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Possible closure conditions in a Kaluza-Klein compactified $M^4\times S^1\times K^{N-1}$ model

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Closure conditions concomitant with dynamical symmetry breaking are proposed for the mass-squared operator m^2 in an $M^4\times S^1\times K^{N-1}$ model for spontaneously compactified space-time. From the functional form of m^2 implied by the closure conditions, one obtains a remarkably accurate mass spectrum for all established leptons and quarks. The next (generation $n=4$) charged lepton δ is predicted to have a mass of 30.23 GeV, the $(n=3)$ top quark t has a mass of 82.89 GeV, and all neutrinos have a theoretical mass of zero.

Considerable interest has been attached recently to generalized Kaluza-Klein[1] models which offer the possibility of unifying gravity with the other fundamental forces while providing extra small-scale spatial dimensions that carry the symmetries of particle physics.[2] In the context of spontaneous compactification of $(4+N)$-dimensional space-time, the extra N dimensions span a compact manifold of characteristic volume $l_P{}^N$ where $l_P\sim\sqrt{G}\sim1.6\times10^{-33}$ cm is the Planck length. The gauge symmetries of strong (color) and electroweak interactions are realized as an $SU(3)_c\otimes SU(2)\otimes U(1)$ Killing group of motions on the compact N-dimensional manifold, and this requires[3] $N\geqq7$. Leptons and quarks receive their masses (which are very small in magnitude compared to the reciprocal of the Planck length) by a dynamical mechanism that breaks the electroweak $SU(2)\otimes U(1)$, perhaps via a Higgs field with a nonzero vacuum expectation value.

If one seeks zero-mass modes for the Dirac operator on the N-dimensional compact space, then a non-Riemannian space (perhaps a manifold with torsion)[4] is required to circumvent Lichnerowicz's theorem[5] (which rigorously precludes zero-mass fermions in the case of a compact homogeneous space with Riemannian geometry). A rather complicated topology for the internal space is also required to accommodate each lepton and quark as a distinct individual particle with the correct fermion quantum numbers. On the other hand, if one of the N compact dimensions, say x_5, is assumed to be canonically conjugate to the *generation quantum number n* or some integral multiple thereof, the task of accommodating all leptons and quarks is greatly simplified, for the $n\geqq2$ fermions can be viewed as x_5 "angular momentum" excitations of the basic e, d, u, and ν_e. This picture leads one to consider a ground-state manifold $M^4\times S^1\times K^{N-1}$, i.e., four-dimensional Minkowski space times a circle of small radius associated with x_5 times an $(N-1)$-dimensional compact space. The latter K^{N-1} sustains the $SU(3)_c$ and broken $SU(2)\otimes U(1)$ symmetries while providing the quantum numbers Q, L, B, and color for all generations of fundamental fermions.

Recently Dolan and Duff[6] have considered the spontaneous compactifications of five-dimensional space-time and shown that all symmetries are spontaneously broken save for Poincaré $\otimes$ U(1), signifying a ground state that is metrically $M^4\times S^1$. The symmetries are described by an infinite-dimensional Kac-Moody hierarchical extension of the Poincaré algebra which incorporates the angular momentum canonical conjugate to x_5 in a manner consistent with the Coleman-Mandula[7] theorem. Spontaneous breaking of the electroweak symmetry $SU(2)\otimes U(1)$ in the K^{N-1} compact spatial manifold is expected to produce closure conditions on the fermion mass-squared operator at the base of the hierarchy. In the absence of detailed dynamical considerations regarding the symmetry-breaking mechanism, is it possible to formulate physical closure conditions on m^2? The correctness of the latter closure conditions would be manifest in accurate mass-squared eigenvalues for all leptons and quarks.

I consider this problem in the present Brief Report. The ground state is assumed to be $M^4\times S^1\times K^{N-1}$, and the wave function for a fermion is expressible correspondingly as a direct product,

$$\Psi=\psi(x_1,x_2,x_3,x_4)\phi_{\hat N}(x_5/l)\xi(x_6,\ldots,x_{N+4}) \; . \tag{1}$$

In (1), ψ is a Dirac spinor depending on the M^4 coordinates; $\phi_{\hat N}$ depends on the S^1 coordinate and is associated with the quantum number n for successive generations of leptons and quarks; ξ depends on the compact K^{N-1} coordinates and is an eigenfunction of simultaneously commuting generators for Q, L, B, and color. Let x_5 have the range $-\pi l\leqq x_5\leqq\pi l$ with topological identification and C^∞ smoothness at $x_5=\pm\pi l$. I make no assumption regarding the magnitude of the constant length l (which may be of the order $l_P\sim10^{-33}$ cm) but eliminate it by employing $\theta\equiv x_5/l$ as a dimensionless coordinate for S^1. Thus any physical field variable, and in particular $\phi_{\hat N}(\theta)$ in (1), is a periodic function of θ with period 2π. If a set of leptons or quarks is characterized by $\phi_{\hat N}(\theta)$'s which all have period $2\pi/\hat N$,

$$\phi_{\hat N}(\theta+2\pi\hat N^{-1})\equiv\phi_{\hat N}(\theta) \; , \tag{2}$$

where $\hat N$ is a positive integer, then the set of leptons or quarks is said to have *subperiod index* $\hat N$.

Concomitant with dynamical symmetry breaking, the components of the fermion wave function (1) satisfy

$$(\partial_\mu\partial^\mu-m^2)\Psi=0 \; , \tag{3}$$

where the mass-squared operator m^2 acts on $S^1\times K^{N-1}$. At the base of the Kac-Moody algebra,[6] the mass-squared operator m^2 must satisfy *closure conditions*, i.e., specified Lie algebraic commutation relations with θ and associated operator variables. As closure conditions on m^2, I propose that the mass-squared operator depends exclusively on Q and

TABLE I. Solutions to Eqs. (10), (11), and (12) for $\phi_{\hat{N}}(\theta)$ with $n \leq 4$ and $\hat{N} \leq 7$.

$\hat{N}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
1	...	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$	$\sin 4\theta$
2	1	...	$\cos 4\theta$	$\cos 6\theta$	$\cos 8\theta$
3	...	$\sin 3\theta$	$\sin 6\theta$	$\sin 9\theta$	$\sin 12\theta$
4	1	...	$\cos 8\theta$	$\cos 12\theta$	$\cos 16\theta$
5	...	$\sin 5\theta$	$\sin 10\theta$	$\sin 15\theta$	$\sin 20\theta$
6	1	...	$\cos 12\theta$	$\cos 18\theta$	$\cos 24\theta$
7	...	$\sin 7\theta$	$\sin 14\theta$	$\sin 21\theta$	$\sin 28\theta$

$P_\theta = -i\partial/\partial\theta$, the angular momentum canonically conjugate to θ

$$m^2 = m^2(Q, P_\theta) , \quad (4)$$

while satisfying

$$[P, m^2] = 0 , \quad (5)$$

$$i[QP_Q, m^2] = 2m^2 , \quad (6)$$

$$[\theta, [m^2, \theta]] = 2m^2 . \quad (7)$$

Here P is the θ-parity operator, $P^{-1}\theta P = -\theta$, $P^{-1}P_\theta P = -P_\theta$, and the operator P_Q in (6) is canonically conjugate to the charge operator Q, $[Q, P_Q] = i$. Equation (5) states that m^2 is invariant under $\theta \to -\theta$, Eq. (6) states that m^2 is proportional to Q^2, and Eq. (7) finally implies the functional form [i.e., the uniquely indicated "general integral" to the conditions (5)–(7)]

$$m^2 = m_1{}^2 Q^2 \cosh(\sqrt{2} P_\theta) , \quad (8)$$

in which m_1 is a disposable positive constant. That the mass-squared operator (8) is positive definite follows from the self-adjointness of Q and P_θ and the positive character of the hyperbolic cosine power series,

$$\cosh(\sqrt{2} P_\theta) \equiv \sum_{k=0}^{\infty} \frac{(2)^k}{(2k)!} P_\theta{}^{2k} . \quad (9)$$

It is assumed that $\phi_{\hat{N}}(\theta)$ is an eigenfunction of $P_\theta{}^2$,

$$P_\theta{}^2 \phi_{\hat{N}} = -\frac{\partial^2 \phi_{\hat{N}}}{\partial\theta^2} = n^2 \hat{N}^2 \phi_{\hat{N}} , \quad (10)$$

where the generation number $n = 0, 1, 2, \ldots$ for a fixed positive integer value of the subperiod index $\hat{N}$. Moreover, solutions to (10) are restricted by requiring definite θ parity equal to $(-1)^{\hat{N}}$,

$$P\phi_{\hat{N}}(\theta) \equiv \phi_{\hat{N}}(-\theta) = (-1)^{\hat{N}} \phi_{\hat{N}}(\theta) , \quad (11)$$

and precluding the $\hat{N}$-even (positive θ-parity) solutions to (10) for $n = 1$ by the subsidiary condition

$$\int_0^{2\pi} (\cos\hat{N}\theta) \phi_{\hat{N}}(\theta) \, d\theta = 0 . \quad (12)$$

Then, to within arbitrary multiplicative constants and for $\hat{N} = 1, \ldots, 7$, admissible $\phi_{\hat{N}}(\theta)$ are $\sin(n\hat{N}\theta)$ or $\cos(n\hat{N}\theta)$ subject to (11) and (12), as shown in Table I. Let the subperiod index values for leptons and quarks be prescribed by the formula[8]

$$\hat{N} = 4 + 3|Q + L| \quad (13)$$

or, equivalently,

$$\hat{N} = 7 - 3|Q - B| , \quad (14)$$

as shown in Table II. Then the successive generations of

TABLE II. Quantum numbers and $\phi_{\hat{N}}(\theta)$ eigenfunctions for leptons and quarks. The eigenfunctions all have the definite θ parity required by (11) with the subperiod index defined by (13) or (14). Notice that $\phi_{\hat{N}} = 1$ corresponds to both e and u and $\phi_{\hat{N}} = \cos 12\theta$ to both τ and c, underscoring the fact that the $\phi_{\hat{N}}(\theta)$ are concomitant with but cannot supplant the fermion quantum numbers carried by ξ in (1).

Q Charge number	L Lepton number	B Baryon number	$\hat{N}$ Subperiod index	Lepton-quark eigenfunctions $\phi_{\hat{N}}(\theta)$ for $n \leq 4$			
-1	1	0	4	e 1	μ $\cos 8\theta$	τ $\cos 12\theta$	δ $\cos 16\theta$
$-\frac{1}{3}$	0	$\frac{1}{3}$	5	d $\sin 5\theta$	s $\sin 10\theta$	b $\sin 15\theta$	h $\sin 20\theta$
$+\frac{2}{3}$	0	$\frac{1}{3}$	6	u 1	c $\cos 12\theta$	t $\cos 18\theta$	g $\cos 24\theta$
0	1	0	7	ν_e $\sin 7\theta$	ν_μ $\sin 14\theta$	ν_τ $\sin 21\theta$	ν_δ $\sin 28\theta$

TABLE III. Masses (units MeV) of leptons and quarks according to the eigenvalue formula $m=m_1|Q|[\cosh(\sqrt{2}n\hat{N})]^{1/2}$ with $m_1=0.521\,72$ MeV.

$\hat{N}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
4	e 0.52	...	μ 105.60	τ 1786.6	δ 30 228
5	...	d 4.22	s 144.79	b 4968.1	h 170 472
6	u 0.35	...	c 1191.1	t 82 889	g 5 768 400
7	...	ν_e 0	ν_μ 0	ν_τ 0	ν_δ 0

leptons and quarks are given by the sets

$$\begin{bmatrix} e & d \\ u & \nu_e \end{bmatrix}, \begin{bmatrix} \mu & s \\ c & \nu_\mu \end{bmatrix}, \begin{bmatrix} \tau & b \\ t & \nu_\tau \end{bmatrix}, \begin{bmatrix} \delta & h \\ g & \nu_\delta \end{bmatrix}, \qquad (15)$$

$$n=0,1 \qquad n=2 \qquad n=3 \qquad n=4$$

in which the $n=0,1$ states combine together in the first generation grouping.

From (10) it follows that the $\phi_{\hat{N}}(\theta)$ are eigenfunctions of the mass-squared operator (8), which has the associated eigenvalues

$$m^2=m_1{}^2Q^2\cosh(\sqrt{2}n\hat{N}) \ . \qquad (16)$$

Table III displays the theoretical lepton and quark masses

$$m=m_1|Q|[\cosh(\sqrt{2}n\hat{N})]^{1/2}$$

given by (16) with the constant parameter set as $m_1\equiv 0.521\,72$ MeV.

As shown in Table III, all of the theoretical masses given by (16) are uniformly consistent with experiment and may indeed be accurate to within a small fraction of a MeV for all leptons and quarks. Observe that the theoretical mass difference between the d and u quarks is given as 3.87 MeV, and the s, c, b masses are all in close accord with well-known production-threshold and quark-model estimates. The most immediate predictions obtained from (16) and displayed in Table III are the $Q=-1$ lepton δ at 30.23 GeV, the top quark t at 82.89 GeV, and the theoretical mass-zero values for all neutrinos.[9]

The fact that all experimentally established lepton and quark masses are given to high accuracy by (16) with a suitable value for the single constant parameter m_1 provides strong experimental support for the closure conditions (4)–(7). In addition to working out the many details of the model described here, it remains to show how the quantum numbers in Table II emerge from K^{N-1} and require $\hat{N}$ to be given by (13) and (14). Moreover, the conditions (11) and (12) for admissible $\phi_{\hat{N}}(\theta)$ clearly interrelate S^1 and K^{N-1}, and this may signify associated topological properties for the complete N-dimensional compact space. Finally, the "robustness" of the closure conditions (4)–(7) must be demonstrated in the context of an ultraviolet-divergent $(4+N)$-dimensional theory of gravity coupled to fermions.

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[3]Since one compact spatial dimension is associated with the generation quantum number in the present theory, we require $N\gtrsim 8$.

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[8]It is clear that there exist compact K^{N-1}'s with torsion (e.g., Ref. 4) and suitably prescribed topology for which the isometry generators Q, L, B satisfy (13) and (14) in a manner consistent with the S^1 conditions (11) and (12).

[9]Scale-transformation considerations applied independently to field-theory models by K. Tennakone and S. Pakvasa [Phys. Rev. Lett. **27**, 757 (1971); Phys. Rev. D **6**, 2494 (1972)] and S. Blaha [Phys. Lett. **84B**, 116 (1979)] suggest a geometric mass spectrum for the charged leptons, and work by the latter author predicts the δ to have a mass of ~ 30.1 GeV; however, the other quark-lepton mass values shown in Table III are believed to be more uniquely associated with the present theory. Since the total cross section for $e^++e^-\rightarrow\delta^++\delta^-$ is about 1.2×10^{-35} cm^2 at the optimum energy of 73 GeV, the δ could be discovered in the present decade.