New ghost-free infrared-soft gauges

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We discuss a new class of infrared-soft, Gribov copy-free gauges in which ghosts decouple. The gauges are natural for gauge theory, allowing a geometric interpretation of the Faddeev-Popov determinant in terms of the gauge-fixing flow on the space of gauge transformations. When a Nicolai-Langevin map is available, the new gauges are seen to be equivalent to Zwanziger's stochastic gauge fixing.

I. INTRODUCTION

In this paper we point out a class of gauges, which we call "flow gauges,"

$$A_0^a = Z^a[\mathbf{A}] , \tag{1.1}$$

for which, with mild technical restrictions on Z^a, ghosts decouple, off-shell infrared singularities are absent, and there is no Grivov ambiguity.[1,2] In Eq. (1.1), the color index a runs from 1 to N^2-1 for SU(N), and we have distinguished a direction called time, which may be chosen arbitrarily. Z^a is a functional of the spatial components A_i^a ($i=1,2,\ldots,D-1$) orthogonal to A_0^a, and contains no time derivatives.

We will focus primarily on the simplest subclass of flow gauges

$$\alpha A_0^a = \nabla\cdot\mathbf{A}^a , \quad 0<\alpha<\infty , \tag{1.2}$$

which, remarkably, interpolates from the temporal gauge at $\alpha=\infty$ to the Coulomb gauge at $\alpha=0$. In this context then, the infrared singularities of axial gauges,[3] as well as the ghosts and Gribov copies of the Coulomb gauge, are seen as aspects of a singular limit. In particular, large α may be used as a defining prescription for the infrared singularities of axial-type gauges.

A conventional treatment of the ghost decoupling and infrared softness is given in Sec. II. The origin of the name flow gauges is explained in Sec. III, where we concentrate on the geometric aspects of the gauges. The gauge-fixing equations for these gauges are first-order differential equations, of the type familiar in nonlinear dynamics, which describe a flow on the space of gauge transformations. It is the first-order nature of the flow that accounts for the absence of Gribov copies, and, ultimately, the ghost decoupling. The flow gauges are natural for gauge theories, allowing a geometric interpretation of the Faddeev-Popov determinant:[4] It is the inverse square root of the volume form of the gauge-fixing flow. Finally, in Sec. IV we note that, when a Nicolai-Langevin map is available, as it is, for example, in the case of four-dimensional QCD (Refs. 5, 6, and 3), the flow gauges correspond to Zwanziger's stochastic gauge fixing.[7] Conclusions and directions are briefly discussed in Sec. V. There is also a technical appendix giving a careful treatment of a geometric identity discussed formally in Sec. III.

II. THE CONVENTIONAL ANALYSIS

For the practical reader, we begin with a conventional discussion of the ghost decoupling for the smeared version of the simple gauge choice Eq. (1.2) in Euclidean or Minkowski space. Using standard textbook methods,[8] the gauge-shell[9] (GS) Faddeev-Popov determinant is easily written in terms of the Grassmann ghost fields ψ^a and $\bar\psi^a$,

$$\Delta_{\rm FP}^{\rm GS}[A] = \det(\alpha D_0^{ab} - \partial_i D_i^{ab}) \tag{2.1a}$$

$$= \int \mathscr{D}\bar\psi\,\mathscr{D}\psi \exp\left[-\int (dx)\bar\psi^a[\delta^{ab}(\alpha\partial_t - \nabla^2) + g f^{abc}(\alpha A_0^c - \nabla\cdot\mathbf{A}^c - A_i^c\partial_i)]\psi^b\right] , \tag{2.1b}$$

where $D_\mu^{ab} \equiv \delta_{ab}\partial_\mu + g f^{abc} A_\mu^c$ is the covariant derivative. It should be kept in mind however that the textbook derivation ignores boundary conditions and Gribov copies, to which we shall return in Sec. III.

The ghost Feynman rules are given in Fig. 1. The crucial observation here is that the inverse ghost propagator is linear in ∂_t and hence E. As a result, for example, the ghost contribution to the spatial gluon vacuum polarization at external momentum $(\mathbf{p},\omega)$,

$$-g^2 f^{adc} f^{bcd} \int (d\mathbf{q}) q_i (q-p)_j \int (dE) \frac{1}{(iE\alpha+\mathbf{q}^2)} \frac{1}{[i(E-\omega)\alpha+(\mathbf{q}-\mathbf{p})^2]} , \tag{2.2}$$

vanishes on integration[10] over E, since all the poles are in the upper half-plane when $\alpha>0$.[11] The same effect is seen in ghost loops with n gluon insertions for all $n\geq 2$. The ghost loop with a single insertion vanishes as well because f^{abc} is antisymmetric, so that the ghosts completely decouple as advertised. In the singular case $\alpha=0$, the argument fails, since the ghost propagator reduces to $\delta^{ab}/\mathbf{q}^2$, and the usual Coulomb ghosts survive.

A similar situation is found in the more general case

$$\alpha A^a_0-\nabla\cdot\mathbf{A}^a-F^a[\mathbf{A}]=0\ , \tag{2.3}$$

for which

$$\Delta^{\rm GS}_{\rm FP}[A]=\int\mathscr{D}\bar\psi\,\mathscr{D}\psi\exp\left[-\int(dx)(dx')\bar\psi^a(x)[\delta^{ab}(\alpha\partial_t-\nabla_x{}^2)\delta^D(x-x')+N^{ab}(x,x')]\psi^b(x')\right]\ . \tag{2.4}$$

Here[12]

$$N^{ab}(x,x')=gf^{abc}(\alpha A^c_0-\partial_i A^c_i-A^c_i\partial^x_i)\delta^D(x-x')+D^{bd}_i(\mathbf{x}',t)\frac{\delta F^a(\mathbf{x},t)}{\delta A^d_i(\mathbf{x}',t)}\delta(t-t')\ , \tag{2.5}$$

and $x\equiv(\mathbf{x},t),x'\equiv(\mathbf{x}',t')$. As above, the first-order nature of the ghost propagator sets to zero all ghost loops with more than one N^{ab} insertion, while the contribution of the ghost loop with a single N^{ab} insertion is

$$\Delta^{\rm GS}_{\rm FP}[A]\simeq\exp\left[\int(dx)(dx')\int(dq)\frac{e^{i\mathbf{q}\cdot(\mathbf{x}'-\mathbf{x})}e^{iE(t'-t)}N^{aa}(x,x')}{i\alpha E+\mathbf{q}^2}\right] \tag{2.6a}$$

$$=\exp\left[\frac{1}{2\alpha}\int(dx)\lim_{\mathbf{y}\to\mathbf{x}}D^{ab}_i(\mathbf{y},t)\frac{\delta F^a(\mathbf{x},t)}{\delta A^b_i(\mathbf{y},t)}\right] \tag{2.6b}$$

$$=\exp\left[\frac{1}{2\alpha}\int(dx)\mathrm{Tr}\left[D\cdot\frac{\delta F}{\delta A}\right]\right]\ . \tag{2.6c}$$

It follows that ghosts decouple for any flow gauge satisfying

$$\mathrm{Tr}\left[D\cdot\frac{\delta Z}{\delta A}\right]=\mathrm{const}\ , \tag{2.7}$$

which includes the simple choice Eq. (1.2). More generally, we may express the Faddeev-Popov determinant Eq. (2.6c) as

$$\Delta^{\rm GS}_{\rm FP}=\exp\left[\tfrac{1}{2}\int(d\mathbf{q})\mathscr{F}(\mathbf{q};\mathbf{0},0)\right]\ , \tag{2.8}$$

where we have defined

$$\frac{1}{\alpha}D^{ab}_i(\mathbf{y},t)\frac{\delta F^a(\mathbf{x},t)}{\delta A^b_i(\mathbf{y},t)}=2^{D-1}\int(d\mathbf{q})(d\mathbf{p})\int(d\omega)e^{i\omega t}e^{i\mathbf{p}\cdot(\mathbf{x}+\mathbf{y})}e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}\mathscr{F}(\mathbf{q};\mathbf{p},\omega)\ . \tag{2.9}$$

Here $(\mathbf{p},\omega)=0$ is the external momentum of the insertion and $\mathbf{q}$ is the internal spatial momentum of the loop. If $F^a[\mathbf{A}]$ is a polynomial in $\mathbf{A}$ and its derivatives, it is straightforward to see that $\mathscr{F}(\mathbf{q};\mathbf{0},0)$ is a polynomial in $\mathbf{q}$, so that the ghost loop with a single insertion always vanishes under dimensional regularization. With this proviso, all flow gauges are ghost-free.

The gauge-field Feynman rules that follow from the Euclidean Lagrangian

$$\mathscr{L}_E=\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\tfrac{1}{2}\xi(\alpha A^a_0-\partial_i A^a_i)^2 \tag{2.10}$$

are given in Fig. 2. Here the gauge choice is the smeared version of the sharp gauge Eq. (1.2), which may be regained at $\xi=\infty$. We call attention to the absence, for finite α, of the E^{-2} infrared singularities of the temporal gauge at $\alpha=\infty$. Large α may therefore be used as a de-

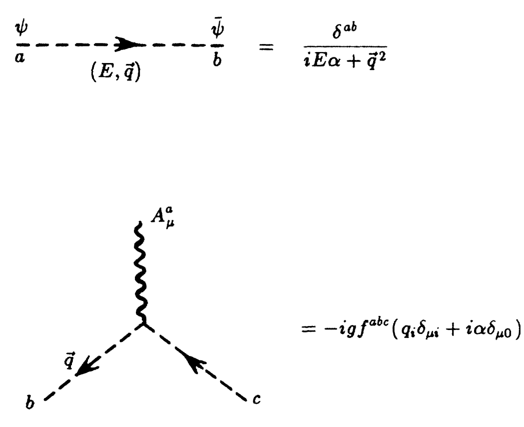

FIG. 1. Ghost Feynman rules.

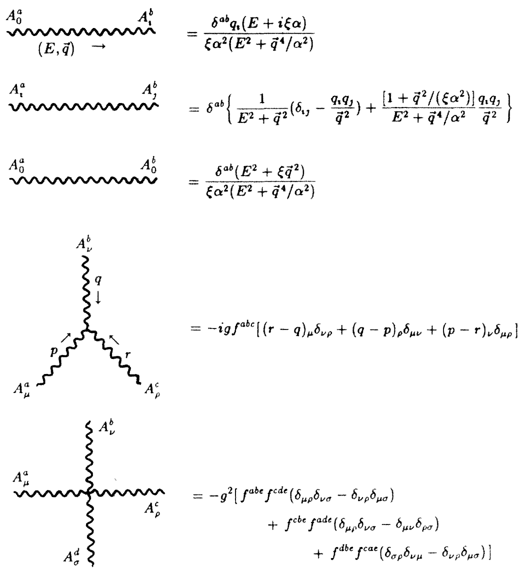

FIG. 2. Feynman rules in Euclidean space.

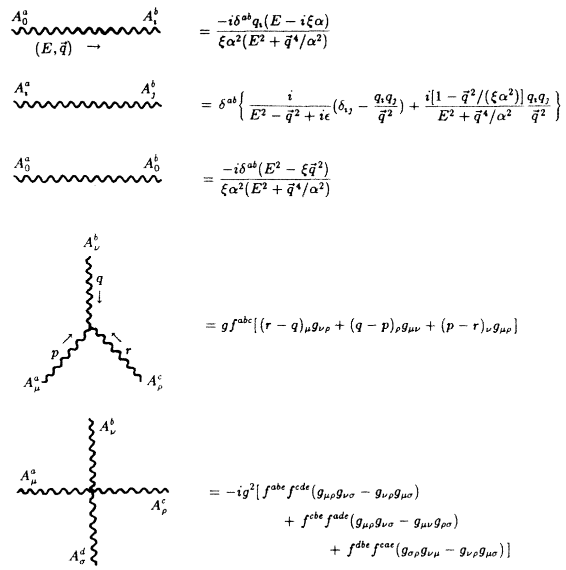

FIG. 3. Feynman rules in Minkowski space.

fining prescription for the singularities of axial-type gauges. The prescription, as easily seen in the Feynman rules, is not a principal value.

We have used these rules and dimensional regularization to compute the one-loop correction to the gluon mass in four dimensions, a nontrivial check because the parameter α has the dimension of inverse length. All integrations are infrared convergent, and the result is that, without ghosts, the gluon remains massless, as it should.

The individual diagrams in this computation are more than quadratically divergent for the simple gauge choice Eq. (2.10), although their sum is only quadratically divergent. In this connection, we mention the simple spatially nonlocal flow gauge

$$\beta A_0^a = \frac{1}{(-\nabla^2)^{1/2}} \nabla \cdot \mathbf{A}^a , \tag{2.11}$$

with dimensionless β. This gauge is also infrared soft, ghost-free (independent of dimensional regularization), and individual diagrams are no more than quadratically divergent. The Feynman rules for this gauge are obtainable from Fig. 2 by the substitution $\alpha \rightarrow \beta(\mathbf{q}^2)^{1/2}$. With dimensional regularization, the gluon remains massless in this gauge as well.

For completeness, we also give in Fig. 3 the gauge-field Feynman rules for the Minkowski Lagrangian

$$\mathscr{L}_M = -\tfrac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - \tfrac{1}{2}\xi(\alpha A_0^a - \partial_i A_i^a)^2 , \tag{2.12}$$

which again is the smeared version of the gauge choice Eq. (1.2). In addition to the infrared softness, we note the curious circumstance that these rules cannot be rotated to Euclidean space in the usual manner, due to complex singularities in the A_0 and longitudinal propagators. This phenomenon is easily traced to the Euclidean rotation property of A_0 ($A_0 \rightarrow iA_0$), which naively requires a compensatory rotation $\alpha \rightarrow -i\alpha$ to maintain the gauge. The topic deserves further study. Alternatively, the phenomenon can be avoided by choosing a spatial direction, instead of time, to define the flow gauge.

III. THE GEOMETRIC ANALYSIS

We turn now to a deeper treatment of the flow gauges, which will lead to a geometric interpretation of the Faddeev-Popov determinant. To reach the general flow gauge Eq. (1.1) from an arbitrary configuration A_μ^a, it is necessary to solve the gauge-fixing equations:

$$A_0^\Omega = Z[\mathbf{A}^\Omega] , \tag{3.1a}$$

$$A_\mu^\Omega \equiv \Omega \left[A_\mu - \frac{i}{g}\partial_\mu \right] \Omega^\dagger , \tag{3.1b}$$

for the gauge transformation $\Omega[A] \in SU(N)$. These first-order differential equations in time

$$\dot{\Omega}^\dagger + ig(A_0\Omega^\dagger - \Omega^\dagger Z[\mathbf{A}^\Omega]) = 0 \tag{3.2}$$

define a flow of the type familiar in nonlinear dynamics. In particular, $\Omega[A]$ is unique given the initial condition $\Omega(\mathbf{x},t_1) = \Omega_1(\mathbf{x})$. The flow gauges are therefore free of Gribov copies for boundary conditions which fix $\Omega(\mathbf{x},t_1)$ (Ref. 13). A situation in which such a boundary condition is natural is the case of a functional integral whose initial condition $A_\mu^a(\mathbf{x},t_1)$ is specified. Then an appropriate definition of the off-gauge-shell[9] Faddeev-Popov determinant is

$$\Delta_{FP}^{-1}[A;t_2,t_1] = \int_{t_1 \le t \le t_2} \mathscr{D}\Omega\, \delta[A_0^\Omega - Z[\mathbf{A}^\Omega]]\, \delta[\Omega(\cdot,t_1)\Omega_1^\dagger(\cdot) - 1] , \tag{3.3}$$

where $\mathscr{D}\Omega$ is Haar measure at each space-time point. Equation (3.3) is gauge invariant under gauge transformations which do not change the initial condition. The determinant contains no Gribov copies because the flow is uniquely determined. Specification of initial condition in this manner is equivalent to a "retarded" boundary condition[14–16] for the determinant, whereas specification of a final condition gives an analogous "advanced" prescription which we will not discuss explicitly.

To bring out the aspects of flow on a manifold, it is convenient to introduce general coordinates $\Lambda^n(\mathbf{x},t)$ on each group manifold $\Omega(\mathbf{x},t)$. In a Cartan-Mauer basis the vielbein and metric[17] are

$$e_{ma}=-2i\,\mathrm{Tr}(\Omega^\dagger T_a \partial_m \Omega)\ , \tag{3.4a}$$

$$e_a^m \partial_m \Omega = iT_a\Omega\ ,\quad e_a^m\partial_m\Omega^\dagger=-i\Omega^\dagger T_a\ , \tag{3.4b}$$

$$g_{mn}=2\,\mathrm{Tr}(\partial_m\Omega^\dagger\partial_n\Omega)\ , \tag{3.4c}$$

$$(T_a)_{rs}(T_a)_{pq}=\frac{1}{2}\left[\delta_{rq}\delta_{sp}-\frac{1}{N}\delta_{rs}\delta_{pq}\right]\ , \tag{3.4d}$$

where $\partial_m \equiv \partial/\partial\Lambda^m$. With the definition $e\equiv\det(e_m^a)$, we may then recast the Faddeev-Popov determinant as the geometric object

$$\Delta_{\mathrm{FP}}{}^{-1}[A;t_2,t_1]=\int_{t_1\le t\le t_2}[\mathscr{D}\Lambda\, e]\delta(e_m^a\dot\Lambda^m-V^a)e^{-1}(t_1)\delta[\Lambda^m(\cdot,t_1)-\Lambda_1^m(\cdot)] \tag{3.5}$$

associated with the generic flow

$$\dot\Lambda^m=V^m(\Lambda)\ ,\quad V^m(\Lambda)=e_a^m(\Lambda)V^a(\Lambda)\ , \tag{3.6}$$

on an arbitrary manifold. In our case the flow is on the space of gauge transformations and, using Eq. (3.2),

$$V^a=2g\,\mathrm{Tr}[T_a(\Omega A_0\Omega^\dagger-Z[\mathbf{A}^\Omega])]\ . \tag{3.7}$$

It is instructive, however, to consider Eq. (3.5), at least temporarily, in the general case.

We may formally evaluate Eq. (3.5) for a generic flow on an arbitrary manifold as follows. The δ functional is exhibited in terms of the unique flow Λ_0^m by expanding $\Lambda^m=\Lambda_0^m+\xi^m$, so that

$$\delta(e_m^a\dot\Lambda^m-V^a)\delta[\Lambda^m(\cdot,t_1)-\Lambda_1^m(\cdot)]=\delta(\Lambda^m-\Lambda_0^m)\det{}_R^{-1}\left[\frac{\delta}{\delta\xi^n(x')}\{e_m^a(\Lambda)\dot\Lambda^m(x)-V^a[\Lambda(x)]\}\right]\Bigg|_{\xi=0}\ , \tag{3.8}$$

where we have noted with the subscript R that the determinant is to be evaluated with retarded boundary conditions.[14] Then

$$\det{}_R\left[\frac{\delta}{\delta\xi^n(x')}\{e_m^a(\Lambda)\dot\Lambda^m(x)-V^a[\Lambda(x)]\}\right]\Bigg|_{\xi=0}$$

$$=(\det{}_R\partial_t)\times\det{}_R\left[e_n^a\delta^D(x-x')-\theta(t-t')\left[(V^m\partial_m e_n^a)(x')\delta^{D-1}(\mathbf{x}-\mathbf{x}')+e_m^a(\mathbf{x},t')\frac{\delta V^m(\mathbf{x},t')}{\delta\Lambda^n(\mathbf{x}',t')}\right]\right]\Bigg|_0 \tag{3.9a}$$

$$\simeq\left[\prod_{t_1<t\le t_2}e(t)\right]\times\det{}_R\left[\delta(t-t')\delta^{D-1}(\mathbf{x}-\mathbf{x}')\delta_n^l-\theta(t-t')\left[(e_a^l V^m\partial_m e_n^a)(x')\delta^{D-1}(\mathbf{x}-\mathbf{x}')+\frac{\delta V^l(\mathbf{x},t')}{\delta\Lambda^n(\mathbf{x}',t')}\right]\right]\Bigg|_0\ , \tag{3.9b}$$

where we have used $\partial_t{}^{-1}\delta(t-t')=\theta(t-t')$ according to the retarded condition, and the subscript zero means to evaluate at the true flow Λ_0. As is well known,[14] only the first power[18] of $\theta(t-t')$ contributes to the final retarded determinant in Eqs. (3.9). We therefore obtain, up to irrelevant constants,

$$\Delta_{\mathrm{FP}}[A;t_2,t_1]=\exp\left[-\tfrac{1}{2}\int_{t_1}^{t_2}(dx)\mathscr{D}\cdot V\Big|_0\right]\ , \tag{3.10}$$

where

$$\mathscr{D}\cdot V=\lim_{\mathbf{y}\to\mathbf{x}}\frac{\mathscr{D}V^m(\mathbf{x},t)}{\mathscr{D}\Lambda^m(\mathbf{y},t)}$$

$$=\lim_{\mathbf{y}\to\mathbf{x}}\delta_m^n\left[\frac{\delta V^m(\mathbf{x},t)}{\delta\Lambda^n(\mathbf{y},t)}+\Gamma_{nr}^m(\Lambda(\mathbf{x},t))V^r(\Lambda(\mathbf{x},t))\times\delta^{D-1}(\mathbf{x}-\mathbf{y})\right] \tag{3.11}$$

is the covariant divergence[19] of the flow, and Γ_{nr}^m is the

Christoffel connection constructed from the metric tensor Eq. (3.4c). Equation (3.10) is an evaluation of the off-gauge-shell Faddeev-Popov determinant in terms of the true flow Λ_0. In the Appendix, the quasilattice method of Ref. 14 is used to check the result Eq. (3.10) in the case of a finite-dimensional manifold.

It is also well known[20] that the volume form (covariant volume element) $\mathscr{D}\omega=\prod_{\mathbf{x}}(d\Lambda e)$ evolves in time according to

$$\left.\frac{\mathscr{D}\omega(t_2)}{\mathscr{D}\omega(t_1)}\right|_A=\exp\left[\int_{t_1}^{t_2}(dx)\mathscr{D}\cdot V\Big|_0\right] \tag{3.12}$$

under the flow Eq. (3.6), which may be driven by a source A. As a result, the flow gauges allow the purely geometric interpretation of the Faddeev-Popov determinant

$$\Delta_{\mathrm{FP}}[A;t_2,t_1]=\left[\left.\frac{\mathscr{D}\omega(t_2)}{\mathscr{D}\omega(t_1)}\right|_A\right]^{-1/2} \tag{3.13}$$

as stated in the Introduction.

We turn now to the special case of flow on the space of gauge transformations. On group manifolds, $\mathscr{D}\cdot e=0$, and the most convenient expression for the covariant divergence Eq. (3.11) of the flow (3.7) is

$$\left.\frac{\mathscr{D}V^m(\mathbf{x},t)}{\mathscr{D}\Lambda^m(\mathbf{y},t)}\right|_0=-ig\,\mathrm{Tr}[\hat{E}(\mathbf{y},t)Z[\mathbf{A}^{\Omega}(\mathbf{x},t)]]\,|_0\;, \tag{3.14}$$

where[21]

$$\begin{aligned}(\hat{E})_{rs}&=-2i(T_a)_{rs}e_a^m\frac{\delta}{\delta\Lambda^m}\\&=\Omega_{rd}\frac{\delta}{\delta\Omega_{sd}}-\frac{\delta_{rs}}{N}\Omega_{cd}\frac{\delta}{\delta\Omega_{cd}}\end{aligned} \tag{3.15}$$

are the (matrix) Killing vectors. It is not difficult to check the gauge invariance of Eq. (3.14).

Finally, we wish to evaluate the gauge-shell Faddeev-Popov determinant. When Eq. (1.1) is satisfied, we say that the flow

$$\Omega\dot{\Omega}^\dagger+ig(\Omega Z[\mathbf{A}]\Omega^\dagger-Z[\mathbf{A}^{\Omega}])=0 \tag{3.16}$$

is on gauge shell. The on-shell flow always has a fixed point at $\Omega(\mathbf{x},t)=1$. If we also choose the initial condition $\Omega(\mathbf{x},t_1)=1$, then this fixed point is the unique flow $\Omega_0=1$. It follows from Eq. (3.14) that

$$\left.\frac{\mathscr{D}V^m(\mathbf{x},t)}{\mathscr{D}\Lambda^m(\mathbf{y},t)}\right|_0^{\mathrm{GS}}=-D_i^{ab}(\mathbf{y},t)\frac{\delta Z^a(\mathbf{x},t)}{\delta A_i^b(\mathbf{y},t)}\;, \tag{3.17}$$

in agreement with the conventional result Eq. (2.6c).

It is amusing to note that, for the simple gauge choice Eq. (1.2), the covariant divergence is a negative constant for $\alpha>0$,

$$\left.\frac{\mathscr{D}V}{\mathscr{D}\Lambda}\right|_0^{\mathrm{GS}}=-\frac{(N^2-1)}{\alpha}\int(d\mathbf{k})\mathbf{k}^2\;, \tag{3.18}$$

so that the fixed point of the on-shell flow is stable, and the flow is a simple contraction mapping. We remind the reader of the fact, discussed in Sec. II, that dimensional regularization sets the covariant divergences to zero, which masks the geometry of the flow; this phenomenon deserves further study.

We should also stress the connection between the "retarded" methods used in this section and the conventional analysis of Sec. II. In Sec. II, the vanishing of ghost loops with more than one insertion followed because the ghost propagator had its poles only in the upper half-plane for $\alpha>0$. This property is precisely the retarded boundary condition discussed in this section. Indeed the momentum space ghost propagator of Fig. 1 is just the Fourier transform of the retarded propagator

$$\begin{aligned}&\Delta_R(\mathbf{x},t;\mathbf{x}',t')\\&\quad=-\frac{i}{\alpha}\theta(t-t')\exp\left[\frac{1}{\alpha}\nabla^2|t-t'|\right]\delta^{D-1}(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{3.19}$$

whose $\theta(t-t')$ factor was used to obtain the simple result Eq. (3.10). The retarded boundary condition can of course be used to write down the on-shell determinant directly,

$$\Delta_{\mathrm{FP}}^{\mathrm{GS}}[A;t_2,t_1]=\det{}_R\left[\delta^{ab}\partial_t\delta^D(x-x')+\delta(t-t')D_i^{ac}(\mathbf{x}',t')\frac{\delta Z^b(\mathbf{x},t')}{\delta A_i^c(\mathbf{x}',t')}\right] \tag{3.20a}$$

$$=\det{}_R[\delta^{ab}\partial_t\delta^D(x-x')]\det{}_R\left[\delta^{ab}\delta^D(x-x')+\theta(t-t')D_i^{ac}(\mathbf{x}',t')\frac{\delta Z^b(\mathbf{x},t')}{\delta A_i^c(\mathbf{x}',t')}\right] \tag{3.20b}$$

$$\simeq\exp\left[\tfrac{1}{2}\int(dx)\lim_{\mathbf{x}'\to\mathbf{x}}D_i^{ab}(\mathbf{x}',t)\frac{\delta Z^a(\mathbf{x},t)}{\delta A_i^b(\mathbf{x}',t)}\right]\;, \tag{3.20c}$$

since only the first power of $\theta(t-t')$ contributes.

IV. THE STOCHASTIC CONNECTION

In this section we discuss the connection of the flow gauges with Zwanziger's stochastic gauge fixing.[7] The connection can be made when there exists a Nicolai-Langevin map of the action gauge theory onto an equivalent stochastic process. In the case of four-dimensional QCD with $A_0=0$, such maps

$$\dot{A}_i^a=\mp B_i^a+\eta_i^a\;, \tag{4.1}$$

with η a Gaussian noise, have recently been discussed in Refs. 5, 6, and 3. The simplicity of the map is due to the fact that the retarded Jacobian of the transformation (from A to η) is a constant.

With a slight generalization of the methods of Ref. 6, it is easily seen that the gauge-invariant action theory can be expressed as the constant Jacobian maps

$$\dot{A}_i^a = \mp B_i^a + D_i^{ab} A_0^b + \eta_i^a \tag{4.2}$$

in which A_0^b acts as a random source: Gaussian η averaging and the original A_0 integration must still be performed. It is therefore not surprising that the smeared ξ-gauge Euclidean action theory Eq. (2.10) can also be expressed as the constant Jacobian maps

$$\dot{A}_i^a = \mp B_i^a + \frac{1}{\alpha} D_i^{ab} \left[\frac{1}{\sqrt{\xi}} \chi^b + \nabla \cdot \mathbf{A}^b \right] + \eta_i^a . \tag{4.3}$$

Here we have made the change of variable

$$\chi^b = \sqrt{\xi} (\alpha A_0^b - \nabla \cdot \mathbf{A}^b) , \tag{4.4}$$

and averaging is over both noise functions η and χ, with the Gaussian Boltzmann factor

$$\exp \left[-\tfrac{1}{2} \int (dx) [\eta_i^a(x) \eta_i^a(x) + \chi^a(x) \chi^a(x)] \right] . \tag{4.5}$$

As in Ref. 6, a final-state constraint on $\mathbf{A}$ is necessary, and the initial and final values of A_μ must be recorded, using Eq. (4.4), as boundary conditions on the χ integration.

The result Eq. (4.3) is a generalized Zwanziger gauge fixing of the original stochastic process (4.1), with χ^a providing an extra random element in the gauge-fixing term. Zwanziger's gauge fixing corresponds to the sharp case $\xi = \infty$, i.e., $\alpha A_0 = \nabla \cdot \mathbf{A}$, in which limit the second noise χ decouples,

$$\dot{A}_i^a = \mp B_i^a + D_i^{ab} \left[\frac{1}{\alpha} \nabla \cdot \mathbf{A}^b \right] + \eta_i^a . \tag{4.6}$$

For $\alpha = \infty$ ($A_0 = 0$), the original map Eq. (4.1) undergoes a random walk in gauge space, in analogy with the Parisi-Wu equation.[22] This is interpreted at the action level as the E^{-2} singularity of the temporal gauge.[3] For $0 < \alpha < \infty$, however, the Zwanziger gauge-fixing term of Eq. (4.6) stops the random walk, replacing it with a longitudinal damping. At the action level this corresponds to the infrared softness discussed in Sec. II.

The longitudinal damping by the Zwanziger term is highlighted by considering the time evolution of the volume form $\mathscr{D}A$ under the stochastic flow Eq. (4.6). The original map Eq. (4.1) describes a constant volume flow,[14] but now, using the methods above,

$$\begin{aligned} \frac{\mathscr{D}A(t)}{\mathscr{D}A(t_1)} &= \det \left[\frac{\delta A_i^a(\mathbf{x},t)}{\delta A_j^b(\mathbf{y},t_1)} \right] \\ &= \det[\delta^{ab}\delta_{ij}\delta(t-t_1)\delta^{D-1}(\mathbf{x}-\mathbf{y})] \\ &\quad \times \exp \left[-(t-t_1) W \frac{(N^2-1)}{\alpha} \int (d\mathbf{k}) \mathbf{k}^2 \right] , \end{aligned} \tag{4.7}$$

where W is the spatial volume of the system. The time dependence of the volume form comes entirely from the Zwanziger gauge fixing, identifying Eq. (4.6) as a simple longitudinal contraction mapping. This longitudinal contraction is essentially the same phenomenon observed for the gauge-fixing flow in Eq. (3.18).

For the general (sharp) flow gauge Eq. (1.1), we obtain the maps Eq. (4.2) with $A_0^a = Z^a[\mathbf{A}]$, which corresponds to Zwanziger's general prescription. There is however an unwanted factor

$$\Delta_{\rm FP}^{\rm GS}[A] \det{}_R \left[\frac{\delta A}{\delta \eta} \right] = \exp \left[\tfrac{1}{2} \int (dx) \lim_{\mathbf{x}' \to \mathbf{x}} [D_i^{ab}(\mathbf{x},t) + D_i^{ba}(\mathbf{x}',t)] \frac{\delta Z^b(\mathbf{x},t)}{\delta A_i^a(\mathbf{x}',t)} \right] \tag{4.8}$$

which must be included with the η averaging. This factor equals unity for the simple gauge choices Eq. (1.2) or Eq. (2.11), and is in general removable by dimensional regularization, as in Sec. II. With this proviso, the flow gauges are isomorphic to Zwanziger's prescription.

The argument above is easily extended to the original fifth-time stochastic quantization. Begin with the fifth-time gauge-invariant Euclidean Lagrangian

$$\mathscr{L}_E = \frac{1}{4} \left[F_{5\mu}^a + \frac{\delta S_{\rm YM}}{\delta A_\mu^a} \right]^2 - \frac{1}{2} \left[\frac{\delta^2 S_{\rm YM}}{\delta A_\mu^a \delta A_\mu^a} \right] , \tag{4.9}$$

where

$$F_{5\mu}^a = \dot{A}_\mu^a - \partial_\mu A_5^a - g f^{abc} A_5^b A_\mu^c \tag{4.10}$$

is the fifth-time field strength, $S_{\rm YM}$ is the four-dimensional Yang-Mills action, and the overdot denotes fifth-time derivative. The Nicolai-Langevin map of this action theory is

$$\dot{A}_\mu^a = -\frac{\delta S_{\rm YM}}{\delta A_\mu^a} + D_\mu^{ab} A_5^b + \eta_\mu^a , \tag{4.11}$$

with A_5^b acting as a random source, as in Eq. (4.2). The "temporal"-gauge choice $A_5^a = 0$ results in the Parisi-Wu equation.[22] The random walk in gauge space exhibited by the Parisi-Wu equation is understood at the action level as the infrared singularities of this temporal gauge. To obtain infrared softness at the action level, choose instead, for example, the sharp flow gauge $A_5^a = Z^a[A_\mu]$. The resulting map

$$\dot{A}_\mu^a = -\frac{\delta S_{\rm YM}}{\delta A_\mu^a} + D_\mu^{ab} Z^b[A_\mu] + \eta_\mu^a \tag{4.12}$$

is the original fifth-time Zwanziger gauge-fixed Langevin equation,[23] with longitudinal damping. As above, the equivalence for the simple flow-gauge choice $\alpha Z^a = \partial \cdot A^a$ proceeds independent of dimensional regularization.

V. CONCLUSIONS

With mild technical restrictions, the flow gauges $A_0=Z[\mathbf{A}]$ are ghost-free, infrared soft, free of Gribov copies, and geometrically natural for gauge theories. These properties are essentially independent of any particular action, following instead from the first-order nature of the gauge-fixing flow, so that retarded (or advanced) boundary conditions apply to Faddeev-Popov determinants. For those actions which possess Nicolai-Langevin maps, the flow gauges are equivalent to Zwanziger's stochastic gauge fixing.

We should also mention connections among flow gauges, stochastic quantization, and Becchi-Rouet-Stora-Tyutin (BRST) symmetry.[24] By keeping the Faddeev-Popov ghosts, even though they decouple, a more or less ordinary BRST symmetry is obtained. This is connected to the fact that, although the ghosts decouple from the gauge fields, the gauge fields do not in general decouple from the ghosts. There has been interest[25,16] in the (scalar) supersymmetry of actions that arise from maps. Our contribution to this subject is that, for Zwanziger gauge-fixed maps, there is an ordinary flow-gauge BRST symmetry present as well at the action level.

It will be interesting to see if analogues of the flow gauges can be found in theories of gravity, supersymmetry, and strings.

Note added in proof. As a special case, we have also explicitly studied light-cone flow gauges (e.g., $\alpha A^+=\nabla_\perp\cdot\mathbf{A}_\perp$), which, as expected, are free of ghosts, Gribov copies, and infrared problems. It is noteworthy that the softening prescription differs from popular $i\epsilon$ prescriptions [see, e.g., S. Mandelstam, Nucl. Phys. **B213**, 149 (1983)].

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APPENDIX: FLOW IDENTITY ON A MANIFOLD

In this appendix, we use the "quasilattice" method of Ref. 14 to check Eq. (3.10) of the text, in the case of a finite-dimensional manifold. Specifically, we shall establish

$$\Delta^{-1}(t_f,t_i)\equiv\int_{t_i\le t\le t_f}[\mathscr{D}x\, e]\delta(e^a_\mu\dot{x}^\mu-V^a)\times e^{-1}(t_i)\delta[x(t_i)-x_i]\simeq\exp\left[\tfrac{1}{2}\int_{t_i}^{t_f}dt\,D\cdot V\right], \quad \text{(A1)}$$

where $D\cdot V$ is the covariant divergence of the flow V on a finite-dimensional manifold, whose coordinates are x^μ.

The basic idea is, as in the text, to define a new variable

$$e^a_\mu\dot{x}^\mu-V^a\equiv\xi^a \quad \text{(A2)}$$

and to consider Eq. (A2) as a flow equation driven by ξ^a. With Ref. 14 we take $\xi^a(t)$ piecewise constant in each of the N time subintervals,

$$\xi^a(t)=\xi^a_{(n)}\ \text{ for } t_{n-1}\le t\le t_n\ , \quad \text{(A3)}$$

$$t_n-t_{n-1}=\epsilon\ ,\ \ t_N=t_f\ ,\ \ t_0=t_i\ , \quad \text{(A4)}$$

where $1\le n\le N$, so that $t_f-t_i=N\epsilon$. Then the integrated form of Eq. (A2)

$$\Delta x^\mu_{(n)}\equiv x^\mu_{(n)}-x^\mu_{(n-1)}=\int_{t_{n-1}}^{t_n}dt(V^\mu+e^\mu_a\xi^a)\ , \quad \text{(A5)}$$

together with the initial condition $x^\mu_{(0)}=x^\mu_i$, selects N points of evolution along the trajectory. Equation (A5) may be expanded as

$$\begin{aligned}\Delta x^\mu_{(n)}=\epsilon[V^\mu_{(n-1)}+e^\mu_{a(n-1)}\xi^a_{(n)}]&+\frac{\epsilon^2}{2}\{[V^\nu\partial_\nu V^\mu]_{(n-1)}+[e^\nu_a\partial_\nu V^\mu]_{(n-1)}\xi^a_{(n)}\\&+[V^\nu_{(n-1)}+e^\nu_{b(n-1)}\xi^b_{(n)}][\partial_\nu e^\mu_{a(n-1)}]\xi^a_{(n)}\}\\&+O(\epsilon^3)\ ,\end{aligned} \quad \text{(A6)}$$

while the inverse expansion

$$\begin{aligned}\xi^a_{(n)}&=e^a_{\mu(n-1)}\left[\frac{\Delta x^\mu_{(n)}}{\epsilon}-V^\mu_{(n-1)}-\frac{\epsilon}{2}[V^\nu\partial_\nu V^\mu]_{(n-1)}-\tfrac{1}{2}[\partial_\nu V^\mu]_{(n-1)}[\Delta x^\nu]_{(n)}+\cdots\right]\\&=e^a_{\mu(n-1)}\left[\frac{\Delta x^\mu_{(n)}}{\epsilon}-V^\mu_{(n-1)}+O(\epsilon)\right],\end{aligned} \quad \text{(A7)}$$

gives $\xi^a_{(n)}$ as a function of $x^\mu_{(m)}$.

To derive Eq. (A1), we begin with the trivial identity

$$1=\int\prod_{n=1}^{N}(d\xi_{(n)})\prod_a\delta(\xi^a_{(n)}) \quad \text{(A8a)}$$

$$=\int\prod_{n=1}^{N}(dx_{(n)})\prod_a\delta\left[e^a_{\mu(n-1)}\left[\frac{\Delta x^\mu_{(n)}}{\epsilon}-V^\mu_{(n-1)}+O(\epsilon)\right]\right]\det\left[\frac{\partial\xi}{\partial x}\right]_{\xi=0}. \quad \text{(A8b)}$$

The determinant of the square matrix $\partial\xi_{(m)}/\partial x_{(n)}$ is easily computed from Eqs. (A6) or (A7), with the result

$$\det\left[\frac{\partial\xi}{\partial x}\right]_{\xi=0}=\left[\prod_{n=0}^{N-1}e_{(n)}\right]\exp\left[-\frac{\epsilon}{2}\sum_{m=1}^{N}(\partial_\mu V^\mu+e_\nu^b V^\mu\partial_\mu e_b^\nu)_{(m-1)}+O(\epsilon^2)\right]. \tag{A9}$$

It follows from Eq. (A8b) that

$$1=\left[\frac{e_{(0)}}{e_{(N)}}\right]\exp\left[-\frac{\epsilon}{2}\sum_{m=1}^{N}(\partial_\mu V^\mu+e_\nu^b V^\mu\partial_\mu e_b^\nu)_{(m-1)}+O(\epsilon^2)\right]\Bigg|_{\xi=0}$$

$$\times\int\prod_{n=1}^{N}(dx_{(n)})e_{(n)}\prod_a\delta\left[e^a_{\mu(n-1)}\left[\frac{\Delta x^\mu_{(n)}}{\epsilon}-V^\mu_{(n-1)}+O(\epsilon)\right]\right]. \tag{A10}$$

As $\epsilon\to0$, the last factor of Eq. (A10) provides a definition of $\Delta^{-1}(t_f,t_i)$, so that

$$\Delta^{-1}(t_f,t_i)=\lim_{\epsilon\to0}\left[\frac{e_{(N)}}{e_{(0)}}\right]\exp\left[\frac{\epsilon}{2}\sum_{m=1}^{N}(\partial_\mu V^\mu+e_\nu^b V^\mu\partial_\mu e_b^\nu)_{(m-1)}+O(\epsilon^2)\right]\Bigg|_{\xi=0} \tag{A11a}$$

$$=\frac{e(t_f)}{e(t_i)}\exp\left[\int_{t_i}^{t_f}dt(\tfrac{1}{2}D_\mu V^\mu-\Gamma^\nu_{\nu\mu}V^\mu)\Big|_0\right] \tag{A11b}$$

$$=\exp\left[\tfrac{1}{2}\int_{t_i}^{t_f}dt\,D\cdot V\Big|_0\right], \tag{A11c}$$

which is the desired result Eq. (A1). In these steps, we have used $\Gamma^\nu_{\nu\mu}=\partial_\mu\ln e$, and the subscript zero denotes evaluation at the true flow $\xi^a=0$, for which $\Gamma^\nu_{\nu\mu}V^\mu=d(\ln e)/dt$.

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[11]The ghosts decouple for either sign of α as long as $\alpha\neq0$. As discussed in Sec. III, α positive (negative) corresponds to retarded (advanced) boundary conditions for the ghost propagator.
[12]The spatial functional derivative $\delta A_i^a(\mathbf{x},t)/\delta A_j^b(\mathbf{x}',t)=\delta^{ab}\delta_{ij}\delta^{D-1}(\mathbf{x}-\mathbf{x}')$ is implied when the functional arguments are at equal time.
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