Time variation of coupling constants in Kaluza-Klein cosmologies reexamined

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We give a revised analysis of the time behavior of the internal radius in a six-dimensional Einstein-Yang-Mills-Higgs model with SO(3) internal symmetry. The two possible solutions are shown to be related if temperature effects are included for the particular case of a negative six-dimensional cosmological constant.

We have recently shown[1] that the correct solutions for the behavior of the internal radius in the six-dimensional Einstein-Yang-Mills-Higgs model of Ref. 2 should be

$$R_2(t)^2=\frac{\Delta-2B\alpha(t)+\alpha(t)^2}{4c\alpha(t)},\quad c>0\ , \tag{1a}$$

$$R_2(t)^2=\frac{-1}{2|c|}[|B|+\sqrt{\Delta}\sin(2\sqrt{-c}\,t+\beta)],\quad c<0\ , \tag{1b}$$

where the various constants in the above are defined by

$$A=\pi G(12e^2+\lambda)/2e^4\ , \tag{2a}$$

$$B=\frac{-\lambda}{2e^2}-k_2-\frac{\lambda\pi G}{e^2}\phi_0{}^2\ , \tag{2b}$$

$$c=2\pi G\left[\frac{\lambda(\phi_0{}^2)^2}{4}+\Lambda\right]+\frac{\lambda}{2}\phi_0{}^2\ , \tag{2c}$$

$$\Delta=B^2-4Ac\ , \tag{2d}$$

$$\alpha(t)=f(R_{20})\exp(\pm 2\sqrt{c}\,t)\ , \tag{2e}$$

$$f(R_{20})=2[c(A+BR_{20}{}^2+cR_{20}{}^4)]^{1/2}+2R_{20}{}^2+B\ , \tag{2f}$$

$$\beta=\sin^{-1}\left[\frac{2cR_{20}{}^2+B}{\sqrt{\Delta}}\right]\ . \tag{2g}$$

As the corrected solution differs substantially from the one originally published, we felt it advisable to expand our analysis. In fact, although the previous solution has some similarity to Eq. (1a), solution (1b) was missing.

In general, solution (1a) will diverge in the limit $t\to\infty$ for both roots in $\alpha(t)$. Nevertheless, for one particular case, we obtain a solution that behaves qualitatively as Eq. (23) of Ref. 2. Namely, if we take the negative root for $\alpha(t)$ and $\Delta=0$, the solution is

$$R_2(t)^2=\frac{|B|}{2c}+\frac{f(R_{20})}{4c}\exp(-2\sqrt{c}\,t)\ . \tag{3}$$

As $t\to\infty$ we again obtain $R_{2\infty}=\text{const}$ as in Ref. 2. We can check that

$$\lim_{t\to\infty}\dot{R}_2=\lim_{t\to\infty}\ddot{R}_2=0$$

in this case. Thus, we may dispense with the extra matter contribution used originally. The $t\to\infty$ limit together with the condition $\Delta=0$ gives

$$A=cR_{2\infty}{}^4=-\frac{BR_{2\infty}{}^2}{2}\ . \tag{4}$$

If we repeat the asymptotic analysis for the field equations we find that

$$R_{2\infty}{}^2=\frac{24\pi G_4}{2e_4{}^2k_2-\lambda_4}\ , \tag{5a}$$

$$\Lambda_4=(2e^2k_2+\lambda)(\lambda-e^2k_2)/576\pi^2G^2e^2\ . \tag{5b}$$

We see that fine-tuning of Λ_4 to zero is equivalent to setting $\lambda=e^2$ ($k_2=1$). For this case the three-dimensional "physical" space would be flat and expanding linearly in time. If we do not set $\Lambda_4=0$ (which seems more natural since from phenomenological results $e_4{}^2\geq 10^2\lambda_4$) the three-space would grow exponentially in time as in the four-dimensional inflationary scenarios.

The asymptotic value $R_{2\infty}$ works as a point attractor in phase space; whatever the initial value R_{20}, the system will always evolve to the attracting solution. The reader can verify that the condition $\Delta=0$ is not a strong restriction in the system. Indeed, as long as $R_{2\infty}>0$ it can be always verified. For $e_4{}^2=0.1$ and $\lambda_4=0.01$, $R_{2\infty}\sim 20l_P$, the time for the asymptotic behavior to set in is now $\sim O(10^2t_P)$, where l_P and t_P are the Planck length and time, respectively. A mechanism to promote a transition to a Friedmann era remains to be found.

Solution (1b) is similar to results found in Kaluza-Klein inflationary scenarios.[3] As $R_2(t)$ must be always positive and $\sqrt{\Delta}>B$, this solution will collapse for a time t_c when $R_2(t_c)=0$, with

$$t_c=\frac{1}{2\sqrt{-c}}\left[\sin^{-1}\left[\frac{|B|}{\sqrt{\Delta}}\right]+\beta\right]\ . \tag{6}$$

If we set the initial time $t_0=0$, we find that $\beta=3\pi/2$.

The above discussion for both solutions can be nicely pictured if we interpret the equation for the internal radius as an equation for a particle of unit mass moving in the potential

$$V(R_2)=-\frac{A}{R_2{}^2}-cR_2{}^2 \tag{7}$$

with total constant energy E given by the constant B defined in Eq. (2b). For solution (1a) ($c>0$) the potential has one maximum at $R_{2\max}{}^2=(A/c)^{1/2}$. We can remove the instability if we set $\Delta=0$. In this case, $R_{2\max}{}^2=R_{2\infty}{}^2$ and the total energy is equal to the potential energy, thus giving a zero net kinetic energy. It is then easy to show that this solution is stable against small perturbations around $R_{2\infty}$.

For solution (1b) ($c<0$), the only extremum point is an inflection point at $R_{2\inf}{}^2=(3A/|c|)^{1/2}$. The total energy is determined by the negative constant B which fixes the maximum value for R_2 as where the potential V is equal to $E=B$. The only possible motion for R_2 is to shrink towards the singularity with increasing velocity, characterizing the collapse. Following the authors of Ref. 3, we may argue that quantum effects will play an important role at this epoch and will stabilize the internal radius at some constant value. From dimensional analysis we can see that $t_c \sim 10 t_P$ or thereabouts for reasonable values of the couplings. The freedom in Λ does not allow us to say anything with confidence at this level.

Finally, if we include a temperature correction to the mass of the Higgs field (i.e., by writing $\mu^2 = a^2 T^2 + \mu_0{}^2$), it is possible to show that the two solutions are only independent if $\Lambda \geqslant 0$. For $\Lambda < 0$, the constant c [Eq. (2c)] will change sign for a certain T_c where $c=0$, while A remains constant and $\beta \rightarrow -\infty$ as $T \rightarrow \infty$. Thus, for sufficiently high temperatures the solution with $c>0$ will dominate, with the behavior being gradually changed until for $T < T_c$ the collapsing solution takes over.

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