Imaginary part in thermo field dynamics

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(Received 10 April 1985)

We investigate the imaginary part of Feynman diagrams in a real-time finite-temperature formalism, thermo field dynamics. Interesting relations are found which lead to a simple derivation of the imaginary part of self-energy diagrams.

There has been a growing interest in the problem of nonequilibrium phenomena in quantum field theory, especially in relation with the quantum effects in the early universe.[1] Unfortunately, at present, there is no well-established theory capable of coping with the general nonequilibrium systems. The best one can do is to formulate a theory which allows one to treat the situations where the deviation from equilibrium is small. This line of approach has been taken by several people to discuss the early universe and some interesting results have been obtained.[1] Among them an important one is the relation between the imaginary part of Feynman diagrams and the transport coefficients used to describe the dissipative processes. Hence developing a systematic and simple method for evaluating the imaginary part of Feynman diagrams at finite temperature is highly desirable.[2]

The present Brief Report is concerned with the calculation of the imaginary part in the perturbation based on the real-time finite-temperature formalism. We shall take thermo field dynamics[3] (TFD) as an example of real-time formalisms.[4] Further, we shall limit our scope and consider only the self-energy diagrams. As regards the foundation of TFD readers may consult the original works,[3] and also works in Ref. 5 for the perturbative calculation of the real part of Feynman diagrams at finite temperature.

TFD perturbation is defined in the Minkowski space and thus the integral contains both real and imaginary parts explicitly. In contrast, the perturbation in the conventional imaginary-time formalism[2,6,7] (ITF) is defined on $S^1\times E^3$ and thus the integral is purely real. Hence the evaluation of the imaginary part in ITF involves an analytic continuation back to real time.[7] One of our purposes is to answer the question raised by Weldon[8] whether TFD reproduces the same imaginary part calculated in ITF. We shall answer this question in the affirmative.

First, let us give a quick review of the ITF calculation.[8] Consider a self-energy diagram in $\lambda\phi^3$ theory (Fig. 1). The imaginary part (or discontinuity) is obtained as follows. The one-loop correction to the self-energy is given by

$$-i\Sigma(p)=\frac{i}{\beta}(-i\lambda)^2\sum_{n=-\infty}^{\infty}\int\frac{d^3k_1}{(2\pi)^3}\frac{i}{k_1{}^2+m_1{}^2}\frac{i}{k_2{}^2+m_2{}^2}\ , \tag{1}$$

$$k_2=p-k_1,\quad k_{10}=\frac{2\pi n}{\beta},\quad p_0=\frac{2\pi N}{\beta}\ ,$$

$$k_i{}^2=k_{i0}{}^2+\mathbf{k}_i{}^2\quad (i=1,2)\ .$$

First, one evaluates the real part of (1). Using the formula

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}f\left(\frac{2\pi n}{\beta}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dz\,f(z)+\frac{1}{\pi}\int_{-\infty+i\epsilon}^{\infty+i\epsilon}dz\,f(z)\frac{1}{e^{-i\beta z}-1}\ , \tag{2}$$

and calculating the residue due to poles at $z=+iE_1$ and $z=p_0+iE_2$, one finds

$$\Sigma=\lambda^2\int\frac{d^3k_1}{(2\pi)^3}\frac{1}{4E_1E_2}\left[[1+n_B(k_1)+n_B(k_2)]\left(\frac{1}{ip_0-E_1-E_2}-\frac{1}{ip_0+E_1+E_2}\right)+[n_B(k_1)-n_B(k_2)]\left(\frac{1}{ip_0+E_1-E_2}-\frac{1}{ip_0-E_1+E_2}\right)\right]\ , \tag{3}$$

$$n_B(k_i)=\frac{1}{e^{\beta E_i}-1},\quad E_i=(\mathbf{k}_i{}^2+m_i{}^2)^{1/2}\quad (i=1,2)\ .$$

Then one continues $ip_0=i2\pi N/\beta\rightarrow p_0(\text{real})+i\epsilon$, and uses the relation

$$\frac{1}{p_0+i\epsilon-A}=\frac{P}{p_0-A}-i\pi\delta(p_0-A) \tag{4}$$

(P is the principal part) to extract the imaginary part. The result is

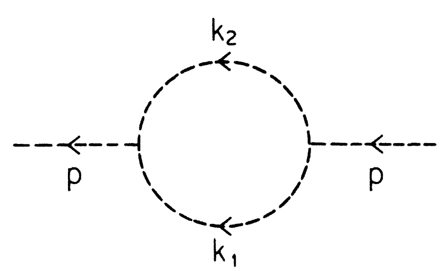

FIG. 1. A one-loop self-energy diagram in $\lambda\phi^3$ theory.

$$\mathrm{Im}\Sigma=-\pi\lambda^2\int\frac{d^3k_1}{(2\pi)^3}\frac{1}{4E_1E_2}\{[\delta(p_0-E_1-E_2)-\delta(p_0+E_1+E_2)][1+n_B(k_1)+n_B(k_2)]$$
$$+[\delta(p_0+E_1-E_2)-\delta(p_0-E_1+E_2)][n_B(k_1)-n_B(k_2)]\}\ . \tag{5}$$

The so-called discontinuity, discΣ, is given by

$$\mathrm{disc}\Sigma=-2i\,\mathrm{Im}\Sigma\ . \tag{6}$$

To see the connection with TFD it is convenient to rewrite the standard thermo propagator, $\Delta(k)^{\alpha\gamma}$, as follows:[9]

$$\Delta(k)^{\alpha\gamma}=[U_B(k_0)\bar{\Delta}(k)U_B(k_0)]^{\alpha\gamma} \tag{7a}$$

$$=\begin{pmatrix}\frac{1}{k^2-m^2+i\epsilon} & 0\\ 0 & \frac{-1}{k^2-m^2-i\epsilon}\end{pmatrix}-2\pi i\delta(k^2-m^2)\frac{1}{e^{\beta|k_0|}-1}\begin{pmatrix}1 & e^{\beta|k_0|/2}\\ e^{\beta|k_0|/2} & 1\end{pmatrix}, \tag{7b}$$

where

$$\bar{\Delta}(k)=\frac{\tau}{(k_0+i\delta\tau)^2-(\mathbf{k}^2+m^2)},\quad \tau=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$
$$U_B(k_0)=\begin{pmatrix}\cosh\theta(k_0) & \sinh\theta(k_0)\\ \sinh\theta(k_0) & \cosh\theta(k_0)\end{pmatrix},\quad \sinh^2\theta(k_0)=\frac{1}{e^{\beta k_0}-1}\ . \tag{8}$$

Then consider the renormalized scalar thermo propagator

$$\Delta'^{-1}(k)^{\alpha\gamma}=(k^2-m^2)\tau^{\alpha\gamma}-\Sigma(k)^{\alpha\gamma}\ , \tag{9}$$

$$\Delta'(k)^{\alpha\gamma}=\left[U_B(k_0)\frac{1}{\bar{\Delta}^{-1}(k)-\bar{\Sigma}(k)}U_B(k_0)\right]^{\alpha\gamma}\ , \tag{10}$$

where $\Sigma(k)$ and $\bar{\Sigma}(k)$ are connected with each other through the following relation:

$$\Sigma(k)^{\alpha\gamma}=[\tau U_B(k_0)\bar{\Sigma}(k)U_B(k_0)\tau]^{\alpha\gamma}\ , \tag{11a}$$

where

$$\bar{\Sigma}^{\alpha\gamma}=\tau^{\alpha\gamma}\,\mathrm{Re}\bar{\Sigma}+i\delta^{\alpha\gamma}\,\mathrm{Im}\bar{\Sigma}\ . \tag{11b}$$

Explicitly, the (1,1) component of the relation (10) reads

$$\Delta'(k)^{11}=\frac{\cosh^2\theta(k_0)}{(k_0+i\Gamma)^2-\omega^2}-\frac{\sinh^2\theta(k_0)}{(k_0-i\Gamma)^2-\omega^2}\ ,$$
$$\omega^2=\mathbf{k}^2+m^2+\mathrm{Re}\bar{\Sigma}-\Gamma^2,\quad \mathrm{Im}\bar{\Sigma}=-2k_0\Gamma\ . \tag{12}$$

From (12) one sees that the imaginary part of our interest is given by $\mathrm{Im}\bar{\Sigma}$. From (11a) and (11b) one easily obtains

$$\mathrm{Im}\Sigma^{11}=\mathrm{Im}\Sigma^{22},\quad \mathrm{Im}\Sigma^{12}=\mathrm{Im}\Sigma^{21},$$
$$\mathrm{Im}\Sigma^{11}/\mathrm{Im}\Sigma^{12}=-\cosh\frac{\beta k_0}{2}\ . \tag{13}$$

Further, one finds

$$\mathrm{Im}\bar{\Sigma}=\frac{e^{\beta k_0}-1}{e^{\beta k_0}+1}\mathrm{Im}\Sigma^{11}\ , \tag{14a}$$

$$\mathrm{Im}\bar{\Sigma}=-\frac{e^{\beta k_0}-1}{2e^{\beta k_0/2}}\mathrm{Im}\Sigma^{12}\ . \tag{14b}$$

The relations (14a) and (14b) are interesting in that one can now obtain $\mathrm{Im}\,\bar{\Sigma}(k)$ in three different ways[10] by calculating directly the imaginary part of either Σ^{11}, Σ^{12}, or Σ^{22}. This is very different from the ITF calculation. What is more interesting is that among the three methods there is a big difference in the labor involved. The easiest way is to evaluate Σ^{12}, which is purely imaginary to begin with; hence no physical real part is involved in the calculation. In the following we exploit the relation (14b) and provide a new and simpler calculation of $\mathrm{Im}\,\Sigma(k)$ in the examples treated previously in ITF.[8,11]

(1) $\lambda\phi^3$ theory (Fig. 1).

$$\mathrm{Im}\bar{\Sigma}=-\frac{e^{\beta p_0}-1}{2e^{\beta p_0/2}}(+\lambda^2)\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-k_1-k_2)2\pi\delta(k_1^2-m_1^2)2\pi\delta(k_2^2-m_2^2)n_b(k_1)n_B(k_2)e^{\beta(|k_{10}|+|k_{20}|)/2}$$
$$=-\pi\lambda^2\frac{e^{\beta p_0}-1}{e^{\beta p_0/2}}\int\frac{d^3k_1}{(2\pi)^3}\frac{e^{\beta(E_1+E_2)/2}}{4E_1E_2}n_B(k_1)n_B(k_2)[\delta(p_0-E_1-E_2)+\delta(p_0-E_1+E_2)$$
$$+\delta(p_0+E_1-E_2)+\delta(p_0+E_1+E_2)]$$
$$=-\pi\lambda^2\int\frac{d^3k_1}{(2\pi)^3}\frac{1}{4E_1E_2}\{\delta(p_0-E_1-E_2)[1+n_B(k_1)+n_B(k_2)]-\delta(p_0+E_1+E_2)[1+n_B(k_1)+n_B(k_2)]$$
$$+\delta(p_0-E_1+E_2)[n_B(k_2)-n_B(k_1)]+\delta(p_0+E_1-E_2)[n_B(k_1)-n_B(k_2)]\}\ . \tag{15}$$

In the above we have made use of, for instance,

$$\frac{e^{\beta p_0}-1}{e^{\beta p_0/2}}e^{\beta(E_1+E_2)/2}n_B(k_1)n_B(k_2)\delta(p_0-E_1-E_2)=(e^{\beta(E_1+E_2)}-1)n_B(k_1)n_B(k_2)\delta(p_0-E_1-E_2)$$
$$=\left[\frac{1}{n_B(k_1)n_B(k_2)}+\frac{1}{n_B(k_1)}+\frac{1}{n_B(k_2)}\right]n_B(k_1)n_B(k_2)\delta(p_0-E_1-E_2)$$
$$=[1+n_B(k_1)+n_B(k_2)]\delta(p_0-E_1-E_2)\ . \tag{16}$$

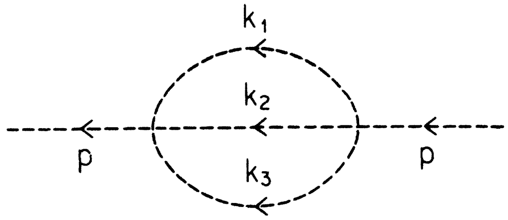

FIG. 2. A two-loop self-energy diagram in $\lambda\phi^4$ theory.

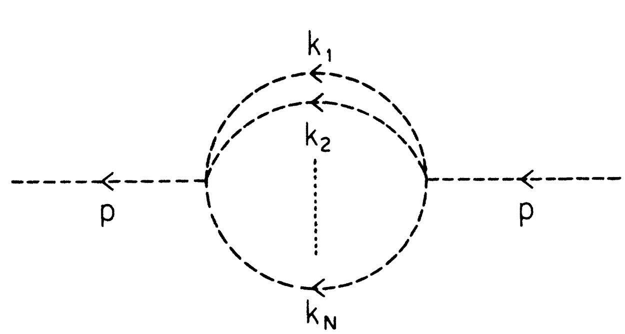

FIG. 3. An $(N-1)$-loop self-energy diagram in $\lambda\phi^{N+1}$ theory.

We have reproduced (5).

(2) $\lambda\phi^4$ theory (Fig. 2).

$$\mathrm{Im}\bar{\Sigma}=-\frac{1}{2}\frac{e^{\beta p_0}-1}{e^{\beta p_0/2}}(-\lambda^2)\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p-k_1-k_2-k_3)\,2\pi\delta(k_1{}^2-m_1{}^2)2\pi\delta(k_2{}^2-m_2{}^2)$$
$$\times 2\pi\delta(k_3{}^2-m_3{}^2)n_B(k_1)n_B(k_2)n_B(k_3)\exp[\beta(|k_{10}|+|k_{20}|+|k_{30}|)/2]$$
$$=-\pi\lambda^2\frac{e^{\beta p_0}-1}{e^{\beta p_0/2}}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{e^{\beta(E_1+E_2+E_3)/2}}{8E_1E_2E_3}n_B(k_1)n_B(k_2)n_B(k_3)$$
$$\times[\delta(p_0-E_1-E_2-E_3)+\delta(p_0-E_1-E_2+E_3)+\delta(p_0-E_1+E_2-E_3)+\delta(p_0+E_1-E_2-E_3)$$
$$+\delta(p_0-E_1+E_2+E_3)+\delta(p_0+E_1-E_2+E_3)+\delta(p_0+E_1+E_2-E_3)+\delta(p_0+E_1+E_2+E_3)]\ . \quad (17)$$

Then, a manipulation similar to (16) leads to a result identical with that of ITF calculation.[11]

(3) $\lambda\phi^{N+1}$ theory in two dimensions (Fig. 3).

The imaginary part of this general diagram is simply given by

$$\mathrm{Im}\bar{\Sigma}=-\pi\lambda^2\frac{e^{\beta p_0}-1}{e^{\beta p_0/2}}\int\prod_{i=1}^{N}\frac{d^2k_i}{(2\pi)^2}(2\pi)^2\delta^{(2)}\left(p-\sum_{i=1}^{N}k_i\right)\exp\left(\frac{\beta}{2}\sum_{i=1}^{N}E_i\right)\prod_{i=1}^{N}[n_B(k_i)2\pi\delta(k_i{}^2-m_i{}^2)]\ . \quad (18)$$

The term proportional to $\delta(p_0-\sum_{i=1}^{N}E_i)$ is

$$-\pi\lambda^2\int\prod_{i=1}^{N-1}\frac{dk_i}{2\pi}\prod_{i=1}^{N}\frac{1}{2E_i}\left(\prod_{i=1}^{N}[1+n_B(k_i)]-\prod_{i=1}^{N}n_B(k_i)\right)\delta\left(p_0-\sum_{i=1}^{N}E_i\right)\ . \quad (19)$$

In comparison with other methods the power of relation (14b) is evident in these examples.

Let us turn to the fermion case. We find relations similar to (14):

$$\mathrm{Im}\bar{\Sigma}=\frac{e^{\beta p_0}+1}{e^{\beta p_0}-1}\mathrm{Im}\Sigma^{11}\quad(\mathrm{Im}\Sigma^{11}=-\mathrm{Im}\Sigma^{22})\ , \quad (20a)$$

$$\mathrm{Im}\bar{\Sigma}=\frac{e^{\beta p_0}+1}{2e^{\beta p_0/2}}\mathrm{Im}\Sigma^{12}\quad(\mathrm{Im}\Sigma^{12}=\mathrm{Im}\Sigma^{21})\ . \quad (20b)$$

As in the scalar case we make use of the second relation.

(4) Fermion self-energy diagram (Fig. 4).

In Fig. 4, the broken line represents a scalar. The fermion propagator is given by

$$S_F(k)=(\not{k}+m)\left[\begin{pmatrix}\frac{1}{k^2-m^2+i\epsilon} & 0\\ 0 & \frac{1}{k^2 m^2-i\epsilon}\end{pmatrix}+2\pi i\delta(k^2-m^2)\frac{1}{e^{\beta|k_0|}+1}\begin{pmatrix}1 & -\epsilon(k_0)e^{\beta|k_0|/2}\\ -\epsilon(k_0)e^{\beta|k_0|/2} & -1\end{pmatrix}\right]\ . \quad (21)$$

With

$$n_F(k)=\frac{1}{e^{\beta E}+1},\quad E=(\mathbf{k}^2+m^2)^{1/2}\ , \quad (22)$$

one finds (f is the Yukawa coupling, $k_2=p-k_1$)

$$\mathrm{Im}\bar{\Sigma}=-\frac{f^2}{2}\frac{e^{\beta p_0}+1}{e^{\beta p_0/2}}\int\frac{d^4k_1}{(2\pi)^4}\frac{(\not{k}_1+m)}{4E_1E_2}(2\pi)^2e^{\beta(E_1+E_2)/2}\epsilon(k_{10})n_F(k_1)n_B(k_2)$$
$$\times[\delta(k_{10}-E_1)+\delta(k_{10}+E_1)][\delta(k_{20}-E_2)+\delta(k_{20}+E_2)]$$
$$=-\pi f^2\frac{e^{\beta p_0}+1}{e^{\beta p_0/2}}\int\frac{d^3k_1}{(2\pi)^3}\frac{e^{\beta(E_1+E_2)/2}}{4E_1E_2}n_F(k_1)n_B(k_2)\{(E_1\gamma_0-\mathbf{k}_1\cdot\boldsymbol{\gamma}+m)[\delta(p_0-E_1-E_2)+\delta(p_0-E_1+E_2)]$$
$$-(-E_1\gamma_0-\mathbf{k}_1\cdot\boldsymbol{\gamma}+m)[\delta(p_0+E_1-E_2)+\delta(p_0+E_1+E_2)]\}\ . \quad (23)$$

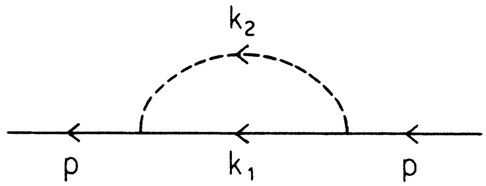

FIG. 4. A one-loop self-energy diagram of a fermion.

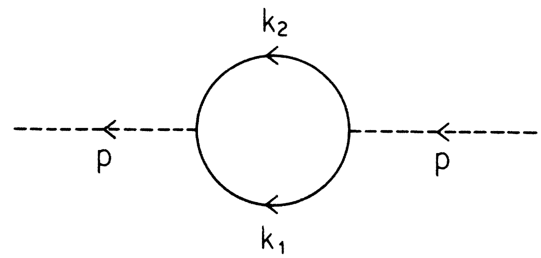

FIG. 5. A one-loop self-energy diagram of a scalar with a fermion loop.

Using

$$e^{\beta(E_1+E_2)}+1=[n_F(k_1)n_B(k_2)]^{-1}+n_F(k_1)^{-1}-n_B(k_2)^{-1} ,$$

etc., one finds

$$\begin{aligned}\mathrm{Im}\bar{\Sigma}=-\pi f^2\int\frac{d^3k_1}{(2\pi)^3}\frac{1}{4E_1E_2}\{&(E_1\gamma_0-\mathbf{k}_1\cdot\boldsymbol{\gamma}+m)[1+n_B(k_2)-n_F(k_1)]\delta(p_0-E_1-E_2)\\&+(E_1\gamma_0-\mathbf{k}_1\cdot\boldsymbol{\gamma}+m)[n_F(k_1)+n_B(k_2)]\delta(p_0-E_1+E_2)\\&-(-E_1\gamma_0-\mathbf{k}_1\cdot\boldsymbol{\gamma}+m)[n_F(k_1)+n_B(k_2)]\delta(p_0+E_1-E_2)\\&-(-E_1\gamma_0-\mathbf{k}_1\cdot\boldsymbol{\gamma}+m)[1+n_B(k_2)-n_F(k_1)]\delta(p_0+E_1+E_2)\} .\end{aligned} \tag{24}$$

This result corrects a minor error of Ref. 8. The final example we present is a fermion loop.

(5) Scalar self-energy diagram with a fermion loop (Fig. 5).

$$\begin{aligned}\mathrm{Im}\bar{\Sigma}&=-\frac{f^2}{2}\frac{e^{\beta p_0}-1}{e^{\beta p_0/2}}\mathrm{Tr}\int\prod_{i=1}^{2}\frac{d^4k_i}{(2\pi)^4}\prod_{i=1}^{2}[\epsilon(k_{i0})(\not{k}_i+m_i)e^{\beta|k_{i0}|/2}n_F(k_i)2\pi\delta(k_i^2-m_i^2)](2\pi)^4\delta^{(4)}(p-k_1-k_2)\\&=-2\pi f^2[p^2-(m_1+m_2)^2]\int\frac{d^3k_1}{(2\pi)^3}\frac{1}{4E_1E_2}\{[\delta(p_0-E_1-E_2)-\delta(p_0+E_1+E_2)][1-n_F(k_1)-n_f(k_2)]\\&\qquad-[\delta(p_0-E_1+E_2)-\delta(p_0+E_1-E_2)][n_F(k_1)-n_F(k_2)]\} .\end{aligned} \tag{25}$$

In summary, we have investigated the imaginary part of Feynman diagrams within the scheme of TFD. We have shown that the real-time formalism allows one to perform a direct calculation of the imaginary part, in a very concise manner, thanks to relations (14b) and (20b).

We wish to thank A. Hosoya, M. Bando, and I. Ojima for discussions. One of us (Y.F.) thanks the Yukawa Foundation for financial support.

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[1]See, for instance, B. L. Hu, Phys. Lett. **108B**, 19 (1982); A. Hosoya and M. Sakagami, Phys. Rev. D **29**, 2228 (1984); A. Hosoya, M. Sakagami, and M. Takao, Ann. Phys. (N.Y.) **154**, 229 (1984); M. Morikawa and M. Sasaki, Prog. Theor. Phys. **72**, 782 (1984); M. Sakagami and A. Hosoya, Phys. Lett. **150B**, 342 (1985).

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[9]Matsumoto *et al.*, in Ref. 5.

[10]In fact, four. One may obtain the imaginary part in TFD in the same manner as in ITF, namely, by calculating only the real part and then replacing p_0 with $p_0+i\epsilon$.

[11]Hosoya *et al.*, in Ref. 1.