P or CP determination by sequential decays: $V_1 V_2$ modes with decays into $\bar{l}_A l_B$ and/or $\bar{q}_A q_B$

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For application at modern colliders, the $\phi\phi$ symmetry tests are generalized to the generic sequential decay $X \to V_1 V_2$ where at least one of the vector bosons decays into a lepton and an antilepton, or a quark and an antiquark. The general decay correlation function $I(\theta_1,\theta_2,\phi)$ and its integrated distributions are simultaneously characterized in convenient α's, β's, and γ's which are simple triquadratic functions of the three sets of helicity amplitudes. From a JJ, $\Upsilon\Upsilon$, or Z^0Z^0 mode the parity η_P or the CP eigenvalue γ_{CP} of X can always be determined if the decay is respectively P or CP invariant, as can $(-)^J$ of X except in certain circumstances. From a W^+W^- mode, γ_{CP} can always be determined. For modes such as ϕJ, $J\Upsilon$, and ϕZ^0, JZ^0, $\eta_P(-)^J$, or $\gamma_{CP}(-)^J$ can be determined provided certain amplitudes do not accidentally vanish. Generalization to the gZ^0 and gg decay channels, $g=$gluon jet, is discussed.

I. INTRODUCTION

It is of fundamental importance to make basic symmetry tests and symmetry measurements as modern colliders are used to study higher-energy regions. By further generalization of the $\phi\phi$ symmetry tests,[1-4] in this series of papers we show what can be learned by measurement of the decay correlation function $I(\theta_1,\theta_2,\phi)$ for the sequential decay

$$X \to V_1 V_2, \quad V_2 \to A_2 B_2, \quad V_1 \to A_1 B_1 \,.$$

Present and proposed detectors at such colliders are able to detect vector bosons such as J/ψ, Υ, and the Z^0 and $W^\pm$ by their decays into a lepton and an antilepton, and/or by their decays into a quark and an antiquark. These new "probes" are especially useful since information carried by the vector meson's polarization is carried forward and displayed kinematically upon the vector meson's decay into AB, or into a three-body channel. Z^0 and $W^\pm$ decays violate P and C separately so their decay distributions provide more information about the first decay $X\to V_1V_2$ than does $J/\psi\to\mu^+\mu^-$, $\phi\to K^+K^-$, or $\omega\to\pi^+\pi^-\pi^0$. This effect is larger when final fermion masses are small versus the magnitude of the decay momentum. It is largest for $W^\pm$ decays and next largest for $Z^0\to\bar{q}q$ where $Q^{\rm EM}=-\frac{1}{3}$ versus $\bar{q}q$ with $Q^{EM}=\frac{2}{3}$ or $\mu^+\mu^-$, because the effect depends on the ratio r of the axial-vector to the vector coupling coefficients between the intermediate vector boson and the fermion current.

Besides providing a generic example and formalism for the study and application of symmetries at high energies through sequential decays, our analysis shows that the advantages of the $\phi\phi$ symmetry tests are generalizable to V_1V_2 modes with $\bar{l}_A l_B$ and/or $\bar{q}_A q_B$ pairs. The tests are rigorous, the signatures are striking, the P/CP determination is independent of the production mechanism, the P/CP determination never fails for X with spin 0 or when there is a $V_1 \leftrightarrow V_2$ exchange property, and finally the tests are normally applicable in semi-inclusive production experiments since many X decay modes yield observed states with only charged particles.

A V_1V_2 decay mode with a sufficient number of events is of course a nontrivial prerequisite for these tests. On the other hand, in some searches for new or anticipated resonance phenomena in domains with complicated competing backgrounds, it may be helpful in excluding spurious signals to have available knowledge of the simple yet nontrivial behavior of the associated decay correlation function $I(\theta_1,\theta_2,\phi)$ for an actual resonance.

Last, because of the promising future[5] of "jet physics," we wish to point out that this same type of P/CP test, for instance, generalizes to gZ^0 and gg decay channels, $g=$gluon jet. While it is difficult to experimentally isolate gluon jets, it should be possible to choose an enriched jet sample with a leading vector meson decaying into a $\bar{l}l$ pair which has been most likely produced from the high-z fragmentation of the gluon. Assuming that this vector meson arises from g, and that its polarization on the average follows that of g, then the polarizations of the leading vector meson and Z^0 would be the same as those of the primary gluon and Z^0. By this factorization argument, the results of this paper for JZ^0 show that the sign of the β parameter will be equal to $\gamma_{CP}(-)^J$ for a gluon jet plus Z^0 that arises from a source X where $X=\bar{X}$, and $\gamma_{CP}=CP$ eigenvalue of X and $J=$spin of X. Similarly, the results for JJ show that $\eta_P=\mathrm{sgn}\beta$ for a gg mode arising from such a source, $\eta_P=$parity of X.

In Sec. II we derive and conveniently characterize the general decay correlation function $I(\theta_1,\theta_2,\phi)$ for the sequential decay $X\to V_1V_2$ followed by $V_{1,2}$ decays into $\bar{l}_A l_B$ and/or $\bar{q}_A q_B$ pairs. Trueman's method[3] is used for the derivations. Some of the results for $J=0$ were obtained earlier (see third paper in Ref. 4) by a covariant coupling calculation and have now been used as checks.

Parameters familiar to us from $\phi\phi$ symmetry tests and here called "the α's, β's, and γ's" are used to conveniently characterize both $I(\theta_1,\theta_2,\phi)$ *and* its associated integrated distributions. In Sec. III for X of any spin we present the results for P/CP determination for the various decay modes, and for possible signature determination. Stronger results for $J=1$ are given in Sec. IV. In a short Appendix we enumerate what part of a complete determination of the helicity amplitudes $a_{\lambda_1\lambda_2}$ describing $X\rightarrow V_1V_2$ is provided by an "ideal" measurement of $I(\theta_1,\theta_2,\phi)$. In several cases with $J=0$ or 1, complete information can in principle be obtained.

In the following paper,[6] the stronger results for $J=0$ are discussed. And in a third paper,[7] we apply this formalism to develop simple tests for CP/P violation by such sequential decays.

II. DERIVATION OF GENERAL DECAY CORRELATION FUNCTION

We consider the following sequential decay of an X system of any spin J:

$$X\rightarrow V_1V_2,\quad V_2\rightarrow A_2B_2,\quad V_1\rightarrow A_1B_1\ .$$

For definitions of the angles, we refer the reader to Trueman's figures, Ref. 3. In the X rest frame the nine helicity amplitudes[8] for the decay $X\rightarrow V_1V_2$ are specified by

$$\langle\Theta,\Phi,\lambda_1,\lambda_2\,|\,JM\rangle_X=D^{J*}_{M\lambda}(\Phi,\Theta,-\Phi)a_{\lambda_1\lambda_2}\ , \tag{1}$$

where $\lambda=\lambda_1-\lambda_2$ and Θ and Φ are the polar and azimuthal angles of V_1. Similarly in the V_1 rest frame the helicity amplitudes for the decay $V_1\rightarrow A_1B_1$ are specified by

$$\langle\theta_1,\phi_1,\mu_1,\mu_2\,|\,1\lambda_1\rangle_{V_1}=D^{1*}_{\lambda_1\mu}(\phi_1,\theta_1,-\phi_1)T(\mu_1\mu_2)\ , \tag{2}$$

where $\mu=\mu_1-\mu_2$ and θ_1 and ϕ_1 are the polar and azimuthal angles of A_1 in the usual helicity coordinate system in the V_1 rest frame. The z axis, labeled z_1, is shown in Fig. 1, and the fixed x axis is not shown in Fig. 1. In this coordinate system the azimuthal angle of A_2 from the decay of V_2 is $(2\pi-\phi_2)$. These angles ϕ_2 and ϕ_1 are invariant under a Lorentz boost along the V_2 three-momentum to the V_2 rest frame (see Fig. 1), in which the helicity amplitudes for the decay $V_2\rightarrow A_2B_2$ are specified by

$$\langle\theta_2,\phi_2,\tau_1,\tau_2\,|\,1\lambda_2\rangle_{V_2}=D^{1}_{\lambda_2\tau}(\phi_2,\theta_2,-\phi_2)W(\tau_1\tau_2)\ , \tag{3}$$

where $\tau=\tau_1-\tau_2$ and θ_2 and ϕ_2 are the polar and azimuthal angles of A_2 in the V_2 rest frame.

The general decay correlation distribution in terms of these six variables is

$$I(\Theta,\Phi,\theta_1,\phi_1,\theta_2,\phi_2)=\sum_{MM'\lambda_1\lambda_2\lambda_1'\lambda_2'}\rho^{\text{prod }X}_{MM'}\rho_{MM'\lambda_1\lambda_2\lambda_1'\lambda_2'}(X\rightarrow V_1V_2)\times\rho_{\lambda_1\lambda_1'}(V_1\rightarrow A_1B_1)\rho_{\lambda_2\lambda_2'}(V_2\rightarrow A_2B_2)\ , \tag{4}$$

where the three decay density matrices are

$$\rho_{MM'\lambda_1\lambda_2\lambda_1'\lambda_2'}=D^{J*}_{M\lambda}(\Phi,\Theta,-\Phi)D^{J}_{M'\lambda'}(\Phi,\Theta,-\Phi)\times a_{\lambda_1\lambda_2}a^*_{\lambda_1'\lambda_2'} \tag{5}$$

with $\lambda'=\lambda_1'-\lambda_2'$,

$$\rho_{\lambda_1\lambda_1'}=\sum_{\mu_1\mu_2}D^{1*}_{\lambda_1\mu}(\phi_1,\theta_1,-\phi_1)D^{1}_{\lambda_1'\mu}(\phi_1,\theta_1,-\phi_1)\times T(\mu_1\mu_2)T(\mu_1\mu_2)^* \tag{6}$$

and

$$\rho_{\lambda_2\lambda_2'}=\sum_{\tau_1\tau_2}D^{1*}_{\lambda_2\tau}(\phi_2,\theta_2,-\phi_2)D^{1}_{\lambda_2'\tau}(\phi_2,\theta_2,-\phi_2)\times W(\tau_1\tau_2)W(\tau_1\tau_2)^*\ . \tag{7}$$

We define the important azimuthal angle ϕ between the A_1 and A_2 decay planes by

$$\phi=\phi_1+\phi_2 \tag{8}$$

and then holding it fixed integrate over all $0\leq\phi_1\leq 2\pi$ which forces $\lambda=\lambda'$ or

$$\Lambda\equiv\lambda_1-\lambda_1'=\lambda_2-\lambda_2'\ . \tag{9}$$

Orthogonality of the $D^J_{M\lambda}(\Phi,\Theta,-\Phi)$ can be used to integrate over all V_1 directions (Θ,Φ) which gives a factor of $4\pi\delta_{MM'}/(2J+1)$. Absorbing the $4\pi/(2J+1)$ we obtain the general decay correlation function for this sequential decay

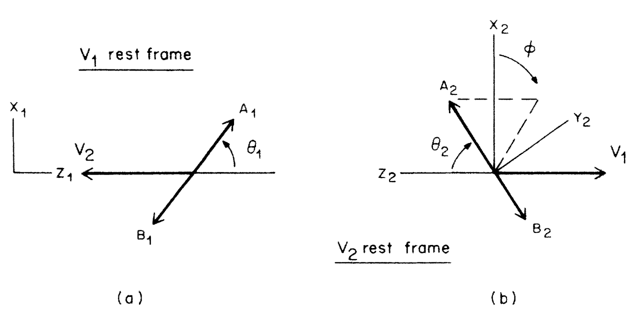

FIG. 1. An illustration showing the three angles θ_1, θ_2, and ϕ describing the decay $X\rightarrow V_1V_2$. A boost along the negative z axis transforms the kinematics from the V_1 rest frame (a) to the V_2 rest frame (b).

$$I(\theta_1,\theta_2,\phi)=(\mathrm{Tr}\rho^{\mathrm{prod}\,X})\sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'\Lambda jj'\mu_1\mu_2\tau_1\tau_2} a_{\lambda_1\lambda_2}a^*_{\lambda_1'\lambda_2'}e^{i\Lambda\phi}(-)^{\lambda_1-\lambda_2}$$
$$\times(2j+1)(2j'+1)d^j_{\Lambda 0}(\theta_1)d^{j'}_{\Lambda 0}(\theta_2)(-)^{\mu+\tau}|T(\mu_1\mu_2)|^2|W(\tau_1\tau_2)|^2$$
$$\times\begin{bmatrix}1 & 1 & j\\ \mu & -\mu & 0\end{bmatrix}\begin{bmatrix}1 & 1 & j\\ \lambda_1 & -\lambda_1' & -\Lambda\end{bmatrix}\begin{bmatrix}1 & 1 & j'\\ \tau & -\tau & 0\end{bmatrix}\begin{bmatrix}1 & 1 & j'\\ \lambda_2 & -\lambda_2' & -\Lambda\end{bmatrix}, \quad (10)$$

where the three angles θ_1, θ_2, and ϕ are as shown in Fig. 1. That is, $I(\theta_1,\theta_2,\phi)$ is a function of the A_1,A_2 acceptances θ_1,θ_2 and the angle ϕ between their associated decay planes.

For the case X is unpolarized (or has spin 0), the six variable decay correlation distribution in Eq. (4) is isotropic in Θ, Φ, and ϕ_1; hence, full Θ, Φ, and ϕ_1 acceptance is not required and Eq. (10) follows for any Θ, Φ, ϕ_1 acceptance. In other cases, the appearance in Eq. (10) of $(\mathrm{Tr}\rho^{\mathrm{prod}\,X})$ as a common factor means that the distribution $I(\theta_1,\theta_2,\phi)$ is independent of the initial X polarization which of course is one of the advantages of this type of decay symmetry test. Nevertheless since the symmetry tests we develop for $J\geq 1$ all follow from Eq. (10), there is the implicit requirement in their empirical application to examine the dependence of the signatures and consistency checks on Θ, Φ, and ϕ_1 cuts when less than full Θ, Φ, and ϕ_1 acceptance is present in the data sample. In other cases, e.g., for reactions at e^+e^- colliders, the X production density matrix may be sufficiently known or calculable so that full Θ, Φ, and ϕ_1 acceptance is not required for the application of analogous signatures and consistency checks.

TABLE I. Parameters characterizing the decay density matrices for J, Z^0, and W^+ decay into a lepton and antilepton, or into a quark and antiquark. Arrows indicate the limiting values when the effect of the final fermion masses is neglected. The important normalized parameters, the $\mathcal{R}$ and $\mathcal{T}$, through Eqs. (36), appear in the master formula, Eq. (20), for the sequential decay correlation function $I(\theta_1,\theta_2,\phi)$. By setting the $\mathcal{R}=1$ and $\mathcal{T}=0$, the master formula for $I(\theta_1,\theta_2,\phi)$ can also be used when $V_1\to A_1B_1$ with A_1 and B_1 spin-0 mesons, or $V_1=\omega$ meson. In the second column, for Z^0 decays the parameter r is the ratio of the axial-vector to the vector coupling coefficients so for $Z^0\to\mu^+\mu^-$, $r=(L_\mu-R_\mu)/(L_\mu+R_\mu)=(1-4\sin^2\theta_W)^{-1}$ and for $Z^0\to\bar{q}q$, $r=(L_q-R_q)/(L_q+R_q)=\tau_3/(\tau_3-4Q_q\sin^2\theta_W)$. For $W^-\to b\bar{t}$, $e^-\bar{\nu}_e$ the third column entries also apply except the sign of the T and $\mathcal{T}$ entries is opposite.

Parameter \ Mode	$J\to\mu^+\mu^-$	$Z^0\to\mu^+\mu^-$ $\to\bar{q}q$	$W^+\to\bar{b}t$ $\to e^+\nu_e$
$R=\|T(++)\|^2+\|T(--)\|^2$	$2\mu^2$ $\to 0$	$2\mu^2$ $\to 0$	$2(E_tE_b-p^2)$ $\to 0$
$S=\|T(+-)\|^2+\|T(-+)\|^2$	m^2	$(1+r^2)m^2-4r^2\mu^2$ $\to(1+r^2)m^2$	$4(E_tE_b+p^2)$ $\to 2m^2$
$T=\|T(-+)\|^2-\|T(+-)\|^2$	0	$-4rpm$ $\to -2rm^2$	$-4pm$ $\to -2m^2$
$\mathcal{R}=\dfrac{R-\frac{1}{2}S}{R+S}$	$-\dfrac{1}{2}\dfrac{m^2-4\mu^2}{m^2+2\mu^2}$ $\to -\frac{1}{2}$	$-\dfrac{p^2}{\left[2p^2+\dfrac{3\mu^2}{1+r^2}\right]}$ $\to -\frac{1}{2}$	$-\dfrac{2p^2}{3E_tE_b+p^2}$ $\to -\frac{1}{2}$
$\mathcal{T}=\dfrac{T}{R+S}$	0	$-\dfrac{2rmp}{(1+r^2)\left[2p^2+\dfrac{3\mu^2}{1+r^2}\right]}$ $\to -\dfrac{2r}{1+r^2}$ -0.237 for $\mu^+\mu^-$ 0.337 for $\bar{u}u$ -0.706 for $\bar{d}d$	$-\dfrac{2pm}{3E_tE_b+p^2}$ $\to -1$

The $\mathscr{R}$, $\mathscr{T}$, $\mathscr{U}$, and $\mathscr{W}$ parameters for $V_{1,2}\to A_{1,2}B_{1,2}$

Previous work has considered the simpler case where $A_1, B_1, \ldots$ are all spin-0 mesons or $V_{1,2}=\omega$ meson. When $V_1\to A_1B_1$ where A_1B_1 are a lepton and an antilepton, $\bar{l}_A l_B$, or a quark and an antiquark, $\bar{q}_A q_B$, as for J/ψ, Z^0, and $W^{\pm}$ decays, the standard decay, density matrix elements give a factor summed over $j=0,1,2$ in Eq. (10) of ($\mu=\mu_1-\mu_2$)

$$\sum_{\mu_1\mu_2}(-)^{\mu}|T(\mu_1\mu_2)|^2\begin{bmatrix}1 & 1 & j\\ \mu & -\mu & 0\end{bmatrix}$$
$$=R\begin{bmatrix}1 & 1 & j\\ 0 & 0 & 0\end{bmatrix}-\tfrac{1}{2}[1+(-)^j]S\begin{bmatrix}1 & 1 & j\\ 1 & -1 & 0\end{bmatrix}+\tfrac{1}{2}[1-(-)^j]T\begin{bmatrix}1 & 1 & j\\ 1 & -1 & 0\end{bmatrix}, \tag{11}$$

where we define

$$R=|T(++)|^2+|T(--)|^2, \tag{12}$$

$$S=|T(+-)|^2+|T(-+)|^2, \tag{13}$$

$$T=|T(-+)|^2-|T(+-)|^2. \tag{14}$$

For $V_2\to A_2B_2$ and $W(\tau_1\tau_2)$ we simply let $R,S,T\to U,V,W$.

There may occur circumstances[9] where it would be of interest to use Eq. (10) to study the decay $V_1\to A_1B_1$. We do not analyze this application in detail in this paper, except implicitly.

By P invariance, for a final $\bar{l}_A l_B$ or a final $\bar{q}_A q_B$

$$T(-\mu_1,-\mu_2)=\eta_V\eta_1\eta_2T(\mu_1,\mu_2) \tag{15}$$

and by C invariance for a final particle-antiparticle pair

$$T(\mu_1,\mu_2)=-C_n(V_1)T(\mu_2,\mu_1), \tag{16}$$

where $C_n=$the charge-conjugation parity of V_1. So T, and the important normalized $\mathscr{T}$ parameter to be defined below, vanish when either symmetry is present. For Z^0 and $W^{\pm}$ decays this is not the case, see Table I, which leads to additional symmetry tests. By CP invariance for a final particle-antiparticle pair

$$T(\mu_1\mu_2)=\gamma_{CP}(V_1)T(-\mu_2,-\mu_1), \tag{17}$$

where $\gamma_{CP}=CP$ eigenvalue of V_1.

Normalizing by the total $V_{1,2}\to A_{1,2}B_{1,2}$ decay rates, we can express $I(\theta_1,\theta_2,\phi)$ in terms of the normalized parameters

$$\mathscr{R}=\frac{R-\frac{1}{2}S}{R+S},\quad \mathscr{T}=\frac{T}{R+S} \tag{18}$$

and

$$\mathscr{U}=\frac{U-\frac{1}{2}V}{U+V},\quad \mathscr{W}=\frac{W}{U+V}. \tag{19}$$

By setting $\mathscr{R}=1$ and $\mathscr{T}=0$, $I(\theta_1,\theta_2,\phi)$ can also be used when $V_1\to A_1B_1$ with A_1 and B_1 spin-0 mesons, or $V_1=\omega$ meson. For a three-body mode, θ_1 labels the direction of the normal to the three-body decay plane. Likewise for $\mathscr{U}$ and $\mathscr{W}$. In these definitions, we identify A_1 as the $\bar{l}$ or $\bar{q}$ for Z^0 and W^+ decays. Note that $\mathscr{R}$ and $\mathscr{T}$, as a function of r, are nonpositive for J/ψ, Z^0 and $W^{\pm}$ decays into $\bar{l}_A l_B$ and $\bar{q}_A q_B$. Other quantities in Table I are specified in the caption except $m=$mass of V_1, $p=$magnitude of final three-momentum in V_1 rest frame, $\mu=$muon mass, $E_{t,b}=$final t,b energy in V_1 rest frame, and a value of $\sin^2\theta_W=0.22$ has been used for numerical values for Z^0 quantities.

The α's, β's, and γ's characterization of $I(\theta_1,\theta_2,\phi)$

The "master formula" for the general decay correlation function when $V_{1,2}\to A_{1,2}B_{1,2}$ where $A_1,B_1,\ldots$ are $\bar{l}_A l_B$ and/or $\bar{q}_A q_B$ can be written as

$$I(\theta_1,\theta_2,\phi)=C(\theta_1,\theta_2)+A_0(\theta_1,\theta_2)\cos\phi+A_x(\theta_1,\theta_2)\sin\phi+B_0(\theta_1,\theta_2)\cos2\phi+B_x(\theta_1,\theta_2)\sin2\phi \tag{20}$$

with coefficients

$$B_0(\theta_1,\theta_2)=\tfrac{1}{4}\mathscr{N}\beta_0\sin^2\theta_1\sin^2\theta_2, \tag{21}$$

$$B_x(\theta_1,\theta_2)=\tfrac{1}{4}\mathscr{N}\beta_x\sin^2\theta_1\sin^2\theta_2, \tag{22}$$

$$A_0(\theta_1,\theta_2)=\tfrac{1}{4}\mathscr{N}(\alpha_0\sin2\theta_1\sin2\theta_2+\alpha^{(3)}\sin\theta_1\sin\theta_2+\alpha^{(5)}\sin2\theta_1\sin\theta_2+\alpha^{(7)}\sin\theta_1\sin2\theta_1)$$
$$=\tfrac{1}{4}\mathscr{N}\sum_{\substack{i=1\\ \text{odd}}}^{7}\alpha^{(i)}A^{(i)}(\theta_1,\theta_2),\quad \alpha^{(1)}\equiv\alpha_0, \tag{23}$$

$$A_x(\theta_1,\theta_2)=\tfrac{1}{4}\mathscr{N}(\alpha_x\sin2\theta_1\sin2\theta_2+\alpha^{(4)}\sin\theta_1\sin\theta_2+\alpha^{(6)}\sin2\theta_1\sin\theta_2+\alpha^{(8)}\sin\theta_1\sin2\theta_2)$$
$$=\tfrac{1}{4}\mathscr{N}\sum_{\substack{i=2\\ \text{even}}}^{8}\alpha^{(i)}A^{(i)}(\theta_1,\theta_2),\quad \alpha^{(2)}\equiv\alpha_x, \tag{24}$$

$$C(\theta_1,\theta_2)=\tfrac{1}{9}\mathcal{N}[1+\gamma^{(20)}P_2(\cos\theta_1)+\gamma^{(02)}P_2(\cos\theta_2)+\gamma^{(22)}P_2(\cos\theta_1)P_2(\cos\theta_2)$$
$$+\gamma^{(11)}P_1(\cos\theta_1)P_1(\cos\theta_2)+\gamma^{(01)}P_1(\cos\theta_2)+\gamma^{(21)}P_2(\cos\theta_1)P_1(\cos\theta_2)$$
$$+\gamma^{(10)}P_1(\cos\theta_1)+\gamma^{(12)}P_1(\cos\theta_1)P_2(\cos\theta_2)]$$
$$=\tfrac{1}{9}\mathcal{N}\sum_{i,j=0}^{2}\gamma^{(ij)}P_i(\cos\theta_1)P_j(\cos\theta_2),\quad \gamma^{(00)}=1\ . \tag{25}$$

The norm for $C(\theta_1,\theta_2)$ is chosen as $\frac{1}{9}$, instead of $\frac{1}{4}$, so the integrated distributions (see below) have simple coefficients. The overall normalization is determined by $\gamma^{(00)}=1$ and

$$\mathcal{N}=d\mathcal{D}\ ,$$

where

$$\mathcal{D}=\sum |a_{\lambda_1\lambda_2}|^2$$

and

$$d=(S+R)(V+U)=\left[\sum_{\mu_1\mu_2}|T(\mu_1\mu_2)|^2\right]\left[\sum_{\tau_1\tau_2}|W(\tau_1\tau_2)|^2\right].$$

The α's, β's, and γ's can be simply determined by measurement of the associated integrated distributions *or equivalently* by measurement of $I(\theta_1,\theta_2,\phi)$. If the entire θ_1,θ_2 acceptance is integrated over

$$F(\phi)\equiv\int_{-1}^{1}d(\cos\theta_1)\int_{-1}^{1}d(\cos\theta_2)I(\theta_1,\theta_2,\phi)\ ,$$
$$F(\phi)=4\bar{C}\left[1+\left[\frac{3\pi}{8}\right]^2\alpha^{(3)}\cos\phi+\left[\frac{3\pi}{8}\right]^2\alpha^{(4)}\sin\phi+\beta_0\cos2\phi+\beta_x\sin2\phi\right], \tag{26}$$

where $\bar{C}\equiv\mathcal{N}/9=d\mathcal{D}/9$. If, instead, the quadrants are separately integrated over

$$F_{11,22}(\phi)=\bar{C}\left\{1+\frac{1}{4}\gamma^{(11)}\pm\frac{1}{2}\gamma^{(01)}\pm\frac{1}{2}\gamma^{(10)}+\left[\alpha_0+\left[\frac{3\pi}{8}\right]^2\alpha^{(3)}\pm\frac{3\pi}{8}\alpha^{(5)}\pm\frac{3\pi}{8}\alpha^{(7)}\right]\cos\phi\right.$$
$$\left.+\left[\alpha_x+\left[\frac{3\pi}{8}\right]^2\alpha^{(4)}\pm\frac{3\pi}{8}\alpha^{(6)}\pm\frac{3\pi}{8}\alpha^{(8)}\right]\sin\phi+\beta_0\cos2\phi+\beta_x\sin2\phi\right\}, \tag{27}$$

where the upper signs give F_{11} for $0\le\theta_{1,2}\le\pi/2$ and the lower signs give F_{22} for $\pi/2\le\theta_{1,2}\le\pi$; and

$$F_{12,21}(\phi)=\bar{C}\left\{1-\frac{1}{4}\gamma^{(11)}\mp\frac{1}{2}\gamma^{(01)}\pm\frac{1}{2}\gamma^{(10)}+\left[-\alpha_0+\left[\frac{3\pi}{8}\right]^2\alpha^{(3)}\pm\frac{3\pi}{8}\alpha^{(5)}\mp\frac{3\pi}{8}\alpha^{(7)}\right]\cos\phi\right.$$
$$\left.+\left[-\alpha_x+\left[\frac{3\pi}{8}\right]^2\alpha^{(4)}\pm\frac{3\pi}{8}\alpha^{(6)}\mp\frac{3\pi}{8}\alpha^{(8)}\right]\sin\phi+\beta_0\cos2\phi+\beta_x\sin2\phi\right\}, \tag{28}$$

where the upper signs give F_{12} for $0\le\theta_1\le\pi/2$, $\pi/2\le\theta_2\le\pi$ and the lower signs give F_{21} for $0\le\theta_2\le\pi/2$, $\pi/2\le\theta_1\le\pi$.

If the azimuthal angle ϕ is integrated over,

$$C(\theta_1,\theta_2)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\,I(\theta_1,\theta_2,\phi) \tag{29}$$

and so the $\gamma^{(ij)}$ are simply obtained from

$$\gamma^{(ij)}=\frac{9}{4}\frac{1}{\mathcal{N}}(2i+1)(2j+1)\int_{-1}^{1}d(\cos\theta_1)\int_{-1}^{1}d(\cos\theta_2)C(\theta_1,\theta_2)P_i(\cos\theta_1)P_j(\cos\theta_2)\ . \tag{30}$$

The α's, β's, and γ's are simple triquadratic functions—that is, they are quadratic functions of the nine $a_{\lambda_1\lambda_2}$ helicity amplitudes for $X\to V_1V_2$, are quadratic in the $T(\mu_1\mu_2)$ amplitudes for $V_1\to A_1B_1$, and are quadratic in the $W(\tau_1\tau_2)$ amplitudes for $V_2\to A_2B_2$. To display this in detail, and simply, we write

$$\beta\equiv\beta_0=b_0\sigma_0\ , \tag{31}$$
$$\beta_x=b_x\sigma_x\ , \tag{32}$$
$$\alpha\equiv\alpha_0=a_0\rho_0\ , \tag{33}$$
$$\alpha^{(i)}\equiv a_i\rho^{(i)}\ , \tag{34}$$

and

$$\gamma^{(ij)} \equiv c_{ij} C^{(ij)} , \tag{35}$$

where $c_{00}=1$ and $C^{(00)}=1$.

The a's, b's, and c's, and the ρ's, σ's, and τ's

Using the $\mathscr{R}$, $\mathscr{T}$, . . . defined by Eqs. (18) and (19), we find the a's, b's, and c's

$$\begin{aligned}
&b_0=b_x=a_0=a_x=c_{22}=\mathscr{R}\mathscr{U} ,\\
&c_{00}=1, \;\; c_{20}=\mathscr{R}, \;\; c_{02}=\mathscr{U} ,\\
&a_3=a_4=c_{11}=\mathscr{T}\mathscr{W} ,\\
&a_5=a_6=-c_{21}=\mathscr{R}\mathscr{W} ,\\
&a_7=a_8=-c_{12}=\mathscr{T}\mathscr{U} ,\\
&c_{01}=-\mathscr{W}, \;\; c_{10}=-\mathscr{T} ,
\end{aligned} \tag{36}$$

when we define the ρ's, σ's, and τ's to be

$$\begin{aligned}
&\sigma_0=2\,\mathrm{Re}(a_{++}a^*_{--})/\mathscr{D} ,\\
&\sigma_x=-2\,\mathrm{Im}(\mathrm{same})/\mathscr{D} ,
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\rho_0=\mathrm{Re}(a_{++}a^*_{00}+a_{00}a^*_{--}-a_{+0}a^*_{0-}-a_{0+}a^*_{-0})/\mathscr{D} ,\\
&\rho_x=-\mathrm{Im}(\mathrm{same})/\mathscr{D} ,\\
&\rho^{(3)}=\mathrm{Re}(a_{++}a^*_{00}+a_{00}a^*_{--}+a_{+0}a^*_{0-}+a_{0+}a^*_{-0})/\mathscr{D} ,\\
&\rho^{(4)}=-\mathrm{Im}(\mathrm{same})/\mathscr{D} ,\\
&\rho^{(5)}=-\mathrm{Re}(a_{++}a^*_{00}-a_{00}a^*_{--}+a_{+0}a^*_{0-}-a_{0+}a^*_{-0})/\mathscr{D} ,\\
&\rho^{(6)}=+\mathrm{Im}(\mathrm{same})/\mathscr{D} ,\\
&\rho^{(7)}=-\mathrm{Re}(a_{++}a^*_{00}-a_{00}a^*_{--}-a_{+0}a^*_{0-}+a_{0+}a^*_{-0})/\mathscr{D} ,\\
&\rho^{(8)}=+\mathrm{Im}(\mathrm{same})/\mathscr{D} ,
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&\tau^{(1)}=(|a_{++}|^2+|a_{--}|^2+|a_{+-}|^2+|a_{-+}|^2)/\mathscr{D} ,\\
&\tau^{(2)}=4|a_{00}|^2/\mathscr{D} ,\\
&\tau^{(3)}=2(|a_{+0}|^2+|a_{-0}|^2)/\mathscr{D} ,\\
&\tau^{(4)}=2(|a_{0+}|^2+|a_{0-}|^2)/\mathscr{D} ,\\
&\tau^{(5)}=(|a_{++}|^2+|a_{--}|^2-|a_{+-}|^2-|a_{-+}|^2)/\mathscr{D} ,\\
&\tau^{(6)}=(|a_{++}|^2-|a_{--}|^2+|a_{+-}|^2-|a_{-+}|^2)/\mathscr{D} ,\\
&\tau^{(7)}=(|a_{++}|^2-|a_{--}|^2-|a_{+-}|^2+|a_{-+}|^2)/\mathscr{D} ,\\
&\tau^{(8)}=2(|a_{+0}|^2-|a_{-0}|^2)/\mathscr{D} ,\\
&\tau^{(9)}=2(|a_{0+}|^2-|a_{0-}|^2)/\mathscr{D} .
\end{aligned} \tag{39}$$

Consequently the magnitudes of all the $|a_{\lambda_1\lambda_2}|$'s can be obtained from the data when one can obtain all the τ's.

The $\gamma^{(ij)}$'s appearing in $C(\theta_1\theta_2)$ depend on these τ's through the $C^{(ij)}$ of Eq. (35). We obtain

$$\begin{bmatrix} 1 \\ C^{(20)} \\ C^{(02)} \\ C^{(22)} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} & 1 \\ -1 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \tau^{(1)} \\ \tau^{(2)} \\ \tau^{(3)} \\ \tau^{(4)} \end{bmatrix} \tag{40}$$

and conversely from, say, empirical $C^{(ij)}$'s

$$\begin{bmatrix} \tau^{(1)} \\ \tau^{(2)} \\ \tau^{(3)} \\ \tau^{(4)} \end{bmatrix} = \frac{4}{9} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ C^{(20)} \\ C^{(02)} \\ C^{(22)} \end{bmatrix} , \tag{41}$$

and

$$\begin{aligned}
&\tau^{(5)}=\tfrac{4}{9}C^{(11)} ,\\
&\tau^{(6)}=\tfrac{1}{9}(-4C^{(10)}+2C^{(12)}) ,\\
&\tau^{(7)}=\tfrac{1}{9}(-4C^{(01)}+2C^{(21)}) ,\\
&\tau^{(8)}=-\tfrac{4}{9}(C^{(10)}+C^{(12)}) ,\\
&\tau^{(9)}=-\tfrac{4}{9}(C^{(01)}+C^{(21)}) .
\end{aligned} \tag{42}$$

Inserting these values for the a's, b's, and c's, the integrated distributions appear in a form convenient for seeing by inspection which of the ρ's, σ's, and τ's can be in principle measured in a chosen decay sequence. Concurrently, by Eqs. (31)–(35), the associated α's, β's, and γ's are obvious. These integrated distributions are

TABLE II. Relations among the helicity amplitudes for the decay $X \to VV$ which follow from invariance under parity or under CP, and from Bose statistics. η_P and γ_{CP} are, respectively, the P and CP eigenvalues of X. J is the spin and $(-)^J$ the signature of X.

Nonvanishing amplitudes	η_P or γ_{CP}	$(-1)^J$
Case I		
$a_{+0}=a_{-0}=-a_{0+}=-a_{0-}$	-1	-1
Case II		
$a_{+0}=-a_{-0}=-a_{0+}=a_{0-}$	$+1$	-1
$a_{+-}=-a_{-+}$		
Case III		
$a_{+0}=a_{0+}=-a_{-0}=-a_{0-}$	-1	$+1$
$a_{++}=-a_{--}$		
Case IV		
$a_{+0}=a_{0+}=a_{-0}=a_{0-}$	$+1$	$+1$
$a_{+-}=a_{-+}$		
$a_{++}=a_{--}$		
a_{00}		

$$F(\phi)=4\bar{C}\left[1+\left[\frac{3\pi}{8}\right]^2\mathcal{T}\mathcal{W}(\rho^{(3)}\cos\phi+\rho^{(4)}\sin\phi)+\mathcal{R}\mathcal{U}(\sigma_0\cos2\phi+\sigma_x\sin2\phi)\right], \tag{43}$$

$$\begin{aligned}F_{11,22}(\phi)=\bar{C}\Bigg\{&1+\tfrac{1}{4}\mathcal{T}\mathcal{W}C^{(11)}\mp\tfrac{1}{2}\mathcal{W}C^{(01)}\mp\tfrac{1}{2}\mathcal{T}C^{(10)}\\&+\left[\mathcal{R}\mathcal{U}\rho_0+\left[\frac{3\pi}{8}\right]^2\mathcal{T}\mathcal{W}\rho^{(3)}\pm\frac{3\pi}{8}\mathcal{R}\mathcal{W}\rho^{(5)}\pm\frac{3\pi}{8}\mathcal{T}\mathcal{U}\rho^{(7)}\right]\cos\phi\\&+\left[\mathcal{R}\mathcal{U}\rho_x+\left[\frac{3\pi}{8}\right]^2\mathcal{T}\mathcal{W}\rho^{(4)}\pm\frac{3\pi}{8}\mathcal{R}\mathcal{W}\rho^{(6)}\pm\frac{3\pi}{8}\mathcal{T}\mathcal{U}\rho^{(8)}\right]\sin\phi+\mathcal{R}\mathcal{U}(\sigma_0\cos2\phi+\sigma_x\sin2\phi)\Bigg\},\end{aligned} \tag{44}$$

$$\begin{aligned}F_{12,21}(\phi)=\bar{C}\Bigg\{&1-\tfrac{1}{4}\mathcal{T}\mathcal{W}C^{(11)}\pm\tfrac{1}{2}\mathcal{W}C^{(01)}\mp\tfrac{1}{2}\mathcal{T}C^{(10)}\\&+\left[-\mathcal{R}\mathcal{U}\rho_0+\left[\frac{3\pi}{8}\right]^2\mathcal{T}\mathcal{W}\rho^{(3)}\pm\frac{3\pi}{8}\mathcal{R}\mathcal{W}\rho^{(5)}\mp\frac{3\pi}{8}\mathcal{T}\mathcal{U}\rho^{(7)}\right]\cos\phi\\&+\left[-\mathcal{R}\mathcal{U}\rho_x+\left[\frac{3\pi}{8}\right]^2\mathcal{T}\mathcal{W}\rho^{(4)}\pm\frac{3\pi}{8}\mathcal{R}\mathcal{W}\rho^{(6)}\mp\frac{3\pi}{8}\mathcal{T}\mathcal{U}\rho^{(8)}\right]\sin\phi+\mathcal{R}\mathcal{U}(\sigma_0\cos2\phi+\sigma_x\sin2\phi)\Bigg\},\end{aligned} \tag{45}$$

and for the $C(\theta_1,\theta_2)$ distribution

$$\begin{aligned}&\gamma^{(20)}=\mathcal{R}C^{(20)},\quad \gamma^{(01)}=-\mathcal{W}C^{(01)},\\&\gamma^{(02)}=\mathcal{U}C^{(02)},\quad \gamma^{(21)}=-\mathcal{R}\mathcal{W}C^{(21)},\\&\gamma^{(22)}=\mathcal{R}\mathcal{U}C^{(22)},\quad \gamma^{(10)}=-\mathcal{T}C^{(10)},\\&\gamma^{(11)}=\mathcal{T}\mathcal{W}C^{(11)},\quad \gamma^{(12)}=-\mathcal{T}\mathcal{U}C^{(12)}.\end{aligned} \tag{46}$$

Note that when the V_1 and/or V_2 decays are both P violating and C violating so $\mathcal{T}\neq0$ and/or $\mathcal{W}\neq0$, the $F_{ab}(\phi)$'s can also be used to determine some of, or all of, $C^{(11)}$, $C^{(01)}$, $C^{(10)}$ since $\gamma^{(11)}$, $\gamma^{(01)}$, $\gamma^{(10)}$ appear in the $F_{ab}(\phi)$'s.

Remarks

(i) Relative to the results obtained previously in the case that $A_1,B_1,\ldots$ are spin-0 bosons, this $I(\theta_1,\theta_2,\phi)$ is complex, though actually, not complicated since a simple basic algebraic structure remains. The choice of V_1 and V_2 determines which of the ρ's, σ's, and τ's can be obtained from, or conversely used to predict, an empirical $I(\theta_1,\theta_2,\phi)$, and so it is convenient to organize the symmetry-test applications according to the specific type of $X\rightarrow V_1V_2$ decay mode. Once a mode is chosen, the different ways of determining the α's, β's, and γ's from the integrated distributions can be evaluated using Eqs. (43)–(46). This organization by V_1V_2 modes is also probably the most useful one to readers interested in applying the results.

(ii) Besides the β_0 and β_x, for Z^0Z^0, $W^{(+)}W^{(-)}$, and $Z^0W^{\pm}$ the $\alpha^{(3)}$ and $\alpha^{(4)}$ can be obtained from the $F(\phi)$ integrated distribution that arises when the entire θ_1,θ_2 acceptance is integrated over, unlike the α_0 and α_x which require study of the four distributions $F_{ab}(\phi)$. So besides the useful appearance of additional α's and γ's in $I(\theta_1,\theta_2,\phi)$ when P and C violating $V_{1,2}\rightarrow A_{1,2}B_{1,2}$ decays occur, some information about $X\rightarrow V_1V_2$ should normally be more accurately obtained from a given data sample than in the comparable case when the decay is P or C conserving. By Table I, α_0 and β_0 signatures can be seen to be not as maximal for vector-boson decays into $\bar{l}_Al_B$ or $\bar{q}_Aq_B$. As has been pointed out, this occurs, for instance, in $J\rightarrow\mu^+\mu^-$ because the muon current in the associated vector-meson rest frame can couple to vector-meson polarizations outside the $\mu^+\mu^-$ decay plane, unlike for a boson current.

(iii) For neutral $X\rightarrow V_1V_2$, the structure of the ρ's and σ's in Eqs. (37) and (38) can often be considerably simplified by defining (for instance, CP) even and odd linear combinations of the $a_{\lambda_1\lambda_2}$ helicity amplitudes. This is particularly useful for enumerating the signatures for violation of the assumed symmetry. In this manner, the magnitudes and the sines and cosines of the relative phases of the (CP-) even and odd amplitudes appear algebraically in the ρ's, σ's, and τ's. This is used in some of the following applications (see, e.g., Ref. 7).

(iv) When each of V_1 and V_2 either decays C invariantly and P invariantly into $\bar{l}_Al_B$ or $\bar{q}_Aq_B$, decays into two spin-0 bosons, or is ω, then $\mathcal{T}=\mathcal{W}=0$ so only the first four terms in $C(\theta_1,\theta_2)$ appear. These terms can be rewritten

$$\begin{aligned}C_0(\theta_1,\theta_2)&\equiv\tfrac{1}{9}\mathcal{N}[1+\mathcal{R}C^{(20)}P_2(\cos\theta_1)+\mathcal{U}C^{(02)}P_2(\cos\theta_2)+\mathcal{R}\mathcal{U}C^{(22)}P_2(\cos\theta_1)P_2(\cos\theta_2)]\\&=\tfrac{1}{4}\mathcal{N}[\gamma^{(1)}\sin^2\theta_1\sin^2\theta_2+\gamma^{(2)}\cos^2\theta_1\cos^2\theta_2+\gamma^{(3)}\sin^2\theta_1\cos^2\theta_2+\gamma^{(4)}\cos^2\theta_1\sin^2\theta_2]\end{aligned} \tag{47}$$

so

$$C_0(\theta_1,\theta_2)=C_0(\pi-\theta_1,\pi-\theta_2)=C_0(\pi-\theta_1,\theta_2)=C_0(\theta_1,\pi-\theta_2)\ . \tag{48}$$

These symmetry properties of $C_0(\theta_1,\theta_2)$ can be used to bin all θ_1,θ_2 events into the square region $0\le\theta_{1,2}\le\pi/2$. The distinct $\gamma^{(1)}$ to $\gamma^{(4)}$ coefficients in Eq. (47), respectively, dominate the $C_0(\theta_1,\theta_2)$ distribution near the four corners; see Fig. 2.

When $\mathscr{R}=\mathscr{U}=1$, the $\gamma^{(i)}=\tau^{(i)}$, $i=1,\dots,4$, and so a visual inspection of the $C_0(\theta_1,\theta_2)$ will directly give the τ's. Otherwise, they are related to the τ's through the $\gamma^{(ij)}$'s by

$$\begin{bmatrix}1\\ \gamma^{(20)}\\ \gamma^{(02)}\\ \gamma^{(22)}\end{bmatrix}=\begin{bmatrix}1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{2}\\ -1 & \frac{1}{2} & -\frac{1}{2} & 1\\ -1 & \frac{1}{2} & 1 & -\frac{1}{2}\\ 1 & 1 & -1 & -1\end{bmatrix}\begin{bmatrix}\gamma^{(1)}\\ \gamma^{(2)}\\ \gamma^{(3)}\\ \gamma^{(4)}\end{bmatrix} \tag{49}$$

and inversely

$$\begin{bmatrix}\gamma^{(1)}\\ \gamma^{(2)}\\ \gamma^{(3)}\\ \gamma^{(4)}\end{bmatrix}=\frac{4}{9}\begin{bmatrix}1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{4}\\ 1 & 1 & 1 & 1\\ 1 & -\frac{1}{2} & 1 & -\frac{1}{2}\\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ \gamma^{(20)}\\ \gamma^{(02)}\\ \gamma^{(22)}\end{bmatrix}\ . \tag{50}$$

(v) Stronger results are obtained using $I(\theta_1,\theta_2,\phi)$ when $J=0,1$ for X because for $J=0$, $a_{\lambda_1\lambda_2}\neq 0$ only if $\lambda_1=\lambda_2$ and for $J=1$, $a_{\lambda_1\lambda_2}\neq 0$ only if $|\lambda|=|\lambda_1-\lambda_2|\le 1$. Finally, for the case of $J\ge 1$ the analysis of Ref. 10, which was to determine the parity and spin of the η_c, supports the idea that an assumption of decay of $X\to V_1V_2$ with a definite J through specific orbital angular momentum channels strongly constrains the ranges of at least some of the α's, β's, and γ's beyond the results presented here, and, second, simultaneously provides a means to determine J.

III. *P*, OR *CP*, DETERMINATION AND POSSIBLE SIGNATURE DETERMINATION

Since the α's, β's, and γ's which characterize $I(\theta_1,\theta_2,\phi)$ appear very simply in the integrated distributions, we will focus our discussion of each of the $X\to V_1V_2$ decay modes around its integrated distributions. The results in this series of three papers for $X\to V_1V_2$ decay followed by decays of V_1 and/or V_2 into $\bar{l}_Al_B$ or $\bar{q}_Aq_B$ apply of course to equivalent modes to the ones listed. For example, ϕ vector meson results also hold if ϕ is replaced by ρ^0 or ω, and similarly, J/ψ can be replaced by $\psi(3685)$, Υ, $\Upsilon(10{,}350)$, all of which have large leptonic branching ratios.

In this paper in which we consider arbitrary J and then $J=1$ (next section), it is convenient for us to present the results for the ρ's, σ's, and τ's so as to avoid complicated formulas. In the following paper we give the stronger results which apply when $J=0$ in the α's, β's, and γ's which directly characterize $I(\theta_1,\theta_2,\phi)$ and the integrated distributions.

JJ or $\Upsilon\Upsilon$ decay mode

When V_1 and V_2 are identical particles, the $X\to VV$ decay helicity amplitudes are related by

$$a_{\lambda_1\lambda_2}=(-)^J a_{\lambda_2\lambda_1}\ , \tag{51}$$

where $J=$spin of X. When P invariance holds for $X\to V_1V_2$, there is the relation

$$a_{-\lambda_1,-\lambda_2}=\eta_P(-)^J a_{\lambda_1\lambda_2}\ , \tag{52}$$

where $\eta_P=P$ quantum number of X. Table II follows from Eqs. (51) and (52).

When instead, CP invariance holds for a VV mode, or for a $V\bar{V}$ mode (i.e., a particle-antiparticle pair),

$$a_{-\lambda_1,-\lambda_2}=\gamma_{CP}\,a_{\lambda_2\lambda_1} \tag{53}$$

and Table III then follows by CP invariance alone. Consequently, for a VV mode when both Bose statistics and CP invariance hold, from Eqs. (51) and (53)

$$a_{-\lambda_1,-\lambda_2}=\gamma_{CP}(-)^J a_{\lambda_1\lambda_2} \tag{54}$$

which is of the same form as Eq. (52) and so it is only necessary to substitute γ_{CP} for η_P in order to convert a P test into a CP test. For instance, here in discussing the JJ and $\Upsilon\Upsilon$ modes, we assume Bose statistics and P invariance holds for $X\to JJ$, $\Upsilon\Upsilon$, and use Table II; however, should P invariance be violated, but CP invariance holds, the same results apply when η_P is replaced by γ_{CP}.

$I(\theta_1,\theta_2,\phi)$ and the integrated distributions are particularly simple for $X\to JJ$ or $\Upsilon\Upsilon$:

$$F(\phi)=4\bar{C}(1+\mathscr{R}^2\sigma_0\cos2\phi)\ , \tag{55}$$

$$F_{ab}(\phi)=\bar{C}(1\pm\mathscr{R}^2\rho_0\cos\phi+\mathscr{R}^2\sigma_0\cos2\phi)\ , \tag{56}$$

with the upper sign (lower sign) for $ab=11,22$ (12,21), respectively, and the nonzero $\gamma^{(ij)}$'s are

$$\gamma^{(20)}=\gamma^{(02)}=\mathscr{R}C^{(20)},\quad \gamma^{(22)}=\mathscr{R}^2C^{(22)} \tag{57}$$

so $\tau^{(1)}$, $\tau^{(2)}$, and $\tau^{(3)}=\tau^{(4)}$ are, in principle, measurable. Versus the $\phi\phi$ symmetry tests, the JJ and $\Upsilon\Upsilon$ signatures

TABLE III. Relations among the helicity amplitudes describing the decay $X\to V_1V_2$ which follow from invariance under CP for a mode where V_1 and V_2 are two identical vector bosons, VV, or are a particle-antiparticle pair of vector bosons, $V\bar{V}$ (e.g., W^+W^-).

Nonvanishing amplitudes	CP eigenvalue
$a_{+0}=-a_{0-}$	$\gamma_{CP}=-1$
$a_{0+}=-a_{-0}$	
$a_{++}=-a_{--}$	
$a_{+0}=a_{0-}$	$\gamma_{CP}=+1$
$a_{0+}=a_{-0}$	
$a_{++}=a_{--}$	
a_{+-},a_{-+},a_{00}	

are not as maximal, e.g., in the massless final fermion limit $\mathscr{R}^2=\frac{1}{4}$ versus 1.

Using Table II, we find

$$\sigma_0=\begin{cases}2\eta_P\,|a_{++}|^2/\mathscr{D}, & (-)^J=+1\ ,\\ 0, & (-)^J=-1\ ,\end{cases} \tag{58}$$

$$\rho_0=[2\,\mathrm{Re}(a_{++}a_{00}^*)-2\eta_P\,|a_{0+}|^2]/\mathscr{D}\ , \tag{59}$$

$$\tau^{(1)}=2(|a_{++}|^2+|a_{+-}|^2)/\mathscr{D}\ ,$$
$$\tau^{(2)}=4\,|a_{00}|^2/\mathscr{D},\quad \tau^{(4)}=\tau^{(3)}=4\,|a_{0+}|^2/\mathscr{D}\ , \tag{60}$$

where

$$\mathscr{D}=|a_{00}|^2+2\,|a_{++}|^2+2\,|a_{+-}|^2+4\,|a_{0+}|^2\ .$$

The P/CP determination is conclusive for the JJ or $\Upsilon\Upsilon$ mode. The parity η_P of X is given by

$$\eta_P=\begin{cases}\mathrm{sgn}\sigma_0\ ;\\ -\mathrm{sgn}\rho_0 \ \text{ if } \sigma_0=0\ ;\\ +1 \ \text{ if } \sigma_0=\rho_0=0\ .\end{cases} \tag{61}$$

The signature $(-)^J$ of X can be determined *to be positive* if both $|a_{++}|$ and $|a_{00}|$ do not vanish (cf. discussion of Ref. 3 and paper 1 of Ref. 4). The signature cannot be shown to be negative by these modes by this method. If $\sigma_0\neq0$ (equivalently $\beta_0\neq0$), then $(-)^J=+1$. If $\sigma_0=0$, then $|a_{++}|=0$. When $\sigma_0=0$, if also $\rho_0\leq0$, then it is worthwhile to use the ρ_0 value and proceed with a two-parameter fit to $C(\theta_1,\theta_2)$ with, e.g., $|a_{+-}|^2$ determined by

$$\rho_0\,|a_{00}|^2+2(2\rho_0+\eta_P)\,|a_{0+}|^2+2\rho_0\,|a_{+-}|^2=0\ .$$

If the fit yields $|a_{00}|\neq0$, then $(-)^J=+1$. For $\rho_0=-\frac{1}{2}$, $|a_{+-}|=|a_{00}|=0$. (If $\rho_0=\frac{1}{2}$, $|a_{00}|=0$.)

ϕJ or $J\Upsilon$ decay mode

We assume $\mathscr{T}=\mathscr{W}=0$. By P invariance, Eq. (52) gives Table IV. Should CP invariance hold instead for a mode where V_1 and V_2 are each their own antiparticle, Eq. (52) holds if η_P is replaced by γ_{CP}. Again, we will display the results assuming P invariance, but they also hold if we let $\eta_P\to\gamma_{CP}$ when only CP invariance holds.

The integrated distributions are (for the ϕJ mode $\mathscr{R}=1$)

$$F(\phi)=4\bar{C}(1+\mathscr{R}\mathscr{U}\sigma_0\cos2\phi)\ , \tag{62}$$

$$F_{ab}(\phi)=\bar{C}(1\pm\mathscr{R}\mathscr{U}\rho_0\cos\phi+\mathscr{R}\mathscr{U}\sigma_0\cos2\phi)\ , \tag{63}$$

with the signs in Eq. (63) as in (56), and the nonzero $\gamma^{(ij)}$'s are

TABLE IV. Relations among the helicity amplitudes describing the decay $X\to V_1V_2$ which follow from invariance under parity, or under CP for a mode where V_1 and V_2 are each their own antiparticle, e.g., ϕJ and ϕZ^0.

Nonvanishing amplitudes	$\eta_P(-)^J$ or $\gamma_{CP}(-)^J$
$a_{+0}=a_{-0}$	$+1$
$a_{0+}=a_{0-}$	
$a_{++}=a_{--}$	
$a_{+-}=a_{-+}$	
a_{00}	
$a_{+0}=-a_{-0}$	-1
$a_{0+}=-a_{0-}$	
$a_{++}=-a_{--}$	
$a_{+-}=-a_{-+}$	

$$\gamma^{(20)}=\mathscr{R}C^{(20)},\quad \gamma^{(02)}=\mathscr{U}C^{(02)}\ ,\quad \gamma^{(22)}=\mathscr{R}\mathscr{U}C^{(22)}\ . \tag{64}$$

Versus $\mathscr{R}\mathscr{U}=1$ for $\phi\phi$ symmetry tests, for ϕJ the $|\mathscr{R}\mathscr{U}|=\frac{1}{2}$ and for $J\Upsilon$ the $|\mathscr{R}\mathscr{U}|=\frac{1}{4}$ in the massless-final-fermion limit so the β_0 and α_0 signatures are not as maximal.

Using Table IV, we have

$$\sigma_0=2\eta_P(-)^J\,|a_{++}|^2/\mathscr{D}\ , \tag{65}$$

$$\rho_0=[2\,\mathrm{Re}(a_{++}a_{00}^*)-2\eta_P(-)^J\,\mathrm{Re}(a_{+0}a_{0+}^*)]/\mathscr{D}\ , \tag{66}$$

$$\tau^{(1)}=2(|a_{++}|^2+|a_{+-}|^2)/\mathscr{D},\quad \tau^{(2)}=4\,|a_{00}|^2/\mathscr{D}\ ,$$
$$\tau^{(3)}=4\,|a_{+0}|^2/\mathscr{D},\quad \tau^{(4)}=4\,|a_{0+}|^2/\mathscr{D}\ , \tag{67}$$

where

$$\mathscr{D}=|a_{00}|^2+2\,|a_{++}|^2+2\,|a_{+-}|^2+2\,|a_{+0}|^2+2\,|a_{0+}|^2\ .$$

When $\beta_0\neq0$ (i.e., $\sigma_0\neq0$),

$$\eta_P(-)^J=\mathrm{sgn}\sigma_0\ . \tag{68}$$

However, when $\sigma_0=0$, the ρ_0 parameter is not sufficient to find $\eta_P(-)^J$. Fortunately $|a_{++}|=0$ would be an accidental occurrence (for $J=0$, $\sigma_0=0$ does imply $\eta_P=+1$). The $C(\theta_1,\theta_2)$ distribution can be used, e.g., via Eq. (47) and Fig. 2, to determine $|a_{00}|$. If $|a_{00}|\neq0$, then $\eta_P(-)^J=+1$.

Z^0Z^0 decay mode

Assuming Bose statistics and CP invariance in $X\to Z^0Z^0$, Table II applies, and the associated integrated distributions are

$$F(\phi)=4\bar{C}\left[1+\left[\frac{3\pi}{8}\right]^2\mathscr{T}^2\rho^{(3)}\cos\phi+\mathscr{R}^2\sigma_0\cos2\phi\right]\ , \tag{69}$$

$$F_{11,22}(\phi)=\bar{C}\left\{1+\tfrac{1}{4}\mathscr{T}^2C^{(11)}+\left[\mathscr{R}^2\rho_0+\left[\frac{3\pi}{8}\right]^2\mathscr{T}^2\rho^{(3)}\right]\cos\phi\pm\left[\frac{3\pi}{4}\right]\mathscr{R}\mathscr{T}\rho^{(6)}\sin\phi+\mathscr{R}^2\sigma_0\cos2\phi\right\}\ , \tag{70}$$

$$F_{12,21}(\phi)=\bar{C}\left\{1-\tfrac{1}{4}\mathscr{T}^2C^{(11)}+\left[-\mathscr{R}^2\rho_0+\left[\frac{3\pi}{8}\right]^2\mathscr{T}^2\rho^{(3)}\right]\cos\phi+\mathscr{R}^2\sigma_0\cos2\phi\right\}\ , \tag{71}$$

and the nonzero $\gamma^{(ij)}$'s are

$$\gamma^{(20)}=\gamma^{(02)}=\mathscr{R}C^{(20)},\ \ \gamma^{(22)}=\mathscr{R}^2C^{(22)},\ \ \gamma^{(11)}=\mathscr{T}^2C^{(11)}\ . \tag{72}$$

Now, unlike for the JJ mode the ρ's, σ's, and τ's are weighted by $\mathscr{T}^2$ and $\mathscr{R}\mathscr{T}$ as well as by $\mathscr{R}^2$ so the choice of $\mu^+\mu^-$ versus $\bar{d}d$ in Z^0 decay effects the relative magnitudes of these weighting factors (see Table I).

Using Table II, we find the ρ's, σ's, and τ's, and $\mathscr{D}$, of Eqs. (58)—(60), plus

$$\rho^{(3)}=[2\,\mathrm{Re}(a_{++}a_{00}^*)+2\gamma_{CP}\,|\,a_{+0}\,|^2]/\mathscr{D}\ , \tag{73}$$

$$\rho^{(6)}=\rho^{(8)}=2\,\mathrm{Im}(a_{++}a_{00}^*)/\mathscr{D}\ , \tag{74}$$

$$\tau^{(5)}=\tfrac{4}{9}C^{(11)}=2(\,|\,a_{++}\,|^2-|\,a_{+-}\,|^2)/\mathscr{D}\ . \tag{75}$$

The CP determination is conclusive, with the CP eigenvalue of X given by

$$\gamma_{CP}=\begin{cases}\mathrm{sgn}\sigma_0\ ;\\ \mathrm{sgn}\rho^{(3)},\ \text{and}\ \rho_0=-\rho^{(3)}\ \text{and}\ \rho^{(6)}=0,\ \ \text{if}\ \sigma_0=0\ ;\\ 1\ \ \text{if}\ \sigma_0=\rho^{(3)}=\rho_0=0\ .\end{cases} \tag{76}$$

Note if $\mathrm{sgn}\gamma^{(11)}>0$, then $\sigma_0\neq 0$.

The discussion given above on signature determination using the JJ mode applies here except that Eqs. (73)—(75) supplement it. From Eq. (75), if $\mathrm{sgn}\gamma^{(11)}>0$, then $(-)^J=+1$. If $\sigma_0=0$, then $\rho_0=-\rho^{(3)}$ and $\rho^{(3)}$ can be used in the analysis in place of, or in combination with, ρ_0.

W^+W^- decay mode

Assuming CP invariance in $X\to W^+W^-$, from Eq. (53) or Table III the relevant integrated distributions are ($\mathscr{T}$ is for W^+ so $\mathscr{T}\leq 0$)

$$F(\phi)=4\bar{C}\left[1-\left[\frac{3\pi}{8}\right]^2\mathscr{T}^2\rho^{(3)}\cos\phi+\mathscr{R}^2\sigma_0\cos2\phi\right]\ , \tag{77}$$

$$F_{11,22}(\phi)=\bar{C}\left\{1-\tfrac{1}{4}\mathscr{T}^2C^{(11)}\pm\tfrac{1}{2}\mathscr{T}C^{(01)}\mp\tfrac{1}{2}\mathscr{T}C^{(10)}+\left[\mathscr{R}^2\rho_0-\left[\frac{3\pi}{8}\right]^2\mathscr{T}^2\rho^{(3)}\mp\frac{3\pi}{4}\mathscr{R}\mathscr{T}\rho^{(5)}\right]\cos\phi+\mathscr{R}^2\sigma_0\cos2\phi\right\}\ , \tag{78}$$

$$F_{12,21}(\phi)=\bar{C}\left\{1+\tfrac{1}{4}\mathscr{T}^2C^{(11)}\mp\tfrac{1}{2}\mathscr{T}C^{(01)}\mp\tfrac{1}{2}\mathscr{T}C^{(10)}+\left[-\mathscr{R}^2\rho_0-\left[\frac{3\pi}{8}\right]^2\mathscr{T}^2\rho^{(3)}\right]\cos\phi\right.$$
$$\left.\mp\frac{3\pi}{4}\mathscr{T}\mathscr{R}\rho^{(6)}\sin\phi+\mathscr{R}^2\sigma_0\cos2\phi\right\}\ , \tag{79}$$

and the nonzero $\gamma^{(ij)}$'s are

$$\begin{aligned}&\gamma^{(20)}=\gamma^{(02)}=\mathscr{R}C^{(20)},\ \ \gamma^{(22)}=\mathscr{R}^2C^{(22)},\ \ \gamma^{(11)}=-\mathscr{T}^{2C^{(11)}}\ ,\\ &\gamma^{(21)}=-\mathscr{R}\gamma^{(10)}=+\mathscr{R}\mathscr{T}C^{(21)},\ \ \gamma^{(12)}=-\mathscr{R}\gamma^{(01)}=-\mathscr{R}\mathscr{T}C^{(12)}\ .\end{aligned} \tag{80}$$

When the final fermion masses are small, the parameter $\mathscr{T}\simeq-1$ so the $\alpha^{(3)}$ and $\gamma^{(11)}$ terms are weighted at the "maximal" level of the $\phi\phi$ test which means that usually they will be more reliably determined than the other parameters.

Using Table III, we obtain

$$\sigma_0=2\gamma_{CP}\,|\,a_{++}\,|^2/\mathscr{D}\ , \tag{81}$$

$$\rho^{(3)}=[2\,\mathrm{Re}(a_{++}a_{00}^*)+\gamma_{CP}(\,|\,a_{+0}\,|^2+|\,a_{0+}\,|^2)]/\mathscr{D}\ , \tag{82}$$

$$\rho_0=[2\,\mathrm{Re}(a_{++}a_{00}^*)-\gamma_{CP}(\,|\,a_{+0}\,|^2+|\,a_{0+}\,|^2)]/\mathscr{D}\ , \tag{83}$$

$$\rho^{(5)}=-\rho^{(7)}=-\gamma_{CP}(\,|\,a_{+0}\,|^2-|\,a_{0+}\,|^2)/\mathscr{D}\ , \tag{84}$$

$$\rho^{(6)}=\rho^{(8)}=+2\,\mathrm{Im}(a_{++}a_{00}^*)/\mathscr{D}\ , \tag{85}$$

and

$$\begin{aligned}
&\tau^{(1)}=(2|a_{++}|^2+|a_{+-}|^2+|a_{-+}|^2)/\mathscr{D}, \quad \tau^{(2)}=4|a_{00}|^2/\mathscr{D},\\
&\tau^{(3)}=\tau^{(4)}=2(|a_{+0}|^2+|a_{0+}|^2)/\mathscr{D},\\
&\tau^{(5)}=(2|a_{++}|^2-|a_{+-}|^2-|a_{-+}|^2)/\mathscr{D},\\
&\tau^{(6)}=-\tau^{(7)}=(|a_{+-}|^2-|a_{-+}|^2)/\mathscr{D},\\
&\tau^{(8)}=\tau^{(9)}=2(|a_{+0}|^2-|a_{0+}|^2)/\mathscr{D},
\end{aligned} \tag{86}$$

with

$$\mathscr{D}=(|a_{00}|^2+2|a_{++}|^2+|a_{+-}|^2+|a_{-+}|^2 +2|a_{+0}|^2+2|a_{0+}|^2).$$

The CP determination is conclusive, with the CP eigenvalue of X given by

$$\gamma_{CP}=\begin{cases}\mathrm{sgn}\sigma_0\,;\\ \mathrm{sgn}\rho^{(3)},\ \rho_0=-\rho^{(3)},\ \rho^{(6)}=0,\ \text{if } \sigma_0=0\,;\\ 1 \ \text{if } \sigma_0=\rho^{(3)}=\rho_0=-\rho^{(5)}=0\,.\end{cases} \tag{87}$$

There are several consistency checks that can be used here: If the $\mathrm{sgn}\gamma^{(11)}>0$ which is one of the terms with maximal level weighting, then $\sigma_0\neq 0$. When $\sigma_0=0$, $\gamma_{CP}=\mathrm{sgn}(\rho^{(3)}-\rho^{(5)})$ and $\gamma_{CP}=\mathrm{sgn}(\rho^{(3)}+\rho^{(5)})$ with the corollary that $\{|\rho^{(3)}|=|\rho^{(0)}|\}\geq|\rho^{(5)}|$. Also $C(\theta_1,\theta_2)$ can be used to show either $|a_{00}|$, $|a_{+-}|$, or $|a_{-+}|$ to be nonzero which would imply $\gamma_{CP}=+1$. When $\sigma_0=0$, knowledge of $\rho^{(3)}=-\rho_0$ and of $\rho^{(5)}$ means $|a_{0+}|/\sqrt{\mathscr{D}}$ and $|a_{+0}|/\sqrt{\mathscr{D}}$ are known if a choice for γ_{CP} is made, and so a three-parameter fit to $C(\theta_1,\theta_2)$ can be attempted and assessed for each choice for γ_{CP}, i.e., vary $|a_{00}|$, $|a_{+-}|$, and $|a_{-+}|$.

ϕZ^0 or JZ^0 decay mode

By CP invariance for a mode where V_1 and V_2 are each their own antiparticle and where $\mathscr{T}=0$, we obtain

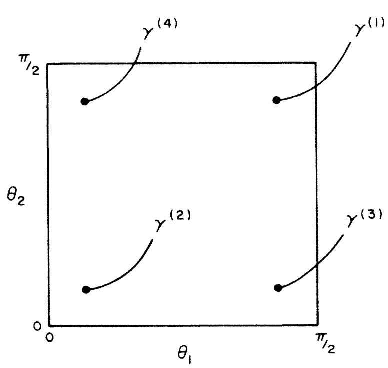

FIG. 2. An illustration showing the distinct coefficients which, respectively, dominate the first four terms of the $C(\theta_1,\theta_2)$ distribution near the four corners. The $C(\theta_1,\theta_2)$ distribution is obtained by integration over the entire ϕ azimuthal acceptance. When only the first four terms occur, symmetry properties of $C(\theta_1,\theta_2)$ (see text), can be used to rebin events outside the displayed region into the displayed square region $0\leq\theta_{1,2}\leq\pi/2$.

from Eq. (52) or Table IV the integrated distributions

$$F(\phi)=4\bar{C}(1+\mathscr{R}\mathscr{U}\sigma_0\cos2\phi)\,, \tag{88}$$

$$F_{11,22}(\phi)=\bar{C}\left[1+\mathscr{R}\mathscr{U}\rho_0\cos\phi\pm\frac{3\pi}{8}\mathscr{R}\mathscr{W}\rho^{(6)}\sin\phi +\mathscr{R}\mathscr{U}\sigma_0\cos2\phi\right], \tag{89}$$

$$F_{12,21}(\phi)=\bar{C}\left[1-\mathscr{R}\mathscr{U}\rho_0\cos\phi\pm\frac{3\pi}{8}\mathscr{R}\mathscr{W}\rho^{(6)}\sin\phi +\mathscr{R}\mathscr{U}\sigma_0\cos2\phi\right], \tag{90}$$

and the nonzero $\gamma^{(ij)}$'s which are

$$\gamma^{(20)}=\mathscr{R}C^{(20)},\quad \gamma^{(02)}=\mathscr{U}C^{(02)},\quad \gamma^{(22)}=\mathscr{R}\mathscr{U}C^{(22)}\,. \tag{91}$$

The weighting for terms with an $\mathscr{R}\mathscr{U}$ factor is the same as for the ϕJ and $J\Upsilon$ case considered above, and the $\mathscr{R}\mathscr{W}$ term will also not be as maximal as in $\phi\phi$ symmetry tests.

Using Table IV, we find ρ's, σ's, and τ's, and $\mathscr{D}$, as in Eqs. (65)–(67) except $\eta_P\to\gamma_{CP}$, plus

$$\rho^{(6)}=[+2\,\mathrm{Im}(a_{++}a_{00}^*)+2\gamma_{CP}(-)^J\,\mathrm{Im}(a_{+0}a_{0+}^*)]/\mathscr{D}\,. \tag{92}$$

When $\beta_0\neq 0$,

$$\gamma_{CP}(-)^J=\mathrm{sgn}\sigma_0\,. \tag{93}$$

When $\sigma_0=0$, the remarks following Eq. (68) apply with $\eta_P\to\gamma_{CP}$.

Summary

Table V compactly reviews previous results on P/CP determination from an $X\to V_1V_2$ decay which is, respectively, P or CP invariant followed by the decays $V_{1,2}\to A_{1,2}B_{1,2}$ with spin-0 mesons $A_1,B_1,\ldots$ or $V_{1,2}=\omega$ meson. For comparison, Table VI summarizes the results obtained in this section.

IV. STRONGER RESULTS FOR $J=1$

For $J=1$, $|a_{+-}|=|a_{-+}|=0$ so stronger results follow.

JJ or $\Upsilon\Upsilon$ mode

Only one amplitude $a_{0+}\neq 0$, so $\beta_0=0$ and $\alpha_0=-\eta_P\mathscr{R}^2/2$ (or when CP is good $\alpha_0=-\gamma_{CP}\mathscr{R}^2/2$). The

TABLE V. Summary of results for P, or CP, determination from an $X\to V_1V_2$ decay which is, respectively, P or CP invariant followed by the decays $V_{1,2}\to A_{1,2}B_{1,2}$ with spin-0 mesons $A_1,B_1,\ldots$ or $V_{1,2}=\omega$ meson. For convenience we frequently let $\beta\equiv\beta_0$ and $\alpha\equiv\alpha_0$, i.e., simply suppress the zero subscript, in the tables and text.

Mode / Measure	Case have $V_1\leftrightarrow V_2$ exchange property	Case no $V_1\leftrightarrow V_2$ but $V_1=\bar{V}_1$ and $V_2=\bar{V}_2$	Case no $V_1\leftrightarrow V_2$ with $V_1\neq\bar{V}_1$ and/or $V_2\neq\bar{V}_2$
X neutral $\alpha,\beta\Longrightarrow$	η_P if P good γ_{CP} if CP good	$\eta_P(-)^J$ if P good $\gamma_{CP}(-)^J$ if CP good	$\eta_P(-)^J$ if P good
X charged $\alpha,\beta\Longrightarrow$	η_P if P good		$\eta_P(-)^J$ if P good
Examples	$\phi,\phi,\rho^0\rho^0$ $K^{*0}\bar{K}^{*0},K^{*+}K^{*-}$ $K^{*+}\bar{K}^{*0},D^{*+}\bar{D}^{*0}$	$\phi\rho^0,\omega\rho^0,\omega\phi$	$\phi K^*,\rho K^*,\omega K^*$, $\rho^{\pm}\rho^0,\phi\rho^{\pm},\omega\rho^{\pm}$ and others $K^*\to D^*$
	Never fails	$J=0$ never fails; $J\geq 1$ inconclusive if both $\lvert a_{++}\rvert=0$ (accidental) and $\lvert a_{00}\rvert=0$	

integrated distribution

$$C(\theta_1,\theta_2)=\tfrac{1}{9}\mathcal{N}\{1+\tfrac{1}{2}\mathcal{R}[P_2(\cos\theta_1)+P_2(\cos\theta_2)]-2\mathcal{R}^2P_2(\cos\theta_1)P_2(\cos\theta_2)\}\ . \quad (94)$$

ϕJ or $J\Upsilon$ decay mode

Now, $\tau^{(1)}=2\,|a_{++}|^2/\mathcal{D}$ so the parity $\eta_P=-\sigma_0/\tau^{(1)}$ when $\tau^{(1)}$ and σ_0 are each nonzero, and otherwise $\tau^{(1)}=\sigma_0=0$. (When CP is good, $\eta_P\to\gamma_{CP}$ here.)

Z^0Z^0 decay mode

Besides the above results for JJ which apply here when $\eta_P\to\gamma_{CP}$, now $\gamma^{(11)}=0$ and $\alpha^{(3)}=\gamma_{CP}\mathcal{T}^2/2$. Unlike for arbitrary J, $\rho^{(6)}=\rho^{(8)}=0$.

W^+W^- decay mode

By Eqs. (81)–(83), we obtain

$$\sigma_0+\rho^{(3)}-\rho_0=-1\ \text{ if }\gamma_{CP}=-1\ , \quad (95)$$

$$1\geq(\sigma_0+\rho^{(3)}-\rho_0)\geq 0\ \text{ if }\gamma_{CP}=+1\ , \quad (96)$$

so if $\gamma_{CP}=+1$, $\tau^{(2)}=4\,|a_{00}|^2/\mathcal{D}=4(1-\sigma_0-\rho^{(3)}+\rho_0)$ provides a consistency check. When $\sigma_0=0$, only a one-parameter fit (i.e., vary $|a_{00}|$) to $C(\theta_1,\theta_2)$ is needed [see discussion following Eq. (87)].

ϕZ^0 or JZ^0 decay mode

Remarks above for ϕJ or $J\Upsilon$ apply when $\eta_P\to\gamma_{CP}$.

TABLE VI. Summary of results for P, or CP, determination from an $X\to V_1V_2$ decay which is, respectively, P or CP invariant followed by decays of V_1 and/or V_2 into a lepton and antilepton or into a quark and antiquark. For the case W^+W^- the equally informative $\alpha^{(3)}$ parameter should often be more reliably obtained from data than α_0, since $\alpha^{(3)}$ is about four times larger in the approximation that final fermion masses are zero, and since $\alpha^{(3)}$ but not α_0 survives when $I(\theta_1,\theta_2,\phi)$ is integrated over the full θ_1 and θ_2 acceptances.

Mode / Measure	Case have $V_1\leftrightarrow V_2$ exchange property	Case no $V_1\leftrightarrow V_2$ but $V_1=\bar{V}_1$ and $V_2=\bar{V}_2$
X neutral $\alpha_0,\beta_0\Longrightarrow$	η_P if P good	$\eta_P(-)^J$ if P good
Examples	JJ, $\Upsilon\Upsilon$	ϕJ, $J\Upsilon$
$\alpha_0,\beta_0\Longrightarrow$	γ_{CP} if CP good	$\gamma_{CP}(-)^J$ if CP good
Examples	Z^0Z^0, W^+W^-	ϕZ^0, JZ^0, ΥZ^0
	Never fails	$J=0$ never fails; $J\geq 1$ inconclusive if both $\lvert a_{++}\rvert=0$ (accidental) and $\lvert a_{00}\rvert=0$

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APPENDIX: EXTENT OF COMPLETE DETERMINATION OF $a_{\lambda_1\lambda_2}$ BY MEASUREMENT OF $I(\theta_1,\theta_2,\phi)$

For modes like Z^0Z^0, W^+W^-, and $Z^0W^\pm$, it is in principle possible to obtain the eight ρ's, two σ's, and nine τ's, so it is of interest to enumerate what part of complete knowledge of the $a_{\lambda_1\lambda_2}$ helicity amplitudes could be obtained from $I(\theta_1,\theta_2,\phi)$ and to point out cases with $J=0$ or 1 where complete knowledge can be obtained. The nine magnitudes follow from the nine τ's since

$$|a_{00}|^2=\mathscr{D}\tau^{(2)}/4, \quad |a_{\pm 0}|^2=\mathscr{D}(\tau^{(3)}\pm\tau^{(8)})/4 , \quad |a_{0\pm}|^2=\mathscr{D}(\tau^{(4)}\pm\tau^{(9)})/4 , \tag{A1}$$

and

$$\begin{bmatrix} |a_{++}|^2 \\ |a_{--}|^2 \\ |a_{+-}|^2 \\ |a_{-+}|^2 \end{bmatrix} = \frac{1}{4}\mathscr{D}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \tau^{(1)} \\ \tau^{(5)} \\ \tau^{(6)} \\ \tau^{(7)} \end{bmatrix} . \tag{A2}$$

The two σ's and eight ρ's provide

$$\sigma_0=2|a_{++}|\,|a_{--}|\cos(\theta_{++}-\theta_{--})/\mathscr{D} , \quad \sigma_x=-2|a_{++}|\,|a_{--}|\sin(\theta_{++}-\theta_{--})/\mathscr{D} , \tag{A3}$$

and

$$\begin{bmatrix} |a_{++}|\,|a_{00}|\cos(\theta_{00}-\theta_{++}) \\ |a_{00}|\,|a_{--}|\cos(\theta_{--}-\theta_{00}) \\ |a_{+0}|\,|a_{0-}|\cos(\theta_{0-}-\theta_{+0}) \\ |a_{0+}|\,|a_{-0}|\cos(\theta_{-0}-\theta_{0+}) \end{bmatrix} = \frac{1}{4}\mathscr{D}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \rho_0 \\ \rho^{(3)} \\ -\rho^{(5)} \\ -\rho^{(7)} \end{bmatrix} . \tag{A4}$$

Equation (A4) also applies for the sines of the listed angles, i.e., insert sines in place of the cosines in Eq. (A4), if ρ_0, $\rho^{(3)}$, $\rho^{(5)}$, $\rho^{(7)}$ are, respectively, replaced by ρ_x, $\rho^{(4)}$, $\rho^{(6)}$, $\rho^{(8)}$.

So, we see that $\sigma_{0,x}$ together provide the phase difference $(\theta_{++}-\theta_{--})$ and provide the product $|a_{++}|\,|a_{--}|$, i.e., one constraint on the τ's. The eight ρ's provide the four phases of Eq. (A4) and also, through the products $|a_{++}|\,|a_{00}|$, etc., of Eq. (A4), provide four constraints on the τ's. When both $\sigma_{0,x}$ and the eight ρ's are known, there is a consistency constraint from $\theta_{++}-\theta_{--}=(\theta_{++}-\theta_{00})-(\theta_{--}-\theta_{00})$ so one knows four relative phases, of the eight physically meaningful ones. For $J=0$, one has only three amplitudes and complete information can frequently be obtained from $I(\theta_1,\theta_2,\phi)$ as is discussed in the third paper of this series. For $J=1$, by Bose statistics, $X\to Z^0Z^0$ has only two amplitudes and complete information can in principle be obtained from $I(\theta_1,\theta_2,\phi)$.

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