larger than that required[2] to escape detection. A very strong argument against a light glueball h is furnished by noting that it would lead to a partial decay width for $\Gamma(\eta'\rightarrow h\eta)$ quite a bit larger than the experimental total η' width. Finally in this section a simple effective-Lagrangian treatment for the two-photon coupling of a light glueball is given and related physics discussed.

The theoretically interesting limit where the low-mass glueball becomes a true zero-mass dilaton is discussed in Sec. V. Although such models were proposed[3,4] many years ago we give an alternative treatment by taking the limit as the heavy scalar mass goes to infinity of a scale-invariant linear σ model. It is pointed out that the presence of a dilaton noticeably changes the classic predictions[5] of the $\pi\pi$ scattering lengths. Furthermore, the true dilaton cannot be a pure glueball but must contain some admixture of quark matter.

Next we return to the main track by examining the more realistic case of a heavier glueball (see Sec. VI). It is reasonable to choose the scale dimension $d=1$ in our model. To be specific we consider the recently announced[6] experimental candidate $G(1590)$ to be the prototype scalar glueball and attempt to explain[7] its somewhat unusual properties. The most serious problem to be faced initially, pointed out in Ref. 8 and also in Ref. 9, is that in a simple model of the present type, the partial widths into $\pi\pi$ and $K\bar{K}$ are expected to be of the order of several GeV, rather than tens of MeV, as for the $G(1590)$. Remarkably the presence of the same derivative-coupling term in the Lagrangian which was required to fit the scalar-nonet widths (see Sec. III) also can naturally suppress the glueball widths to the correct order of magnitude.

The most conspicuous experimental property of the $G(1590)$ is the dominance of the $\eta\eta'$ decay mode. This feature, as reviewed in Sec. VII, was already expected from the construction of a Lagrangian to satisfy the $U(1)_A$ anomaly in addition to the trace anomaly. We find the $\eta\eta'$ partial width to be 66 MeV, which is within the large experimental uncertainty, but possibly too low. Finally we note that the Lagrangian satisfying the two anomaly conditions is still more general than required to predict the η' mass. A symmetrical ansatz for the anomalous terms which takes account of the requirement of periodicity in the QCD vacuum angle[10] θ leads to the reasonable prediction $(m_{\eta'})^2 \approx 6\langle H\rangle/5F_\pi^2$.

II. MASS BOUND IN THE SCALAR-SINGLET CHANNEL

First consider, for orientation, a simple effective Lagrangian[11,12] for pure QCD (no "matter" fields present) designed to mock up the trace anomaly[13] equation:

$$\theta_{\mu\mu} = -[\beta(g)/g]\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) ,$$

where $\theta_{\mu\nu}$ is the improved energy-momentum tensor, $\beta(g)$ the renormalization-group function, and $F_{\mu\nu}$ the Yang-Mills field strength tensor. A natural choice of scalar "order parameter" field is the right-hand side (RHS) of this equation:

$$H \equiv -[\beta(g)/g]\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) .$$

With the restriction that terms with at most two derivatives appear, the unique answer is

$$\mathcal{L} = -\tfrac{1}{2}aH^{-3/2}(\partial_\mu H)^2 - \tfrac{1}{4}H\ln(H/\Lambda^4) , \tag{2.1}$$

where Λ is a constant of mass dimension one and a is a dimensionless constant. Even though it is clearly an oversimplificiation to approximate the entire spectrum of QCD by a single scalar glueball field, Eq. (2.1) already possesses a couple of reasonable features. First, the second term $\equiv -V$ corresponds to a potential which has a minimum at a nonzero value of H: $\langle H\rangle = \Lambda^4/e$. This agrees with the "bag model" picture[14] of confinement in which $\langle H\rangle$ is identified as (-4) times the outside vacuum energy density. The QCD sum-rule determination[15]

$$\langle H\rangle = 0.0135\ \mathrm{GeV}^4 \tag{2.2}$$

can be reconciled with the bag model if it is assumed that the inside energy density, while less negative than the outside energy density, is not zero. We shall adopt (2.2) for purposes of making numerical estimates, though it should be recognized that this value may be subject to change.

To give a particle interpretation of the model we may set

$$H = \langle H\rangle + Zh , \tag{2.3}$$

with h being the conventionally normalized glueball field. (An equivalent alternate approach is to write $H=\langle H\rangle e^{\chi}$.) The resulting Lagrangian is completely specified if in addition to (2.2) we fix the glueball mass $m_h = Z/2\langle H\rangle^{1/2}$. Furthermore, $a=\langle H\rangle^{3/2}/Z^2$. Even though only a single glueball, rather than an infinite family, is present, the amplitudes derived from (2.1) and (2.3) will satisfy the $1/N_c$ counting rules.[16] Specifically, setting $a=O(N_c)$ and $Z=O(N_c)$ will give

$$\langle H\rangle = O(N_c{}^2),\quad m_h = O(N_c{}^0) ,$$

as well as a suppression factor $O(1/N_c)$ for emission of an extra glueball. To cubic order in the fluctuation field h (2.1) becomes

$$-\tfrac{1}{2}m_h{}^2h^2 + \frac{3}{2}\frac{m_h}{\langle H\rangle^{1/2}}h(\partial_\mu h)^2 + \frac{1}{3}\frac{m_h{}^3}{\langle H\rangle^{1/2}}h^3 + \cdots .$$

Clearly the next step is to add matter fields. The natural order parameter for a low-energy description would seem to be the 3×3 nonet matrix

$$M_{ab} \equiv S_{ab} + i\phi_{ab} \sim \bar{q}_{Rb}q_{La} , \tag{2.4}$$

where q_{La} is the left-handed quark field of flavor type a, for example. Under a chiral transformation (U_L, U_R)

$$M \to U_L M U_R^\dagger \tag{2.5}$$

while under an infinitesimal scale transformation $\delta x_\mu = -\rho x_\mu$,

$$\delta M(x) = -\rho(d + x\cdot\partial)M(x) . \tag{2.6}$$

In (2.6) d is the *effective* (mass) dimension of M. For a canonical free boson field, d would be unity, but here we would like to leave d as a parameter which may contain some dynamical information.

First consider the nonderivative (or potential) terms which are allowed by chiral symmetry and correct scale anomaly. Scale-invariant terms like $H^{(1-d)/2}\mathrm{Tr}(MM^\dagger)$, $H^{(1-d)}\mathrm{Tr}(MM^\dagger MM^\dagger)$, etc., are of course, possible. To satisfy the scale-anomaly condition one might retain just the second term of (2.1) or more generally allow a sum of terms[11]

$$H\sum_m \frac{C_m}{m}\ln\frac{R_m(M,H)}{\Lambda^m},\quad \sum_m C_m = 1 ,$$

where the R_m are homogeneous functions of the fields of order m. Actually it is unnecessary to give the explicit form of V. All of our results follow from the fact that scale invariance is broken only through the anomaly term; this is equivalent to the equation

$$H=-4V+4H\frac{\partial V}{\partial H}+d\,\mathrm{Tr}\left[S\frac{\partial V}{\partial S}+\phi\frac{\partial V}{\partial \phi}\right] . \tag{2.7}$$

Differentiating (2.7) with respect to the fields will yield the needed relations among masses and coupling constants of various order. Equations analogous to (2.7) which express the chiral symmetry of V have been discussed elsewhere.[17] Initially we shall employ the simplest scale- and chiral-invariant derivative term for the matter fields

$$-\frac{1}{2}\left[\frac{H}{\langle H\rangle}\right]^{(1-d)/2}\mathrm{Tr}(\partial_\mu M\,\partial_\mu M^\dagger) . \tag{2.8}$$

Note that this reduces to the ordinary kinetic term only for $d=1$. Initially we shall not include the chiral-symmetry-breaking "quark mass terms." The potential V is assumed to be of sufficient complexity to allow for a spontaneously broken solution

$$\langle S_{ab}\rangle=\frac{F_\pi}{2}\delta_{ab} ,$$

with $F_\pi \simeq 132$ MeV. If V were to be written explicitly, the value of F_π would be proportional to the quantity Λ. We wish to examine at the tree level the effective theory defined by

$$\mathscr{L}=-\frac{1}{2}aH^{-3/2}(\partial_\mu H)^2 -\frac{1}{2}\left[\frac{H}{\langle H\rangle}\right]^{(1-d)/2}\mathrm{Tr}(\partial_\mu M\,\partial_\mu M^\dagger)-V(M,H) , \tag{2.9}$$

where V satisfies (2.7). We note at once that the glueball field h is allowed by SU(3) and parity invariance to mix with the scalar singlet

$$\sigma'\equiv\frac{1}{\sqrt{3}}(S_{11}+S_{22}+S_{33}) .$$

Differentiating (2.7) with respect to both H and S and using the nonet symmetry formula (derived e.g., by $1/N_c$ arguments)

$$\left\langle\frac{\partial^2 V}{\partial S_{aa}S_{bb}}\right\rangle=\delta_{ab}\left\langle\frac{\partial^2 V}{\partial \sigma'^2}\right\rangle\equiv\delta_{ab}A ,$$

we find the two equations

$$\begin{aligned}1&=\frac{4\langle H\rangle}{Z^2}\left\langle\frac{\partial^2 V}{\partial h^2}\right\rangle+\frac{\sqrt{3}dF_\pi}{2Z}\left\langle\frac{\partial^2 V}{\partial h\,\partial\sigma'}\right\rangle ,\\ 0&=\frac{4\langle H\rangle}{\sqrt{3}Z}\left\langle\frac{\partial^2 V}{\partial h\,\partial\sigma'}\right\rangle+\frac{dF_\pi}{2}\left\langle\frac{\partial^2 V}{\partial\sigma'^2}\right\rangle .\end{aligned} \tag{2.10}$$

This gives the σ'-h mass-squared mixing matrix:

$$\begin{bmatrix} A & \dfrac{-\sqrt{3}Z\,dF_\pi A}{8\langle H\rangle} \\ \dfrac{-\sqrt{3}Z\,dF_\pi A}{8\langle H\rangle} & \dfrac{Z^2}{4\langle H\rangle}+\left[\dfrac{\sqrt{3}ZF_\pi d}{8\langle H\rangle}\right]^2 A \end{bmatrix} . \tag{2.11}$$

Here A is a bare-quarkonium mass squared [around (1.5 GeV)^2] and $Z^2/4\langle H\rangle$ is like a bare gluonium mass squared. If one were to arbitrarily delete the trace anomaly the quantity $Z^2/4\langle H\rangle$ would be set to zero. Then the determinant of (2.11) vanishes and we have a Goldstone boson—the "dilaton." Of course the trace anomaly is intrinsic and cannot be deleted. Nevertheless, there is some reminder of a light particle in (2.11)—we can find an upper bound for the smallest mass eigenvalue. This situation corresponds to the limit where A and Z are both very large. One then finds[9]

$$m_<^{\,2}\le\frac{16}{3}\frac{\langle H\rangle}{d^2F_\pi^{\,2}}\approx\left[\frac{2.03}{d}\ \mathrm{GeV}\right]^2 , \tag{2.12}$$

where (2.2) was used in the last step. If the operator M_{ab} in (2.4) scales, at the relevant low energy, like a product of two free quark fields (zero anomalous dimension) we have $d=3$ and (2.12) becomes rather restrictive. In that case this model would require a scalar particle (glueball?) lighter than 670 MeV. On the other hand, it might be argued that $d=1$ is a reasonable choice. This is because the composite field representing a quark-antiquark bound state is likely not to differ too much in its scaling properties from a free boson field. In this case (2.12) is not very restrictive.

It seems worthwhile to compare the low-lying mass spectrum in the present 0^{++} gluonium channel with that in the 0^{-+} gluonium channel. In that channel (corresponding to the operator $F\tilde{F}$ rather than FF) it appears that the important low-lying particle is the quarkonium state $\eta'(960)$ rather than a gluonium-type excitation. Such a situation is accommodated in the effective-Lagrangian framework[18] by deleting the kinetic term (and all derivative terms) for the effective operator $G\sim F\tilde{F}$. Then, by the Lagrangian equation of motion G becomes proportional to the η' field. Formally, the 0^{++} and 0^{-+} channels look very similar although there is the important difference that the 0^{++} anomaly remains in the large-N_c limit while the 0^{-+} anomaly [U(1) anomaly] goes to zero. Nevertheless, unless there is a hidden scalar glueball at extremely low energies, the experimental data seem to favor FF being dominated at low energies by a quarkonium state. Thus one might also like to explore the possibility that the scalar channel should be formulated without the h kinetic term. [A still higher-mass scalar glueball might be added directly, as was done[19] to accommodate the $\iota(1440)$ in the U(1) effective Lagrangian.] This situation corresponds to the choice of Lagrangian (setting $d=1$)

$$\mathscr{L}=-\tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu M\,\partial_\mu M^\dagger)-V(M,H) , \tag{2.13}$$

where $V(M,H)$ satisfies (2.7). Remarkably, the mass bound (2.12) is just the same for the Lagrangian in (2.13). To see this first notice that the equation of motion

$$\frac{\partial}{\partial H} V(\sigma',H)=0$$

determines H as a function of σ': $H=H(\sigma')$. Then the equation

$$\frac{\partial}{\partial H} V(\sigma',H(\sigma'))=0$$

holds for all σ'. Differentiating it with respect to σ' and evaluating the result in the ground state yields

$$\left\langle \frac{\partial H}{\partial \sigma'} \right\rangle = -\left\langle \frac{\partial^2 V}{\partial H\, \partial \sigma'} \right\rangle \Big/ \left\langle \frac{\partial^2 V}{\partial H^2} \right\rangle . \tag{2.14}$$

Now we require

$$\left\langle \frac{d^2 V}{d\sigma'^2} \right\rangle = \left\langle \frac{\partial^2 V}{\partial \sigma'^2} \right\rangle + \left\langle \frac{\partial^2 V}{\partial \sigma' \partial H} \right\rangle \left\langle \frac{\partial H}{\partial \sigma'} \right\rangle$$
$$= \left\langle \frac{\partial^2 V}{\partial \sigma'^2} \right\rangle - \left\langle \frac{\partial^2 V}{\partial H\, \partial \sigma'} \right\rangle^2 \Big/ \left\langle \frac{\partial^2 V}{\partial H^2} \right\rangle . \tag{2.15}$$

The quantities $\langle \partial^2 V/\partial H\, \partial\sigma' \rangle$ and $\langle \partial^2 V/\partial H^2 \rangle$ can both be written in terms of $\langle \partial^2 V/\partial \sigma'^2 \rangle$ by using (2.7) or (2.10). Then we finally have

$$\left\langle \frac{d^2 V}{d\sigma'^2} \right\rangle = \left\langle \frac{\partial^2 V}{\partial \sigma'^2} \right\rangle \left[1 + \frac{3F_\pi{}^2}{16\langle H \rangle} \left\langle \frac{\partial^2 V}{\partial \sigma'^2} \right\rangle \right]^{-1} . \tag{2.16}$$

The bound on the physical σ' mass squared is obtained by letting the "bare"-quarkonium mass squared $A=\langle \partial^2 V/\partial \sigma'^2 \rangle$ go to infinity. It is amusing to notice that the physical σ' mass is always reduced from the bare-quarkonium value by taking account of the matter-glue "duality" expressed above. This is the exact opposite of the situation in the 0^{-+} channel where the η' has zero bare-quarkonium mass [U(1) problem] and gets a positive value by matter-glue duality. It is conceivable that this provides part of the explanation for the unnaturally low mass of the $S(975)$ scalar singlet.

III. PHENOMENOLOGICAL PROBLEMS OF THE SCALARS

If, as suggested in the last section, a low-lying scalar glueball is likely to mix with scalar states formed out of quark fields, one should examine the existing scalar-meson candidates with an eye to determining whether a (small?) admixture of gluonium would improve the agreement of predicted masses and widths with experiment. Unfortunately, as outlined below, the low-lying scalar candidates present a pattern which seems very far from the canonical one (e.g., the vector or tensor nonets). Explaining their masses seems to require paying attention to complicated unitarity corrections, as recently discussed by Törnqvist[20] and by Achasov, Devyanin, and Shestakov.[21] On the other hand, as we will show, the pattern of widths may be readily understood using a chiral Lagrangian approach. We might remark that the general chiral Lagrangian does not predict the scalar-meson mass spectrum and in fact suggests a pattern which is sensitively dependent on the details of SU(3) symmetry breaking.

In experiments the scalar mesons do not show up as easily identifiable resonances but have to be obtained through complicated phase-shift analyses which result in large uncertainties. The known features[1] are listed below (mass and width in MeV):

particle	i spin	mass	width
S	0	975 ± 4	33 ± 6 ($\pi\pi$, $78\pm3\%$; $K\bar{K}$, $22\pm3\%$)
ϵ	0	≈ 1300	200–600 ($\pi\pi$, 90%; $K\bar{K}$ 10%; $\eta\eta$ seen)
δ	1	980	54 ± 7 for $\eta\pi$; $K\bar{K}$ seen
κ	$\frac{1}{2}$	≈ 1350	≈ 250 ($K\pi$ seen)

It is noteworthy that the δ and the S couple strongly to $K\bar{K}$ even though their masses are below threshold.

We will briefly review the interpretation of the scalars in the simple quark model.[22] In the simple quark model the scalar is a $q\bar{q}$ state. Its quantum numbers J^{PC} imply that the constituents are in a P wave and that $S=1$. Since the spin-orbit interaction is small one is led to the assumption $m(0^{++})\approx m(2^{++})$. The physical scalars depart appreciably from this result, as one observes that the tensors analogous to $(S,\epsilon,\delta,\kappa)$ are (f',f,A_2,K^*) with masses (1525,1270,1320,1430). The tensor mesons follow the usual rule that particles containing strange quarks are heavier than those made up out of light quarks. The lightest scalar meson, however, presumably contains two strange quarks and also the large splitting between the isosinglet ϵ and the triplet δ remains unexplained in the quark model.

These difficulties led to the proposal that the scalars should be regarded as $\bar{q}\bar{q}qq$ states.[23] Since the binding energy due to one-gluon exchange is larger for an $\bar{s}s\bar{u}u$ state than for a state containing $(\bar{s}u)(\bar{u}s)$, i.e., two kaons, one easily concludes that the δ and the S lie just below $K\bar{K}$ threshold and decay strongly into two kaons. But this approach predicts too many particles. The δ and the S belong to a multiplet whose lighter members have not been observed. The ϵ and κ have to be part of another multiplet whose other states are unaccounted for. What happened to the scalar $q\bar{q}$ states predicted by the quark model?

On the other hand, there is a physical process which leads to the distortion of the mass pattern—the coupling of the scalars to pairs of pseudoscalars. For other nonets this effect will be small since the pseudoscalars have to be in a higher partial wave leading thus to decay. Following Ref. 20, we make the phenomenological ansatz for the coupling constant

$$g=\gamma\times[\mathrm{SU}(3)_{\mathrm{Clebsch}}]\times F(k) , \tag{3.1}$$

where γ is an overall constant and $F(k)=e^{-k^2/k_c^2}$ is a form factor (k being the momentum of the pseudoscalars in the rest frame of the scalar), corresponding to our intuitive notion that a pair of pseudoscalars with high relative momentum will lead to particle decay. The one-loop correction to the propagator is then

$$\pi(s)\propto\gamma^2\int_{4\mu^2}^{\infty}F^2(k)\frac{k}{\sqrt{s'}}\frac{ds'}{s-s'}\ , \tag{3.2}$$

where $s'=4(k^2+\mu^2)$ and μ is the pseudoscalar mass. $\pi(s)$ has a cusp at the threshold for pair production. The physical mass is then identified with the pole of the propagator $\{s-[m^2+\pi(s)]\}^{-1}$, which differs from the bare value m. Since the spin-orbit interaction is small and since this effect is negligible for tensor mesons the masses of the tensor mesons can be used as the bare mass parameters. Adding the contributions from all channels one can fit the scalar masses with two parameters. While this approach seems reasonable to us, it does not lend itself to a convincing calculation of the widths and (in common with most constituent models) does not incorporate the symmetries and anomalies of QCD.

Now let us discuss the width predictions for the scalar nonet on the basis of chiral symmetry.[24] A convenient way (and the original one) to include the scalars as well as the pseudoscalars is to use a linear σ model with the field $M=S+i\phi$, given in (2.4). It has been known for many years that the standard form of such a model described by the Lagrangian[17]

$$-\tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu M\,\partial_\mu M^\dagger)-V(M,M^\dagger)\ , \tag{3.3}$$

V being an arbitrary chiral-invariant function, gives widths at the tree level much larger than those of the experimental candidates. For example, the coupling constant for $\kappa\to K\pi$ is predicted to be, neglecting the pseudoscalar masses,

$$g_{\kappa K\pi}=\frac{1}{F_K}m^2(\kappa)\ . \tag{3.4}$$

Here F_K is the kaon weak decay constant and is very roughly the same as F_π (actually about $1.2F_\pi$). Equation (3.4) leads to a width prediction $\Gamma(\kappa\to K\pi)\approx 1.7$ GeV, which is very much greater than the experimental one. A reasonable way out of this dilemma is to include chiral-invariant interaction terms like $\mathrm{Tr}(\partial_\mu M\,\partial_\mu M^\dagger M M^\dagger)$ in $\mathcal{L}$. This will then yield a derivative interaction $g'_{\kappa K\pi}\kappa^+\partial_\mu\bar{K}^0\partial_\mu\pi^-$, whose effect should be added to that of (3.4). (A detailed treatment will be given in the next section.) Now notice that the derivative interaction will also yield an amplitude proportional to $m^2(\kappa)$. Thus we reach the conclusion that the general chiral-symmetric interaction for $S\to\phi\phi$ is proportional to the squared mass of the initial scalar. Equivalently, we may regard the general $S\phi\phi$ interaction to be of derivative type with *arbitrary* coupling constant:

$$\mathcal{L}_{\mathrm{eff}}(S\phi\phi)=-\gamma\,\mathrm{Tr}(S\partial_\mu\phi\,\partial_\mu\phi)\ . \tag{3.5}$$

This result, of course, may also be gotten from the general formulation[25] of the nonlinear realization of chiral symmetry. A possible term of the type $\mathrm{Tr}(S)\mathrm{Tr}(\partial_\mu\phi\,\partial_\mu\phi)$ violates the Okubo-Zweig-Iizuka (OZI) rule and would be suppressed, for example, in the large-N_c limit. With (3.5) the scalar width is predicted to be

$$\Gamma(S\to\phi\phi)=\frac{|\mathbf{q}|}{8\pi m_S{}^2}g_{\mathrm{eff}}{}^2,\quad g_{\mathrm{eff}}=\frac{\gamma m_S{}^2}{2}C_{\mathrm{CG}}\ , \tag{3.6}$$

where $|\mathbf{q}|$ is the magnitude of a daughter particle momentum in the scalar's rest frame. C_{CG} is a Clebsch-Gordan coefficient equal to unity for $\kappa^+\to K^0\pi^+$. The use of (3.5) rather than the standard SU(3)-invariant interaction $\mathrm{Tr}(S\phi\phi)$ neatly overcomes one of the puzzles associated with the scalar-meson widths. Consider the ratio $\Gamma(\kappa\to K\pi)/\Gamma(\delta\to\eta\pi)$ which is unaffected by possible mixing with (isosinglet) gluonium and by the ambiguity of being close to threshold. Using (3.5) and (3.6) we obtain

$$\frac{\Gamma(\kappa\to K\pi)}{\Gamma(\delta\to\eta\pi)}\simeq\frac{3}{2}\frac{m^2(\kappa)}{m^2(\delta)}\frac{|\mathbf{q}_\kappa|}{|\mathbf{q}_\delta|}\left[\frac{2}{\sqrt{6}}\cos\theta_p-\frac{2}{\sqrt{3}}\sin\theta_p\right]^2$$
$$=3.97\ , \tag{3.7}$$

where we used a value $-18°$ for the η-η' mixing angle θ_p. Taking $\Gamma(\delta\to\eta\pi)$ to be its experimental value, 54 ± 7 MeV yields $\Gamma(\kappa\to K\pi)=215\pm30$ MeV, in nice agreement with the experimental picture in which the total width is 250 MeV and $K\pi$ is the dominant mode. For contrast, the standard nonderivative interaction would lead to the bad result in which the ratio in (3.7) is reduced by a factor $[m(\delta)/m(\kappa)]^4=0.28$. The quantity γ in (3.5) has a magnitude of about 4.0 GeV^{-1} obtained by fitting the $\delta\to\eta\pi$ partial width. Knowing γ we can predict the width of ϵ in terms of the partial width for $S^*\to\pi\pi$ which is unaffected by the ambiguities associated with the $K\bar{K}$ threshold. The σ-σ' mixing, in the absence of a scalar glueball, is described by an angle θ_S:

$$\sigma=\epsilon\cos\theta_S+S^*\sin\theta_S\ ,$$
$$\sigma'=-\epsilon\sin\theta_S+S^*\cos\theta_S\ , \tag{3.8}$$

where σ and σ' are the SU(3)-octet number and singlet, respectively. From the experimental value $\Gamma(S^*\to\pi\pi)=26$ MeV we find $\theta_S\approx-32°$. This is to be contrasted with the ideal mixing value $\theta_S\approx-55°$. Using the above values for γ and θ_S then yields the predictions for the $\epsilon(1300)$ partial widths

$$\Gamma(\epsilon\to\pi\pi)=357\text{ MeV}\ ,$$
$$\Gamma(\epsilon\to K\bar{K})=13\text{ MeV}\ , \tag{3.9}$$
$$\Gamma(\epsilon\to\eta\eta)=1\text{ MeV}\ .$$

This is in reasonable agreement with the experimental total width which lies in the range 200–600 MeV and with the dominance of the $\pi\pi$ mode.

To sum up, it seems that the scalar-meson widths can be easily understood in a chiral-symmetric framework. This gives us some confidence that it is reasonable to go on and add the effects of mixing with scalar gluonium. However, the manifest uncertainties associated with the scalar-singlet channel prevent us from claiming that we will be doing any more in this regard than presenting some qualitatively interesting toy models.

IV. IS THERE A VERY-LOW-MASS GLUEBALL?

The simple model of Sec. II does not *a priori* predict the scalar glueball's mass but rather relates the width and mixing with quarkonium to the mass. Lattice gauge theory calculations[26] indicate a glueball mass in the neighborhood of 1 GeV. Using the mass bound (2.12), this would seem to be an argument in favor of choosing the scale dimension $d \approx 1$ for a quarkonium bound-state field rather than $d \approx 3$. Here we would like to see if the case $d \approx 3$ can be ruled out from the fact that it is only consistent with a rather low glueball mass. If the glueball mass were between zero and 354 MeV $[m(K^+)-m(\pi^+)]$ it would most likely have been observed in the sensitive experiments designed to look at $K^+ \to \pi^+ +$ anything. Thus it is interesting to investigate glueball candidates in the mass range of or slightly greater than 354 MeV. A recent analysis[2] of $\pi\pi$ scattering indicates that a $\pi\pi$ resonance in this range might have escaped detection if its width were less than about 2 MeV. We shall now calculate the width of such a possible glueball, taking account of its mixing with an SU(3)-singlet quarkonium scalar σ'. For simplicity we neglect SU(3)-breaking effects and do not try to constrain σ' to be a given combination of $S(975)$ and $\epsilon(1300)$.

A suitable effective Lagrangian for the present purpose is

$$\mathcal{L} = \frac{-e}{2}\left[\frac{H}{\langle H\rangle}\right]^{(1-d)/2} \mathrm{Tr}(\partial_\mu M\, \partial_\mu M^\dagger) - \tfrac{1}{2} a H^{-3/2}(\partial_\mu H)^2 - \frac{f}{2}\left[\frac{H}{\langle H\rangle}\right]^{1/2-d} \mathrm{Tr}(\partial_\mu M\, \partial_\mu M^\dagger M M^\dagger) - V(M,H) . \quad (4.1)$$

The matter-field derivative terms, with coefficients e and f, differ from the derivative term in (2.9) in order that one obtains, as indicated in Sec. III, the correct value of the width for $\kappa \to K\pi$, etc. This necessitates a wave-function renormalization for M. Now we should write [instead of (2.4)] $M_{ab} = X_{ab} S_{ab} + i Y_{ab} \phi_{ab} = \bar{q}_{Rb} q_{La}$. For simplicity we shall work in the nonet limit $X_{ab}=X$ and $Y_{ab}=Y = \langle \bar{q}q\rangle / F_\pi$, where $\langle \bar{q}q\rangle = -0.012$ GeV^3 from current-algebra arguments. Furthermore, in the case of (4.1) (but not in general[27]) one has $X=Y$. The correct normalization of the kinetic terms implies then

$$\left[\frac{\langle \bar{q}q\rangle}{F_\pi}\right]^2 \left[e + \frac{f}{4}\langle \bar{q}q\rangle^2\right] = 1 . \quad (4.2)$$

The value of f can be gotten from $\kappa \to K\pi$:

$$g_{\rm eff}(\kappa^+ \to K^0 \pi^+) = \frac{m_\kappa^2}{F_\pi^2}\left[F_\pi + \frac{f}{4}\frac{\langle \bar{q}q\rangle^4}{F_\pi}\right] \approx \pm 3.65 \text{ GeV} , \quad (4.3)$$

where $g_{\rm eff}$ is defined in (3.6). We define the "physical" fields σ'_p and h_p by

$$\sigma' = \sigma'_p \cos\theta + h_p \sin\theta , \qquad h = -\sigma'_p \sin\theta + h_p \cos\theta . \quad (4.4)$$

θ is obtained by diagonalizing (2.11) as in Ref. 9. The relevant $h_p\pi\pi$ and $\sigma_p\pi\pi$ terms from (4.1) are found to be, using[9] chiral invariance and broken scale invariance,

$$\frac{c_p}{2} h_p (\partial_\mu \boldsymbol{\pi})^2 - \frac{\tilde{c}_p}{2} h_p \boldsymbol{\pi}^2 + \frac{b_p}{2}\sigma'_p (\partial_\mu \boldsymbol{\pi})^2 - \frac{\tilde{b}_p}{2}\sigma'_p \boldsymbol{\pi}^2 , \quad (4.5)$$

where

$$c_p = c\cos\theta + b\sin\theta, \quad \tilde{c}_p = \tilde{c}\cos\theta + \tilde{b}\sin\theta , \qquad b_p = -c\sin\theta + b\cos\theta, \quad \tilde{b}_p = -\tilde{c}\sin\theta + \tilde{b}\cos\theta , \quad (4.6)$$

and

$$c = \frac{Z}{\langle H\rangle}\frac{\langle \bar{q}q\rangle^2}{F_\pi^2}\left[\left[\frac{d-1}{2}\right]e + \left[d - \frac{1}{2}\right] f \frac{\langle \bar{q}q\rangle^2}{4}\right] , \qquad \tilde{c} = -\frac{d}{4}\frac{ZA}{\langle H\rangle}, \quad b = -\frac{f}{\sqrt{3}}\frac{\langle \bar{q}q\rangle^4}{F_\pi^3} , \qquad \tilde{b} = \frac{2A}{\sqrt{3}F_\pi} . \quad (4.7)$$

Recall that A is the bare σ' squared mass in (2.11) and Z is defined in (2.3). From (4.5) we notice that the effective coupling constant for $h_p \to \pi^+\pi^-$, for example, is

$$g_{\rm eff}(h_p \to \pi^+\pi^-) = -\tilde{c}_p + c_p \left[\frac{m^2(h_p)}{2} - m_\pi^2\right] . \quad (4.8)$$

This results in the following pionic widths for h_p in the mass range of interest ($A = 1.5$ GeV^2):

$$m(h_p) = \begin{Bmatrix} 305\text{ MeV}, & 2.2\text{ MeV} \\ 333\text{ MeV}, & 4.0\text{ MeV} \\ 358\text{ MeV}, & 6.1\text{ MeV} \\ 381\text{ MeV}, & 8.3\text{ MeV} \\ 402\text{ MeV}, & 10.7\text{ MeV} \end{Bmatrix} = \Gamma(h_p) . \quad (4.9)$$

These numbers correspond to taking the plus sign on the left-hand side of (4.3) (which implies $f = -2.48 \times 10^6$ GeV^{-10}); they do not change much if the minus sign is taken and do not depend sensitively on A. The corresponding σ'_p masses and widths are around 1.47 GeV and 290 MeV, respectively. The σ'_p mass is somewhat high. From (4.9) we see that a low-mass scalar glueball is improbable since the widths are predicted to be greater than 2 MeV. However the widths are small enough to make it reasonable to pursue the analysis of low-energy $\pi\pi$ scattering data in more detail. An amusing point is that the light-mass glueball is not pure h but contains a sizable amount of "quarkonium" field σ'; the angle θ ranges from 29° to 41° for the masses in (4.9). This feature persists even to the exact dilaton limit, as we will see in the next section.

Another way to rule out a low-mass glueball makes use of the unusual feature that it should have, in our model, a very strong coupling to the η and η' mesons, as will be discussed in Sec. VII. The coupling constant is given in (7.3). This leads (remembering that $Z\approx 0.1$ for a low-mass glueball) to rather large partial widths for $\eta'\rightarrow\eta+h$. For example, taking $m_h=300$, 360, and 400 MeV yields partial widths of about 22, 15, and 7 MeV, respectively. These are almost two orders of magnitude larger than the experimental total width of η' which is 0.29 MeV.

Still another means of detecting a low-mass glueball would be through its two-photon coupling. Although, of course, the glueball does not have a direct electromagnetic interaction, virtual quark loops can, in analogy to the case of the π^0, mediate such an interaction. Actually, the two-photon coupling of any object which is postulated to dominate the energy-momentum tensor at low energies was computed[28] a long time ago based on the electromagnetic trace anomaly. This can be formulated easily in the present case by constructing a suitable effective Lagrangian. It is only necessary to replace the anomaly term in (2.1) by

$$-\tfrac{1}{4}(H+H_{\rm em})\ln\frac{H}{\Lambda^4}\ , \tag{4.10}$$

where

$$H_{\rm em}=\frac{-e^2}{24\pi^2}N_c\left[\sum_i Q_i{}^2\right]a_{\mu\nu}a_{\mu\nu}\ , \tag{4.11}$$

is the electromagnetic contribution to the trace anomaly. Here $e^2/4\pi\simeq\frac{1}{137}$, Q_i is the electric charge number for each relevant quark, and $a_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu$ is the electromagnetic field strength tensor. A more general interaction obtained by replacing $\ln H/\Lambda^4$ by

$$(1-\rho)\ln H/\Lambda^4+(2\rho/d)\ln[{\rm Tr}(MM^\dagger)/\Lambda^{2d}]$$

with ρ an arbitrary constant, for example, also satisfies the trace anomaly but would not correspond to complete glueball dominance. For simplicity we will use (4.10) and interpret the result as an upper limit. Thus we have the effective $h2\gamma$ interaction

$$\mathscr{L}(h\gamma\gamma)=\frac{e^2N_c}{48\pi^2}\frac{m_h}{\langle H\rangle^{1/2}}\sum_i Q_i{}^2 h a_{\mu\nu}a_{\mu\nu}\ , \tag{4.12}$$

which by a simple calculation yields[29] the width

$$\Gamma(h\rightarrow 2\gamma)=\frac{\alpha^2 m_h{}^5 N_c{}^2}{576\pi^3\langle H\rangle}\left[\sum_i Q_i{}^2\right]^2 \approx 0.88\left[\frac{m_h}{1\ {\rm GeV}}\right]^5\ {\rm keV}\ . \tag{4.13}$$

For a glueball mass of about 350 MeV the two-photon width would be at most about 5 eV. One might hope to see a particle of this type produced by the Primakoff mechanism. It might also be seen in a "two-photon" experiment like the one used to see $\eta(560)$ at the Crystal Ball detector. A simple estimation[30] shows that the production rate $e^+e^-\rightarrow e^+e^-h$ for h about 350 MeV could be as high as 0.07 that of the production rate $e^+e^-\rightarrow e^+e^-\eta$ in that experiment.

V. NONLINEAR MODEL OF PIONS PLUS DILATON

The situation discussed in the last section, where the gluonium scalar is much lighter than the quarkonium singlet, is, though probably unrealistic, related to the theoretically interesting modification of the model in which the gluonium state has zero mass. This case must be achieved by deleting the anomaly term so that scale invariance is exact; then the "glueball" will be the Goldstone boson (dilaton) associated with the spontaneous breakdown of scale invariance. Lagrangians containing both zero-mass pions and a dilaton were discussed[3,4] a long time ago. Recently this approach was updated by Ellis and Lanik[8] to include the terms (2.1), thus considering the trace anomaly as a kind of perturbation. The resulting model is similar to the present one with neglect of gluonium-quarkonium mixing.

In the present section we will give an alternative formulation of the old nonlinear model of zero-mass pions plus a dilaton. Our formulation has the advantage that it permits a more direct calculation of the $\pi\pi$ scattering lengths. Indeed we note that the presence of a true dilaton has a nontrivial effect on these quantities. Several other related points are discussed.

First let us review how the ordinary nonlinear σ model[31] may be obtained from the linear one. The latter, working for simplicity with SU(2) rather than SU(3), is described by the Lagrangian

$$\mathscr{L}=-\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi})^2-\tfrac{1}{2}(\partial_\mu\sigma)^2-c\left[\boldsymbol{\pi}^2+\sigma^2-\frac{F_\pi{}^2}{2}\right]^2\ , \tag{5.1}$$

where c is a positive constant. Clearly the field σ develops a vacuum value $\langle\sigma\rangle=F_\pi/\sqrt{2}$ and its mass squared is $4cF_\pi{}^2$. Physically the nonlinear model corresponds to this mass going to infinity ($c\rightarrow\infty$). In this limit the potential part dominates the Lagrangian equation of motion for σ which becomes the constraint

$$\boldsymbol{\pi}^2+\sigma^2-\frac{F_\pi{}^2}{2}=0\ . \tag{5.2}$$

We regard (5.2) as an expression for σ in terms of $\boldsymbol{\pi}$: $\sigma=+(F_\pi{}^2/2-\boldsymbol{\pi}^2)^{1/2}$. Substituting this back gives the nonlinear Lagrangian

$$\mathscr{L}=-\tfrac{1}{2}(\partial_\mu\boldsymbol{\pi})^2-\frac{1}{2}\left[\partial_\mu\left[\frac{F_\pi{}^2}{2}-\boldsymbol{\pi}^2\right]^{1/2}\right]^2\ , \tag{5.3}$$

A point transformation can be used to present $\mathscr{L}$ in the form $-(F_\pi{}^2/8){\rm Tr}(\partial_\mu U\,\partial_\mu U^\dagger)$, $U=\exp(\sqrt{2}i\boldsymbol{\phi}\cdot\boldsymbol{\tau}/F_\pi)$. The Lagrangians of Eq. (5.1) and therefore also (5.3) are not scale invariant. In order to modify (5.3) to achieve scale invariance one may introduce an additional field and also require $\boldsymbol{\pi}$ to have scale transformation zero, as noted by Ellis,[4] who studied the Jacobi identity of the dilaton operator, axial charge, and pion field. Then the quantity $(F_\pi{}^2/2-\boldsymbol{\pi}^2)^{1/2}$ will transform homogeneously under scale transformation and

$$\mathscr{L}=-\tfrac{1}{2}(\partial_\mu\psi)^2 -\frac{1}{2}\frac{\psi^2}{\langle\psi\rangle^2}\left[(\partial_\mu\boldsymbol{\pi})^2+\left[\partial_\mu\left[\frac{F_\pi{}^2}{2}-\boldsymbol{\pi}^2\right]^{1/2}\right]^2\right]\ , \tag{5.4}$$

where ψ is the dilaton field with scale dimension 1, is properly scale invariant.

Although it appears peculiar that the pion should have scale dimension zero it is well known[3] that one can make an equivalence transformation to a new field $\pi'=\pi\psi/\langle\psi\rangle$ which has canonical scale dimension one. Thus, in a simple theory of type (5.4) the actual pion scale dimension is irrelevant, at least for computing the tree amplitudes. On the other hand, in the model of Sec. II the observable masses were clearly related to the choice of dimension d for $M=S+i\phi$. For example, the mass bound (2.11) depends directly on d. This dependence arises[32] from the fact that the field S mixes with the glueball field H as dictated by Eq. (2.7).

Instead of introducing ψ to make the nonlinear Lagrangian (5.3) scale invariant, it seems natural[33] to directly obtain a nonlinear model from a scale-invariant linear model containing ψ. Evidently the most direct analog of (5.1) is the following:

$$\mathscr{L}=-\tfrac{1}{2}(\partial_\mu\pi)^2-\tfrac{1}{2}(\partial_\mu\sigma)^2-\tfrac{1}{2}(\partial_\mu\psi)^2 -c(R\psi^2-\sigma^2-\pi^2)^2\,, \tag{5.5}$$

where c and R are positive dimensionless constants. Equation (5.5) is manifestly scale invariant. The vacuum values $\langle\psi\rangle$ and $\langle\sigma\rangle$ will obey

$$R\langle\psi\rangle^2=\langle\sigma\rangle^2\equiv F_\pi{}^2/2\,, \tag{5.6}$$

although the numerical value of $\langle\sigma\rangle$ cannot be found by minimizing the potential. For this purpose a scale-breaking piece could be added. Note that for making contact with Sec. II, we should set $\psi=4a^{1/2}H^{1/4}$. Equation (5.5) is diagonalized by the rotated fields $\hat\sigma$ and $\hat\psi$:

$$\begin{aligned}\sigma&=(1+R)^{-1/2}(\hat\sigma+R^{1/2}\hat\psi)\,,\\ \psi&=(1+R)^{-1/2}(-R^{1/2}\hat\sigma+\hat\psi)\,.\end{aligned} \tag{5.7}$$

$\hat\psi$ is the massless Goldstone field while $\hat\sigma$ has a squared mass

$$m^2(\hat\sigma)=4c(1+R)F_\pi{}^2\,. \tag{5.8}$$

As in the treatment of (5.1) the constraint is gotten by sending $c\to\infty$. This yields

$$R\psi^2-\sigma^2-\pi^2=0\,. \tag{5.9}$$

We must eliminate the heavy field $\hat\sigma$; substituting (5.7) into (5.9) yields,[34] after choosing a sign convention,

$$\hat\sigma(\pi^2,\hat\psi)=\frac{1}{1-R}\{-R^{1/2}\hat\psi+[R\hat\psi^2-(1-R)\pi^2]^{1/2}\}\,. \tag{5.10}$$

The nonlinear Lagrangian is then given by

$$\mathscr{L}=-\tfrac{1}{2}(\partial_\mu\pi)^2-\tfrac{1}{2}(\partial_\mu\hat\psi)^2 -\frac{1}{2}\frac{1}{(1-R)^2}(\partial_\mu\{-R^{1/2}\hat\psi +[R\hat\psi^2-(1-R)\pi^2]^{1/2}\})^2\,. \tag{5.11}$$

The vacuum value $\langle\hat\psi\rangle$ should be chosen to be $(1+R)^{1/2}R^{-1/2}F_\pi/\sqrt{2}$. One may see the physical significance of the ratio R from (5.6): it measures the ratio "(matter condensate)/(glue condensate)" in the toy model; explicitly

$$R=\frac{F_\pi{}^2}{32a\langle H\rangle^{1/2}}=0.075\,, \tag{5.12}$$

where we used the typical value $a\approx0.06$ and (2.2). Notice that (5.7) shows the dilaton state $\hat\psi$ to contain about an $R^{1/2}=0.27$ admixture of "matter." To extract physical amplitudes at the tree level from (5.11) we may expand

$$\hat\psi=\langle\hat\psi\rangle+\chi\,. \tag{5.13}$$

Substituting this into (5.11), we find the interaction terms up to quintic order in the fields

$$\mathscr{L}_{\text{int}}=-\frac{1}{4F_\pi{}^2(1+R)}(\partial_\mu\pi^2)(\partial_\mu\pi^2) +\frac{R^{1/2}}{\sqrt{2}F_\pi{}^3(1+R)^{3/2}}(\partial_\mu\pi^2)[\partial_\mu(\chi\pi^2)]+\cdots\,. \tag{5.14}$$

Notice that the $\chi\pi^2$ terms have all canceled out in agreement with the result of Ref. 9. In fact every interaction vertex must have at least four pion fields.

The tree-level $\pi\pi$ scattering amplitudes can be read off from the first term of (5.14). We observe the new feature that the canonical results[5] are multiplied by a factor $(1+R)^{-1}\approx0.93$. It is interesting that this deviation by itself is perhaps large enough to suggest that the existence of a true (zero-mass) dilaton is ruled out experimentally. In the equivalent presentation of the pion-dilaton theory (5.4), the deviation from the canonical scattering lengths may not be immediately obvious. However, inclusion of the dilaton pole diagrams together with the predicted $\chi\,\partial_\mu\pi\cdot\partial_\mu\pi$ interaction (which gives zero $\chi\to\pi\pi$ on shell) is easily seen to result in the same factor $(1+R)^{-1}$. Actually, there exists a point transformation of the field variables which transforms the Lagrangian (5.4) into (5.11).

We remark that the model (5.11) is consistent with the Jacobi identity of the dilaton operator, axial charge, and pion field even though π_i has the canonical scale dimension one.

A possible different application of the toy model presented in this section might be to discuss the response of the Skyrmion[35] to scale instabilities.

VI. PROPERTIES OF A HIGHER-MASS GLUEBALL

Most theoretical calculations[26] lead one to expect the mass of the lowest-lying scalar glueball to be in the 0.8 to 2 GeV range. There are a number of experimental candidates; perhaps the one for which the most forceful claims have been made is the $G(1590)$.[6,7] This particle (mass=1592±25 MeV, width=210±40 MeV) has an interesting signature—the dominant decay channel appears to be $\eta\eta'$ and the $\pi\pi$ and $K\bar{K}$ widths are rather small compared to usual expectations. Specifically

$$\frac{G\to\pi\pi}{G\to\eta\eta}<1,\quad \frac{G\to K\bar{K}}{G\to\eta\eta}<0.6,\quad \frac{G\to\eta\eta'}{G\to\eta\eta}=2.7\pm0.8\ . \tag{6.1}$$

Note, however, that the $\pi\pi$ coupling of the $G(1590)$ cannot vanish since it is produced in the reaction $\pi^-+p\to G(1590)+n$.

It seems natural to try to accommodate this particle in a simple effective Lagrangian of the present type. The large partial width of the order of 100 MeV into $\eta\eta'$ can easily be understood when the Lagrangian is modified to satisfy the U(1) anomaly as well as the trace anomaly. More discussion of this feature will be given in the next section. What at first seems harder to understand is the small partial width into $\pi\pi$ (and $K\bar{K}$). For example, in their analysis of a similar model containing only a glueball, Ellis and Lanik[8] find a pionic partial width of about 6 GeV for a 1.59-GeV glueball. A similarly large width was found in Ref. 9. Now recall from Sec. III that the problem of very large widths also exists for the quarkonium scalars when no derivative-interaction terms are included. A simple choice of derivative term was found sufficient to explain the quarkonium-nonet scalar widths. We will point out here that this same derivative-coupling term also reduces the glueball widths into $\pi\pi$ and $K\bar{K}$ to the appropriate order of magnitude.

One can roughly understand this effect in the following way. By scale symmetry one finds the ratio of the nonderivative-coupling constants $g_{h\phi\phi}/g_{\kappa K\pi}$ to be $-ZF_\pi/(4\langle H\rangle)$. For $d=1$ the only contribution to the derivative-coupling constants for these processes comes from the third term of (4.1) and yields for the analogous effective ratio $-ZF_\pi/(4\langle H\rangle)\times(m_h{}^2/m_\kappa{}^2)$. Since the strength of the derivative-coupling term has been chosen to reduce the $\kappa\to K\pi$ width to its physical value the width of the bare glueball state is similarly reduced for comparable masses. This effect is further modified by the mass difference of the physical singlet particles and their mixing.

Since our goal is to demonstrate a qualitative result, we shall simplify the model by representing the scalar-quarkonium states by a single SU(3) singlet, σ_p' (i.e., neglect of σ_8-σ_0 mixing). One would expect the mass of this effective singlet to be very roughly around 1.1 GeV and its with into $\pi\pi$ to be rather substantial (in the 200-MeV range). The glueball-bare singlet mixing should result in a σ_p' with these properties as well as another—physical glueball—state, h_p whose mass is around 1590 MeV and whose $\pi\pi$ and $K\bar{K}$ widths [see (6.1)] are of the order of 20 MeV rather than several GeV.

The Lagrangian is taken basically to be Eq. (4.1). In order to more accurately model the quarkonium-gluonium mixing we shall introduce a chiral- and scale-invariant term which yields "kinetic" mixing in addition to the usual kind. This term turns out not to change the qualitative nature of our results but it can be used for "fine-tuning" if desired. Thus we write

$$\mathscr{L}=[\text{Eq.}(4.1)] + K\left[\frac{\langle H\rangle}{H}\right]^{(d+1)/2}\partial_\mu H\,\partial_\mu \text{Tr}(MM^\dagger)\ , \tag{6.2}$$

where K measures the strength of this new term. The treatment of (6.2) is similar to that for (4.1). Notice that the additional term makes no contribution to the axial-vector currents, as one sees by a partial integration. However, the existence of both kinetic and mass term mixing results in a nonorthogonal mixing matrix. Furthermore, we will use the scale dimension $d=1$ here, which gives more realistic masses. The parameters e and f are found from (4.2) and from (4.3) (where the + sign is required on the RHS to get acceptable results). The parameters which are subject to variation are $Z=\langle H\rangle^{3/4}/a$ (Sec. II shows that $Z\approx0.37$ GeV^3 when glue-matter mixing is neglected), $A=$ the bare-quarkonium-singlet squared mass and $Q=-\sqrt{3}KZ\langle\bar{q}q\rangle^2/F_\pi$ (a convenient dimensionless quantity to be used instead of K).

Diagonalizing the quadratic terms of (6.2) in the SU(3) singlet channel results in the following relation between bare and physical fields ψ_i:

$$\begin{bmatrix}\sigma'\\ h\end{bmatrix}=\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}, \tag{6.3}$$

where the p_{ij} are given by

$$\begin{aligned}
p_{11}&=\frac{1}{\sqrt{2}}\left[\frac{\cos\omega}{\sqrt{1-Q}}-\frac{\sin\omega}{\sqrt{1+Q}}\right],\\
p_{12}&=\frac{1}{\sqrt{2}}\left[\frac{\sin\omega}{\sqrt{1-Q}}+\frac{\cos\omega}{\sqrt{1+Q}}\right],\\
p_{21}&=-\frac{1}{\sqrt{2}}\left[\frac{\cos\omega}{\sqrt{1-Q}}+\frac{\sin\omega}{\sqrt{1+Q}}\right],\\
p_{22}&=\frac{1}{\sqrt{2}}\left[-\frac{\sin\omega}{\sqrt{1-Q}}+\frac{\cos\omega}{\sqrt{1+Q}}\right].
\end{aligned} \tag{6.4}$$

Here the angle ω is determined from the formula

$$\tan 2\omega=\frac{(C+B^2/A-A)(1-Q^2)^{1/2}}{Q(A+C+B^2/A)+2B}\ , \tag{6.5}$$

where $B=\sqrt{3}Z\,dF_\pi A/8\langle H\rangle$ and $C=Z^2/4\langle H\rangle$. In the limit $Q\to0$, $\omega\to\theta+\pi/4$ [see (4.4)]. Furthermore, the physical masses are gotten from

$$m_l{}^2=\frac{1}{2}\frac{1}{1-Q^2}\left[\left[A+C+\frac{B^2}{A}+2QB\right]+\frac{(-1)^l(1-Q^2)^{1/2}(A-C-B^2/A)}{\sin 2\omega}\right]. \tag{6.6}$$

Finally the pionic decay widths are given by

$$\Gamma(\psi_i \to \pi\pi) = 3g_i{}^2/32\pi m_i \ ,$$
$$g_i = (m_i{}^2/2 - m_\pi{}^2)(cp_{2i} + p_{1i}) - \tilde{c}p_{2i} - \tilde{b}p_{1i} + \frac{2Qm_i{}^2 p_{2i}}{\sqrt{3}F_\pi} \ , \quad (6.7)$$

where b, $\tilde{b}$, c, and $\tilde{c}$ are given in (4.7). The formula for $\Gamma(\psi_i \to K\bar{K})$ can be obtained by a trivial modification of (6.7).

First let us fix $Q=0$ and $A=1.5$ GeV^2, which is about what one would expect the bare-quarkonium squared mass to be in the quark model. Then both masses and all widths depend only on the single parameter Z. A qualitatively good overall fit is found for $Z=0.25$ GeV^3. In this case, denoting h_p as the heavier and σ'_p as the lighter particle,

$$m(h_p)=1.53 \text{ GeV}, \quad m(\sigma'_p)=0.86 \text{ GeV} \ ,$$
$$\Gamma(h_p \to \pi\pi)=63 \text{ MeV}, \quad \Gamma(\sigma'_p \to \pi\pi)=307 \text{ MeV} \ , \quad (6.8)$$
$$\Gamma(h_p \to K\bar{K})=33 \text{ MeV} \ .$$

This reproduces the desired pattern in which the heavier scalar meson (glueball) has fairly small widths into $\pi\pi$ and $K\bar{K}$ while the lighter scalar meson has an appreciable $\pi\pi$ width. For the values in (6.8) the states h_p and σ'_p are close to 50-50 mixtures of bare quarkonium and bare gluonium. Thus the small width into $\pi\pi$ is not an indication that h_p is "pure glue." Equation (6.8) gives a better description of the glueball properties than it does for the effective scalar singlet which comes out too low and too wide. The low value for the mass of the lighter particle is related to the mass bound discussed in Sec. II. By including the effect of kinetic term mixing we can raise this mass at the expense of substantially increasing its width. Two typical results are ($Z=0.26$ GeV^3, $A=1.7$ GeV^2, $Q=-0.1$)

$$m(h_p)=1.56 \text{ GeV}, \quad m(\sigma'_p)=0.94 \text{ GeV} \ ,$$
$$\Gamma(h_p \to \pi\pi)=78 \text{ MeV}, \quad \Gamma(\sigma'_p \to \pi\pi)=465 \text{ MeV} \ , \quad (6.9)$$
$$\Gamma(h_p \to K\bar{K})=19 \text{ MeV} \ ,$$

and ($A=2.2$ GeV^2, $Z=0.25$ GeV^3, $Q=-0.3$)

$$m(h_p)=1.57 \text{ GeV}, \quad m(\sigma'_p)=1.07 \text{ GeV} \ ,$$
$$\Gamma(h_p \to \pi\pi)=6 \text{ MeV}, \quad \Gamma(\sigma'_p \to \pi\pi)=841 \text{ MeV} \ , \quad (6.10)$$
$$\Gamma(h_p \to K\bar{K})=238 \text{ MeV}, \quad \Gamma(\sigma'_p \to K\bar{K})=546 \text{ MeV} \ .$$

Finally, we remark that it is possible to construct in our framework a model which reproduces all the known properties of the $G(1590)$ if we introduce it as a new field with a vanishing vacuum value and treat H as an auxiliary field, as discussed at the end of Sec. II. Then G will not modify any of the successful predictions for the widths of the scalar mesons in Sec. IV. The properties of the G can be accommodated by a suitable choice of additional parameters, but there are no predictions.

VII. $\eta\eta'$ MODES AND AN ANSATZ FOR THE POTENTIAL

Several years before the $G(1590)$ was reported it was suggested[11] that the $\eta\eta'$ mode would predominate for a sufficiently heavy-scalar glueball. This prediction was based on an effective Lagrangian for the U(1) anomaly[18] of QCD modified to include the trace anomaly discussed in detail here. That model featured the pseudoscalar field

$$G = (-3ig^2/16\pi^2)\epsilon_{\mu\nu\alpha\beta}\text{Tr}(F_{\mu\nu}F_{\alpha\beta}) \ , \quad (7.1)$$

where g is the QCD coupling constant and $F_{\mu\nu}$ the field strength tensor, in addition to the scalar field H. The U(1) anomaly requires a term in the Lagrangian

$$\frac{iG}{12}(\ln \det M - \ln \det M^\dagger) \ , \quad (7.2)$$

which is evidently scale invariant. In order for the bare pseudoscalar singlet state η'_0 to develop a mass, G must behave as an auxiliary field and get eliminated by its equation of motion in terms of η'_0. To lowest order in the $N_c{}^{-1}$ expansion a term like G^2 is required. But G^2 is not scale invariant. The most natural way to make it scale invariant is to replace it by G^2/H, which upon using (2.3) and the G-η'_0 equivalence leads to an $h\eta'_0\eta'_0$ interaction proportional to $m^2(\eta')$. Taking account of η-η' mixing (with an angle $\theta_p \approx -18°$) and the σ'-h mixing [measured by θ defined by (4.4) in the $Q=0$ case] finally yields the prediction for the $\eta\eta'$ partial width:

$$\Gamma(h_p \to \eta\eta') = \frac{g_{h_p\eta\eta'}{}^2 |\mathbf{q}|}{8\pi m^2(h_p)} \ ,$$
$$g_{h_p\eta\eta'} = \left[\frac{Zm_{\eta'}{}^2}{2\langle H\rangle}\right]\cos\theta \sin 2\theta_p \ . \quad (7.3)$$

For typical values of Z and θ describing $h_p = G(1590)$, Eq. (7.3) gives a partial $\eta\eta'$ width[36] of about 66 MeV. This is considerably larger than what would normally be expected for an OZI rule and SU(3)-violating term and is in qualitative agreement with (6.1). The ratio $\Gamma(h_p \to \eta\eta')/\Gamma(h_p \to \eta\eta)$ is about 8 if just the $h\eta'_0\eta'_0$ term discussed in this section is taken into account for the $\eta\eta$ mode. However the $\eta\eta$ mode should receive important contributions from other terms, as discussed in the last section, so the exact value of this ratio is sensitive to the choice of parameters.

In the estimate above we assumed that a term like $G^2/\text{Tr}(MM^\dagger MM^\dagger)$, which (for $d=1$) could also be a suitable modification of G^2, is not present. While this assumption appears natural one might like to find a model in which it holds. In general, the problem of finding the low-energy QCD effective Lagrangian involving spin-0 matter as well as gluonium fields seems to be a very interesting one. Although chiral symmetry, the U(1) anomaly, the trace anomaly, and the $1/N_c$ expansion give important constraints there is still a good deal of freedom. Perhaps a useful clue is the similar structure of the terms for the U(1) and trace anomalies. The theory of QCD is labeled by two parameters: the vacuum angle θ and a (complex) "angle" $i\tau$ defined in terms of the scale parameter Λ by $\Lambda = e^{\tau}$. The U(1) anomaly term gives a piece like θG in the effective Lagrangian while the scale anomaly [see (2.1), for example] gives a piece like τH. If the two parameters are represented by a single complex number $Z = \tau + i\theta$, the extra terms may be written as proportional to

$$x^{-1}(\tau + i\theta)(H - iGx) + \text{H.c.} ,$$

where x is a real parameter to be determined from a self-consistency requirement. This suggests searching for a form of the Lagrangian expressed in terms of the combination[37]

$$\Phi = H - iGx . \tag{7.4}$$

One interesting possibility is to choose a potential $V = V_1(H, M, M^\dagger) + V_2$ where V_1 does not contain G and does not contribute to either anomaly. V_2 is postulated to have the form

$$V_2 = f(\Phi, M) + \text{H.c.} , \tag{7.5}$$

where f is some function. The requirement of correct U(1) anomaly is satisfied with

$$V_2 = \frac{1}{4N_F x}\Phi\left[\ln\frac{\det M}{\Lambda^{dN_F}} - i\theta\right] - \rho\Phi\ln\frac{\Phi}{\Lambda^4} + \text{H.c.} , \tag{7.6}$$

where N_F is the number of flavors. The real number ρ is determined from the requirement of correct scale anomaly to be

$$\rho = \frac{d - 2x}{16x} . \tag{7.7}$$

Eliminating G by its equation of motion yields

$$\begin{aligned} G &= -\frac{H}{x}\tan\frac{\phi}{8N_F x\rho} , \\ \phi &= -i\ln\frac{\det M}{\det M^\dagger} - 2\theta , \end{aligned} \tag{7.8}$$

which on substitution back into (7.6) gives

$$\begin{aligned} V_2 = {} & \frac{H}{4N_F x}\ln\frac{\det(MM^\dagger)}{\Lambda^{2dN_F}} - 2\rho H\ln\frac{H}{\Lambda^4} \\ & + \rho H\ln\cos^2\left[\frac{\phi}{8N_F x\rho}\right] . \end{aligned} \tag{7.9}$$

So far, only the parameter x is not fixed. This can be done in the following manner. When matter fields are present they force periodicity[38] (with period 2π) in the vacuum angle θ since a shift of θ by 2π can be canceled out by an otherwise harmless redefinition of the quark fields (or the composite quantity M). But we can require the Lagrangian to make sense when the field M is deleted. Then the θ dependence comes from the last term of (7.9):

$$\rho H\ln\cos^2\left[\frac{\theta}{4N_F x\rho}\right] . \tag{7.10}$$

Demanding a periodicity of 2π then determines x from

$$2N_F x\rho = \pm 1 . \tag{7.11}$$

For scale dimension $d = 1$, as indicated by the discussion of earlier sections, and for $N_F = 3$, we find

$$x = -\tfrac{5}{6} \text{ or } x = +\tfrac{11}{6} . \tag{7.12}$$

Now that the complete part of the Lagrangian involving η_0' is known we can predict its mass, remembering that $\phi \approx 4\sqrt{3}\eta_0'/F_\pi + \cdots$ yields, by expanding the last term of (7.9),

$$m_{\eta'}^2 = \frac{\langle H\rangle}{|x|F_\pi^2} . \tag{7.13}$$

Here the relatively small contribution due to nonzero quark masses has been neglected.

The choice $x = -\frac{5}{6}$ gives a reasonable prediction

$$m_{\eta'}^2 = \frac{6}{5}\frac{\langle H\rangle}{F_\pi^2} = (964\ \text{MeV})^2 . \tag{7.14}$$

Notice that $m_{\eta'}$ is not predicted if one just demands correct trace anomaly and U(1) anomaly. An additional assumption, like that of (7.5) is required. Notice also that with the present ansatz the $h\eta\eta'$ coupling strength is given by (7.3) without any further assumptions.

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[28]See Chanowitz and Ellis (Ref. 13).

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