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One-loop-order renormalization of the massless Wess-Zumino model in anti–de Sitter space

S. Bellucci*
Department of Applied Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
and Physics Department, Brandeis University, Waltham, Massachusetts 02254

J. González
Physics Department, Brandeis University, Waltham, Massachusetts 02254

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We apply the renormalization program to the massless Wess-Zumino model in anti–de Sitter space, using the version of Bunch and Parker of the adiabatic expansion. We consider an extrapolation of the dimensional-reduction prescription taking only the momentum of the expansion in D dimensions, while all the other tensors are kept in $D=4$, and show that the supersymmetry Ward identities are satisfied by the divergent one-loop corrections, but are violated by finite local terms proportional to the curvature of the space.

Recently attention has been paid[1] to the Wess-Zumino model generalized to anti–de Sitter four-dimensional space (AdS_4). In the fixed background of AdS_4 the simple supersymmetry group is OSp(1,4). In this Rapid Communication we are interested in exploring the possible modifications induced from the curved background in the renormalization of the model, in the case of massless fields.

The adiabatic expansion in momentum space[2] has been successfully applied, up to the two-loop level, to the scalar self-interacting theory in a general conformally flat background space;[3] the one-loop renormalization of grand-unified models has been considered as well.[4] Since the ultraviolet divergences are guaranteed to be local, the adiabatic expansion is sufficient to carry the renormalization program at the one-loop level. Actually, expanding the gravitational tensors in the neighborhood of a given point of the background space allows one to extract all the local corrections (both divergent and finite) in a relatively easy computable way. However, one should make sure to maintain supersymmetry in computing those local corrections. We find that the Ward identities are satisfied by the divergent one-loop corrections, but are violated by finite terms proportional to the curvature of the space, indicating either that our regularization prescription is not suitable to maintain supersymmetry at the level of those finite terms, or that the expansion cannot be applied to compute them, for the massless model.

We consider the following kinetic and interaction Lagrangians:[5]

$$L_{\rm kin}=\tfrac{1}{2}(D^\mu A D_\mu A+D^\mu B D_\mu B+i\bar\psi \not{D}\psi+F^2+G^2)+a(AF+BG)+\tfrac{3}{2}a^2(A^2+B^2)\ , \tag{1}$$

$$L_{\rm int}=g(A^2F-B^2F-2ABG)+a(A^3-3AB^2)-g\bar\psi(A+i\gamma_5B)\psi\ . \tag{2}$$

Both Lagrangians separately transform into a total derivative under the supersymmetry transformations

$$\begin{aligned}&\delta A=\bar\epsilon\psi,\ \ \delta B=-i\bar\epsilon\gamma_5\psi\ ,\\ &\delta\bar\psi=i\bar\epsilon\not{\partial}(A+i\gamma_5B)+\bar\epsilon(F+i\gamma_5G)\ ,\\ &\delta F=-i\bar\epsilon\not{D}\psi-a\bar\epsilon\psi,\ \ \delta G=\bar\epsilon\gamma_5\not{D}\psi+ia\bar\epsilon\gamma_5\psi\ .\end{aligned} \tag{3}$$

In the above equations a is the contraction parameter [with dimensions $(\text{length})^{-1}$]. We use the conventions $\gamma_5{}^2=1$ and $(-++)$ in the notation of Ref. 6. We have then $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}=a^2(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha})$.

The next step is the derivation of the Ward identity for the effective action of the model. We must allow for a generating functional depending on the respective sources J_A, J_B, η, J_F, and J_G,

$$Z(J)=\int\mathscr{D}\phi\exp\Big\{i\int dx\sqrt{-g}\,[\mathscr{L}(\phi)+J_AA+J_BB+\bar\psi\eta+J_FF+J_GG]\Big\}\ ,$$

where the J's are supposed to form a supermultiplet in order to maintain the invariance of the generating functional under supersymmetry transformations. The transformation rules are easily found to be

$$\begin{aligned}&\delta J_A=-iD_\mu(\bar\eta\gamma^\mu\epsilon),\ \ \delta J_B=-iD_\mu(\bar\eta i\gamma_5\gamma^\mu\epsilon)\ ,\\ &\delta\eta=-J_A\epsilon+J_Bi\gamma_5\epsilon+i\not{D}(J_F\epsilon)+aJ_F\epsilon+i\not{D}(J_Gi\gamma_5\epsilon)-aJ_Gi\gamma_5\epsilon\ ,\\ &\delta J_F=-\bar\eta\epsilon,\ \ \delta J_G=-\bar\eta i\gamma_5\epsilon\ .\end{aligned} \tag{4}$$

The Ward identity for the generating functional expresses its invariance under a supersymmetry transformation

$$-iD_\mu(\bar\eta\gamma^\mu\epsilon)\frac{\delta Z}{\delta J_A}-iD_\mu(\bar\eta i\gamma_5\gamma^\mu\epsilon)\frac{\delta Z}{\delta J_B}+\frac{\delta Z}{\delta\eta}[-J_A\epsilon+J_Bi\gamma_5\epsilon+i\not{D}(J_F\epsilon)+aJ_F\epsilon+i\not{D}(J_Gi\gamma_5\epsilon)-aJ_Gi\gamma_5\epsilon]-\bar\eta\epsilon\frac{\delta Z}{\delta J_F}-\bar\eta i\gamma_5\epsilon\frac{\delta Z}{\delta J_G}=0\ , \tag{5}$$

and, obviously, a similar identity holds for the generating functional of the connected Green's functions $W=-i\ln Z(J)$.

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By performing the usual Legendre transformation

$$\frac{\delta W}{\delta J_l}=\hat{\phi}_l,\quad \Gamma(\hat{\phi})=W-\int dx\sqrt{-g}J_l\hat{\phi}_l$$

and taking into account that $\delta\Gamma/\delta\hat{\phi}_l=-J_l$, one can translate the previous Ward identity into the corresponding expression for the effective action $\Gamma(\hat{\phi})$,

$$i\frac{\delta\Gamma}{\delta\hat{\psi}}\gamma^\mu\epsilon\partial_\mu\hat{A}+i\frac{\delta\Gamma}{\delta\hat{\psi}}i\gamma_5\gamma^\mu\epsilon\partial_\mu\hat{B}-\frac{\delta\Gamma}{\delta\hat{A}}\hat{\bar{\psi}}\epsilon+\frac{\delta\Gamma}{\delta\hat{B}}\hat{\bar{\psi}}i\gamma_5\epsilon-i\hat{\bar{\psi}}\overleftarrow{\not{D}}\epsilon\frac{\delta\Gamma}{\delta\hat{F}}$$
$$+a\hat{\bar{\psi}}\epsilon\frac{\delta\Gamma}{\delta\hat{F}}-i\hat{\bar{\psi}}\overleftarrow{\not{D}}i\gamma_5\epsilon\frac{\delta\Gamma}{\delta\hat{G}}-a\hat{\bar{\psi}}i\gamma_5\epsilon\frac{\delta\Gamma}{\delta\hat{G}}-\frac{\delta\Gamma}{\delta\hat{\psi}}\epsilon\hat{F}-\frac{\delta\Gamma}{\delta\hat{\psi}}i\gamma_5\epsilon\hat{G}=0\ . \tag{6}$$

This is nothing but the statement of the invariance of the effective action with respect to a supersymmetry transformation, if we assume that the classical fields $\hat{\phi}_l$ transform in the same way as their quantum counterparts.

By taking functional derivatives of this expression, one can obtain successive Ward identities relating the different n-point functions of the model. By differentiating with respect to $\hat{\bar{\psi}}(y)$ and $\hat{A}(x)$, for example, and then setting all the sources equal to zero, we get

$$i\frac{\delta^2\Gamma}{\delta\hat{\bar{\psi}}(y)\delta\hat{\psi}(z)}\gamma^\mu\left[\partial_\mu\frac{1}{\sqrt{-g}}\delta(x-z)\right]\epsilon(z)-\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{A}(z)}\frac{1}{\sqrt{-g}}\delta(y-z)\epsilon(z)$$
$$-i\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{F}(z)}\left[\frac{1}{\sqrt{-g}}\delta(y-z)\overleftarrow{\not{D}}\right]\epsilon(z)+a\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{F}(z)}\frac{1}{\sqrt{-g}}\delta(y-z)\epsilon(z)=0\ . \tag{7}$$

By differentiating once more the original Ward identity with respect to $\hat{A}(u)$, we obtain a relation among the three-point functions of the model,

$$i\frac{\delta^3\Gamma}{\delta\hat{A}(x)\delta\hat{\bar{\psi}}(y)\delta\hat{\psi}(z)}\gamma^\mu\left[\partial_\mu\frac{1}{\sqrt{-g}}\delta(u-z)\right]\epsilon(z)+i\frac{\delta^3\Gamma}{\delta\hat{A}(u)\delta\hat{\bar{\psi}}(y)\delta\hat{\psi}(z)}\frac{1}{\sqrt{-g}}\delta(x-z)\epsilon(z)$$
$$-\frac{\delta^3\Gamma}{\delta\hat{A}(u)\delta\hat{A}(x)\delta\hat{A}(z)}\frac{1}{\sqrt{-g}}\delta(y-z)\epsilon(z)-i\frac{\delta^3\Gamma}{\delta\hat{A}(u)\delta\hat{A}(x)\delta\hat{F}(z)}\left[\frac{1}{\sqrt{-g}}\delta(y-z)\overleftarrow{\not{D}}\right]\epsilon(z)$$
$$+a\frac{\delta^3\Gamma}{\delta\hat{A}(u)\delta\hat{A}(x)\delta\hat{F}(z)}\frac{1}{\sqrt{-g}}\delta(y-z)\epsilon(z)=0\ .$$

In what follows we will focus our attention on the first of these Ward identities. It can be shown that the local parts of all the three-point functions of the model are identically zero to the one-loop order (see below), which makes the last expression trivially satisfied at this level.

The expression (7) can be cast in a more convenient form by making use of integration by parts (after replacing the integral symbol),

$$\int dz\left[-i\bar{\epsilon}(y)\frac{\delta^2\Gamma}{\delta\hat{\bar{\psi}}(y)\delta\hat{\psi}(z)}\gamma_\mu\epsilon(z)\vec{D}_z^\mu\delta(x-z)-\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{A}(z)}\bar{\epsilon}(y)\epsilon(z)\delta(y-z)\right.$$
$$\left.-i\bar{\epsilon}(y)\gamma^\mu\epsilon(z)D_\mu\left[\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{F}(z)}\right]\delta(y-z)+3a\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{F}(z)}\bar{\epsilon}(y)\epsilon(z)\delta(y-z)\right]=0\ ,$$

where use has been made of the definition of $\epsilon(z)$, $D_\mu\epsilon(z)=-(ia/2)\gamma_\mu\epsilon(z)$. Finally, we get

$$-i\bar{\epsilon}(y)\frac{\delta^2\Gamma}{\delta\hat{\bar{\psi}}(y)\delta\hat{\psi}(x)}\gamma_\mu\epsilon(x)\vec{D}_x^\mu-\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{A}(y)}+3a\frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{F}(y)}=0\ . \tag{8}$$

It can be shown that this expression is satisfied at the tree level in the sense of operators, with

$$\frac{\delta^2\Gamma}{\delta\hat{\bar{\psi}}(y)\delta\hat{\psi}(x)}=i\not{D}_y\frac{1}{\sqrt{-g}}\delta(y-x),\quad \frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{A}(y)}=(-\Box_x+3a^2)\frac{1}{\sqrt{-g}}\delta(x-y),\quad \frac{\delta^2\Gamma}{\delta\hat{A}(x)\delta\hat{F}(y)}=a\frac{1}{\sqrt{-g}}\delta(x-y)\ .$$

By applying, for example, Eq. (8) to a test function $\phi(x)$ we get

$$\int dx\sqrt{-g}\left[-i\bar{\epsilon}(y)i\left[\not{D}_y\frac{1}{\sqrt{-g}}\delta(y-x)\right]\gamma_\mu\epsilon(x)\overleftarrow{D}_x^\mu\phi(x)-\phi(x)(-\Box_x+3a^2)\frac{1}{\sqrt{-g}}\delta(x-y)+\phi(x)3a^2\frac{1}{\sqrt{-g}}\delta(x-y)\right]\ ,$$

which is identically zero after integration by parts.

Since (1) and (2) are invariant with respect to the supersymmetry transformations in AdS_4 [Eq. (3)], we take the adiabatic expansion of the propagators in the four-dimensional background space,

$$\langle A(x)A(x')\rangle = i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')}\left[\frac{1}{k^2+i0}+\frac{1}{3}R\frac{1}{(k^2+i0)^2}-\frac{2}{3}R_{\rho\nu}(x')\frac{k^\rho k^\nu}{(k^2+i0)^3}-2a^2\frac{1}{(k^2+i0)^2}+O\left(\frac{a^4}{k^6}\right)\right]=\langle B(x)B(x')\rangle ,$$

$$\langle A(x)F(x')\rangle = -a\langle A(x)A(x')\rangle = \langle B(x)G(x')\rangle ,\quad \langle F(x)F(x')\rangle = i\delta(x,x')+a^2\langle A(x)A(x')\rangle = \langle G(x)G(x')\rangle , \tag{9}$$

$$\langle \psi(x)\bar\psi(x')\rangle = -i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')}\left[\frac{\gamma^{\hat a}k_{\hat a}}{k^2+i0}+\frac{R}{12}\frac{\gamma^{\hat a}k_{\hat a}}{(k^2+i0)^2}-\frac{2}{3}R_{\rho\nu}(x')\frac{k^\rho k^\nu}{(k^2+i0)^3}\gamma^{\hat a}k_{\hat a}+\frac{1}{4}\gamma^{\hat a}\gamma^{\hat b}\gamma^{\hat c}\frac{R_{\hat b\hat c\hat a\nu}k^\nu}{(k^2+i0)^2}+O\left(\frac{a^4}{k^5}\right)\right] ,$$

with $R_{\mu\nu}=3a^2g_{\mu\nu}$ and $R=12a^2$. In the propagators (9) the Feynman boundary condition is denoted by adding to k^2 a small positive imaginary part. This is understood as a prescription in momentum space needed to make contact with the quantum model in the flat–space-time limit. It has no relation with the boundary conditions that the large-scale features of anti–de Sitter space impose on the Green's functions defined in x space, which are extensively discussed in the literature.[7] Although the adiabatic expansion cannot reflect the global structure of the space-time, the ultraviolet behavior of a quantum model is not sensitive to these global features nor to any particular choice of the vacuum in the curved space-time background. The adiabatic expansion is then adequate to our purposes as long as we are interested in renormalization coefficients that depend on the short-distance behavior of the model.

In what follows we are going to take a regularization procedure in which only the momenta are analytically continued to D dimensions, while keeping all the remaining tensors in four dimensions. This is, in fact, an extrapolation of the usual dimensional-reduction method adopted in a flat–space-time background,[8] since we have to give now a definite prescription for the gravitational tensors $R_{\mu\nu\alpha\beta}$, $R_{\mu\nu}$. The solution adopted here in which these tensors are considered in four dimensions, while the momenta are considered in D, seems to us the closest in spirit to the usual dimensional-reduction prescription, connecting at the same time with the classical invariance of the Lagrangian for the four-dimensional anti–de Sitter background space.

The calculation of the Feynman diagrams contributing to the one-particle-irreducible two-point functions Γ_{AF}, Γ_{AA}, and $\Gamma_{\psi\bar\psi}$ to the one-loop order gives the results

$$\Gamma_{AF}(x,x')=\frac{4iu^2}{16\pi^2}a(\Delta+2)\delta(x,x')+\text{nonlocal } O(a^3) \text{ terms},$$

$$\Gamma_{AA}(x,x')=\frac{-4iu^2}{16\pi^2}[(\Box_x-3a^2)\Delta +2(\Box_x-4a^2)]\delta(x,x')+O(a^4) , \tag{10}$$

$$\Gamma_{\psi_\alpha\bar\psi_\beta}(x,x')=\frac{-4u^2}{16\pi^2}[(\gamma_{\hat a})_{\alpha\beta}(\Delta+2)D_x^{\hat a}\delta(x,x') -\langle\psi_\alpha(x)\bar\psi_\beta(x')\rangle a^2]+O(a^4) ,$$

where $g=u\mu^{\epsilon/2}$ with u dimensionless, $\epsilon=4-D$, and

$$\Delta=\frac{2}{\epsilon}+\left.\frac{d\Gamma(z)}{dz}\right|_{z=1}+\ln 4\pi-\ln\frac{-p^2}{\mu^2}$$

with $\Gamma(z)$ being the Euler function. In deriving $\Gamma_{AA}(x,x')$ we have used the relation between $\partial^\mu\partial_\mu$ and $D^\mu D_\mu$ in normal coordinates,[4]

$$\eta^{\mu\nu}\frac{\partial^2}{\partial y^\mu\partial y^\nu}\delta(y)=(D^\mu D_\mu-\tfrac{1}{3}g^{\alpha\beta}g^{\mu\nu}R_{\mu\beta\nu\alpha})\delta(x,x') .$$

Substituting (10) into (8) and using the relation

$$i\bar\epsilon(x')\Gamma_{\psi_\alpha\bar\psi_\beta}(x,x')\gamma_{\hat b}\epsilon(x)\overleftarrow{D}_x^{\hat b}=\frac{-4iu^2}{16\pi^2}[(\Delta+2)\delta(x,x')\Box_x-a^2\delta(x,x')+2ia^3\bar\epsilon(x')\langle\psi_\alpha(x)\bar\psi_\beta(x')\rangle\epsilon(x)] , \tag{11}$$

one readily checks that the Ward identity is satisfied for the divergent part including terms proportional to a^2. Equation (8) is also satisfied for the flat–space-time finite terms, but violated by finite terms, of order a^2. In fact,

$$i\bar\epsilon(x')\Gamma_{\psi_\alpha\bar\psi_\beta}(x,x')\gamma_{\hat a}\epsilon(x)\overleftarrow{D}_x^{\hat a}+[-\Gamma_{AA}(x,x')+3a\Gamma_{AF}(x,x')]\delta_{\alpha\beta}=-\frac{4iu^2}{16\pi^2}a^2\delta(x,x')+\text{nonlocal } O(a^3) \text{ terms} . \tag{12}$$

The corresponding Ward identity for the three-point functions is trivially satisfied for the local corrections at the one-loop level, since they vanish by explicit cancellation between the virtual (B,G) contribution and the virtual (A,F) one,[9]

$$\Gamma_{AAA}(x,y,z)|_{\text{local}}=\Gamma_{FAA}(x,y,z)|_{\text{local}}=\Gamma_{A\psi_\alpha\bar\psi_\beta}(x,y,z)|_{\text{local}}=0 .$$

To summarize, we have carried out the renormalization of the massless Wess-Zumino model in AdS_4 space using the adiabatic expansion of the propagators in momentum space. The expansion allows us to extract all the local contributions (both divergent and finite), and we have shown that it maintains supersymmetry in the results at the one-loop level for the divergent part including terms proportional to the curvature, using a regularization scheme in which the momenta in the expansion are taken in D dimensions and the other tensor indices are taken in $D=4$. While in our case it is not obvious that the counting of the fermionic degrees of freedom equals that of the bosonic ones, the finite terms proportional to the curvature do not satisfy the supersymmetry Ward identity (8) between two-point functions, and we believe that a manifestly supersymmetric regularization scheme is required. This would tell us whether the violation of the above supersymmetry Ward identity is indicating here the nonsupersymmetric character of our choice of the regulator or rather is a signal that the adiabatic expansion is inadequate to calculate the finite one-loop corrections to the massless model. A natural choice seems to be Pauli-Villars regularization, although one should also consider the possibility of regularizing by the method of point splitting. Work on this is presently in progress and will be reported elsewhere.[10]

From Eq. (10) we can extract the common renormalization factor Z that multiplied times L_{kin} gives the effective action for both the divergent and the flat-space-time finite parts,

$$Z = 1 + \frac{4u^2}{16\pi^2}(\Delta + 2) \quad .$$

The vanishing of the three-point functions at the one-loop level indicates that the statement of the renormalizability of the model and the absence of renormalization of L_{int} (Ref. 11) are still valid in the anti-de Sitter background.

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*On leave from International School for Advanced Studies, Strada Costiera 11, I-34014 Trieste, Italy.

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