Foldy-Wouthuysen representation and the relativistic equations of motion for a classical colored, spinning particle

H. Arodź and K. Golec*
Institute of Physics, Jagellonian University, Reymonta 4 PL-30-059 Cracow 16, Cracow, Poland
(Received 9 April 1985)

Nonrelativistic classical equations of motion for a colored, spinning particle obtained within the framework of the Foldy-Wouthuysen representation are analyzed with respect to the possibility of a relativistic generalization. To this end a new color operator is introduced by means of an operatorial gauge transformation. Consequently, a relativistic generalization of the classical equations of motion is proposed.

I. INTRODUCTION

Investigations of classical, colored particles have been initiated by Wong in a well-known paper.[1] The topic has been continued in a number of papers by other authors.[2,3] At present one can find in the literature a whole variety of approaches to the classical mechanics of colored particles. In particular, Wong's equations have been generalized to the case of a spinning particle. Also, the equations have been rederived by many different techniques.

We would like to present yet another approach to obtaining classical equations of motion for colored and spinning particles. Our starting point is the observation that the natural prototype of a particle with color charge is a quark, i.e., a spin-$\frac{1}{2}$ particle obeying the Dirac equation and belonging to the fundamental representation of the SU(n) group. Therefore, we would like to obtain classical equations of motion for spin-$\frac{1}{2}$ particles belonging to the fundamental representation of the SU(n) gauge group. In our approach the classical spin and color are defined as expectation values of spin and color operators, i.e., as polarization vectors. Thus, the classical mechanics we are looking for is, by its construction, nothing more than merely an approximation to quantum mechanics of the Dirac particle. This is why spin and color operators play the central role in our considerations. Investigations along this line of thought have been initiated in another paper.[4] We use the Foldy-Wouthuysen (FW) representation because this representation for Dirac particles,[5] and for other particles,[6] is well known for its virtue of clarifying the physical content of relativistic wave equations. The equations obtained in Ref. 4 have a nonrelativistic form. Also, they contain essentially a new classical observable, not considered in Refs. 1–3, which describes a "mixing" of spin and color, see Ref. 4 for details.

Let us remark here that also in Ref. 1 and 3 the starting point is just the Dirac equation. However, in these papers the classical quantities were obtained by a formal replacement of operators by c numbers, with no reference to wave functions describing the physical state of the particle. Moreover, in Ref. 1 one can find the statement that the classical limit consists of $\hbar\to 0$, $S\to\infty$, $\hbar S=$fixed constant. This is not consistent with the fixed value of spin, $S=\frac{1}{2}$, for the Dirac particle.

The aim of the present paper is to propose a relativistic generalization of the classical equation of motion derived in Ref. 4. For this purpose we introduce a new color operator and a new kinetic momentum operator in the FW representation, Sec. III. The reason is that the operators of color and kinetic momentum used in Ref. 4 obey Heisenberg equations of motion of a form unsuitable for a relativistic generalization (see Sec. II).

The problem of a relativistic generalization by itself might seem less than important when one recalls the difficulties in extracting from quantum mechanics the notion of a classical trajectory for colored particles.[7] Consideration of this problem, however, leads to new color and kinetic momentum operators. Because of this we think that the problem of the relativistic generalization is in fact interesting.

The contents of the paper are the following. In Sec. II we recall the Heisenberg equations of motion in the FW representation for the operators of position, kinetic momentum, spin and color obtained in Ref. 4, and we present in detail the problem of the relativistic generalization. In Sec. III we introduce the new operators of color and kinetic momentum and we write Heisenberg equations of motion for them. In Sec. IV we take a formal classical limit and we propose the relativistic generalization of the classical equations of motion for a colored, spinning particle. For simplicity we consider only the degenerate case with no mixing between spin and color.[4] In this degenerate case, the classical equations obtained in present paper agree in form with equations obtained other approaches, except for the term describing the coupling of spin to derivatives of gauge-field strength (see Sec. IV). In our approach this term is present in the Heisenberg equation of motion for the operator of kinetic momentum; however, this term is negligible in the classical limit for the Dirac equation (see the discussion in Sec. IV). The lack of this term has no influence on the problem of relativistic covariance of equations of motion.

II. EQUATIONS OF MOTION FOR OPERATORS IN FOLDY-WOUTHUYSEN REPRESENTATION

The Dirac Hamiltonian in the FW representation, calculated up to the order $(mc)^{-2}$, has the form[5,4]

$$H_2=mc^2\beta\delta^{-1}+g\hat{A}_0+\frac{1}{2m}\beta\hat{\Pi}^i\hat{\Pi}^i\delta$$
$$-\frac{g\hbar}{mc}\beta\hat{S}^i\hat{B}^i\delta-\frac{g\hbar^2}{8m^2c^2}D_i\hat{E}^i\delta^2$$
$$-\frac{g\hbar}{4m^2c^2}\epsilon_{iks}\hat{S}^s(\hat{E}^i\hat{\Pi}^k+\hat{\Pi}^k\hat{E}^i)\delta^2+O(\delta^3)\ . \quad (1)$$

Here δ is an auxiliary dimensionless, book-keeping parameter we shall set equal to 1 at the very end of the calculations. Roughly speaking, counting powers of δ is equivalent to counting powers of $(mc)^{-1}$. $\hat{\Pi}^i$ is the kinetic momentum

$$\hat{\Pi}^i=\hat{p}^i-\frac{g}{c}\hat{A}^i,\ \ p^i=-i\hbar\frac{\partial}{\partial x^i}\ ,$$

$\hat{A}^i=A^{ai}\hat{T}^a$, where $\hat{T}^a$ are the Hermitian generators of the SU(n) group, and $\hat{S}^i=\frac{1}{2}\sigma^i$ is the spin operator. We also have

$$\hat{F}_{\mu\nu}=\partial_\mu\hat{A}_\nu-\partial_\nu\hat{A}_\mu+\frac{ig}{\hbar c}[\hat{A}_\mu,\hat{A}_\nu],\ \ \hat{E}^i=\hat{F}_{0i}\ ,$$
$$D_\rho\hat{F}_{\mu\nu}=\partial_\rho\hat{F}_{\mu\nu}+\frac{ig}{\hbar c}[\hat{A}_\rho,\hat{F}_{\mu\nu}]\ .$$

The color magnetic field is

$$\hat{B}^i=-\tfrac{1}{2}\epsilon_{ikl}\hat{F}_{kl},\ \ \text{i.e.,}\ \ \hat{F}_{kl}=-\epsilon_{kli}\hat{B}^i\ .$$

The higher orders (in δ) of the FW transformation have been calculated in Ref. 8.

Now let us turn to the observables for the colored Dirac particle. It is well known[5] that the position operator (in the Schrödinger picture) of the particle should be taken to be the coordinate x^i, acting by multiplication of the wave function in the FW representation. This wave function we shall denote by ψ'. The spin operator is given by $\hat{S}^i=\sigma^i/2$, where σ^i act on the spinor indices of ψ'. Notice that we do not include the $\hbar$ factor into $\hat{S}^i$. Also, the momentum operator $\hat{p}^i$ is given by $-i\hbar\partial/\partial x^i$, where the derivative acts on ψ'. These operators, called, respectively, the mean position, the mean spin, and the mean momentum, become quite complicated when transformed back to the initial representation of the Dirac equation.

The Heisenberg equations of motion for $\hat{x}^i,\hat{\Pi}^k,\hat{S}^i$ are

$$\frac{d\hat{x}^i}{dt}=\frac{i}{\hbar}[H_2,x^i]=\frac{1}{m}\beta\hat{\Pi}^i\delta-\frac{g\hbar}{2m^2c^2}\epsilon_{tis}\hat{S}^s\hat{E}^t\delta^2\ , \quad (2)$$

$$\frac{d\hat{\Pi}^k}{dt}=\frac{g}{2c}\epsilon_{kil}\left[\hat{B}^l\frac{d\hat{x}^i}{dt}+\frac{d\hat{x}^i}{dt}\hat{B}^l\right]$$
$$+g\hat{E}^k+\frac{g\hbar}{mc}\beta\hat{S}^iD_k\hat{B}^i\delta$$
$$+\frac{g\hbar}{4mc^2}\epsilon_{ils}\beta\hat{S}^s\left[D_k\hat{E}^i\frac{d\hat{x}^l}{dt}+\frac{d\hat{x}^l}{dt}D_k\hat{E}^i\right]\delta$$
$$+\frac{g\hbar^2}{8m^2c^2}D_k(D_i\hat{E}^i)\delta^2\ , \quad (3)$$

$$\frac{d\hat{S}^t}{dt}=\frac{g}{mc}\beta\epsilon_{tkl}\hat{B}^l\hat{S}^k\delta$$
$$+\frac{1}{4}\frac{g}{mc^2}\beta\hat{S}^p\left[\hat{E}^t\frac{d\hat{x}^p}{dt}-\hat{E}^p\frac{d\hat{x}^t}{dt}\right.$$
$$\left.+\frac{d\hat{x}^p}{dt}\hat{E}^t-\frac{d\hat{x}^t}{dt}\hat{E}^p\right]\delta\ . \quad (4)$$

Here $D_i\hat{E}^i$ means

$$(\partial_i\hat{E}^i)_H+\frac{ig}{\hbar c}[\hat{A}_i,\hat{E}^i]_H\ .$$

For the color operator we have at least two possibilities, namely $\hat{T}^a=\frac{1}{2}\sigma^a$ with σ^a acting on the color indices of either the original Dirac bispinor ψ or the bispinor in the FW representation ψ'. The first possibility is not appealing, because this color operator when transformed to the FW representation by standard formulas given in Ref. 4 becomes a rather lengthy expression containing, in particular, Dirac matrices α^i. Also, one can see that this operator in the FW representation has a rather unpleasant transformation law under the gauge group.

The second possibility, i.e., $\hat{T}^a$ acting on ψ', was used in Ref. 4, and it is used as the starting point in this paper. However, we find it useful to consider instead of this $\hat{T}^a$ the quantity

$$\hat{\Phi}(x)=\Phi^a(x)\hat{T}^a\ ,$$

where $\Phi^a(x)$ is a fixed number-valued function belonging to the adjoint representation of the gauge group, e.g., $\Phi^a=F^a_{\mu\nu}$. For $\Phi^a=\delta^{ac}$ we recover $\hat{T}^c$. The Heisenberg equation of motion for $\hat{\Phi}$ has the form

$$\frac{d\hat{\Phi}}{dt}=cD_0\hat{\Phi}+\frac{1}{2m}\beta(\hat{\Pi}^iD_i\hat{\Phi}+D_i\hat{\Phi}\hat{\Pi}^i)\delta$$
$$-\frac{ig}{mc}\beta\hat{S}^s[\hat{B}^s,\hat{\Phi}]\delta$$
$$-\frac{ig}{4m^2c^2}\epsilon_{iks}\hat{S}^s[\hat{E}^i\hat{\Pi}^k+\hat{\Pi}^k\hat{E}^i,\hat{\Phi}]\delta^2$$
$$-\frac{ig\hbar}{8m^2c^2}[D_i\hat{E}^i,\hat{\Phi}]\delta^2\ , \quad (5)$$

where

$$D_\mu\hat{\Phi}=(\partial_\mu\hat{\Phi})_H+\frac{ig}{\hbar c}[\hat{A}_\mu,\hat{\Phi}]_H\ ,$$

the subscript H means "in the Heisenberg picture."

Now we would like to describe in detail the problem of relativistic generalization. We adopt the following rather formal rules for the classical limit:

$$\hat{\Pi}^i\to m\dot{x}^i,\ \ \hat{S}^i\to s^i,\ \ \hat{\Phi}\to\Phi^a(x)I^a\ ,$$
$$D_i\Phi\to[\partial_i\Phi^a-g(\hbar c)^{-1}f_{abc}A_i^b\Phi^c]I^a\ , \quad (6)$$

etc., where f_{abc} are the structure constants of the SU(n) group. Also, one should abandon in the equations all the terms which are multiplied by the positive powers of $\hbar$, and to set $\beta=1,\delta=1$. On the right-hand side (RHS) of (6) there are the classical quantities: the classical spin $\mathbf{s}$, the classical color charge (I^a), the velocity of the particle $\dot{\mathbf{x}}$. For a justification and for a discussion of the rules (6) see Ref. 4, where a more refined approach to the classical limit is presented.

It is easy to see that the rules (6) applied to Eqs. (4) and (5) yield classical equations of motion which do not allow for an immediate relativistic generalization. Precisely these equations were obtained in Ref. 4, the only difference being that in that paper we considered $\hat{T}^a$ operators instead of $\hat{\Phi}$.

The classical equation for spin obtained in this manner can be recognized as the small-velocity limit of the Thomas equation.[9] This requires that we interpret $\mathbf{s}$ as the rest-frame classical spin, in full accordance with earlier considerations of spin polarization operators for the Dirac particle in the external electromagnetic field, see, e.g., Ref. 10. The well-known substitution[9]

$$\mathbf{w}=\mathbf{s}+\tfrac{1}{2}c^{-2}\dot{\mathbf{x}}(\dot{\mathbf{x}}\mathbf{s}) , \tag{7a}$$

where $\mathbf{w}$ is the laboratory-frame classical spin of the particle in the small-$\dot{\mathbf{x}}$ limit, gives a nonrelativistic equation for $\mathbf{w}$. This equation possesses immediate relativistic generalization in the form of the Bargmann-Michel-Telegdi (BMT) equation.

Motivated by these classical considerations we introduce on the quantum level the laboratory-frame mean spin operator

$$\hat{W}^i=\hat{S}^i+\frac{1}{4m^2c^2}[\hat{\Pi}^i(\hat{\Pi}^k\hat{S}^k)+(\hat{\Pi}^k\hat{S}^k)\hat{\Pi}^i]\delta^2 . \tag{7b}$$

The Heisenberg equation of motion for W^i reads

$$\frac{d\hat{W}^t}{dt}=\frac{g}{mc}\epsilon_{tkl}\beta\hat{B}^l\hat{W}^k\delta+\frac{g}{2mc^2}\beta\hat{W}^p\left[\hat{E}^t\frac{d\hat{x}^p}{dt}+\frac{d\hat{x}^p}{dt}\hat{E}^t\right]\delta . \tag{8}$$

Here we have neglected the terms of order higher than δ^2 because the Hamiltonian (1) is given only up to this order. The rules (6) applied to (8) give the small-velocity limit of the BMT equation. The precise form of this classical equation for the case of the colored, spinning particle we will write in Sec. IV.

The classical equation obtained from (5) by use of the rules (6) also does not have the form of the small velocity limit of a relativistic equation. Inspired by the case of spin presented above, we propose to introduce a new color operator $\tilde{\Phi}$, such that the Heisenberg equation of motion for it would give on the classical level the small velocity limit of a Lorentz-covariant classical equation.

Closer inspection of the problem shows that the difficulty with the relativistic generalization is due to the factor $\frac{1}{4}$ present in the last term of the Hamiltonian (1). There would be no problem with the relativistic equation for $\hat{\Phi}$ if this was $\frac{1}{2}$ instead of $\frac{1}{4}$. This $\frac{1}{4}$ instead of $\frac{1}{2}$ is due to the Thomas precession of spin.[9] Thus, the Thomas precession term, necessary for the right time evolution of the spin, gives an undesirable contribution to the equation of motion for the color charge. The fact that the equation for $\hat{\Phi}$ does not have the form of a relativistic equation (in the small velocity limit) indicates that the color operators $\hat{T}^a$ are not Lorentz scalars.

Let us also notice that the rules (6) applied to Eq. (3), with the $\hbar\to 0$ prescription suspended, also give an equation which does not have the form of a small-velocity limit of a Lorentz-covariant equation. Again the difficulty is due to the $\frac{1}{4}$ factor in the last term of the Hamiltonian (1). However, here the problem can be regarded as not so important because the unwanted term is annihilated by the $\hbar\to 0$ prescription. This applies also to the Abelian counterpart of Eq. (3). On the other hand, Eq. (5) does not have the Abelian counterpart.

III. THE NEW COLOR AND MOMENTUM OPERATORS

We shall make the rather natural assumption that the new color operator $\tilde{\Phi}$ is related to the old one $\hat{\Phi}$ by an operatorial gauge transformation Ω. That is,

$$\begin{gathered}\tilde{\Phi}=\Omega\hat{\Phi}\Omega^{-1},\quad \tilde{A}_\mu=\Omega\hat{A}_\mu\Omega^{-1}+\frac{i\hbar c}{g}\partial_\mu\Omega\Omega^{-1} ,\\ \tilde{F}_{\mu\nu}=\Omega\hat{F}_{\mu\nu}\Omega^{-1},\quad \tilde{\Pi}^i=\Omega\hat{\Pi}^i\Omega^{-1} ,\end{gathered} \tag{9}$$

etc. The spin operator $\hat{S}^i$ is not transformed because it does not contain color degrees of freedom.

We adjust Ω in such a way that the Heisenberg equation for $\tilde{\Phi}$ becomes on the classical level a Lorentz-covariant equation. The new color operator $\tilde{\Phi}$ is assumed to be a Lorentz scalar. Precisely, we assume that the equation for $\tilde{\Phi}$ has the form (5), with the exception for the fourth term on the RHS of (5) which should have the factor $\frac{1}{2}$ instead of the present $\frac{1}{4}$. In this way the operatorial gauge transformation (9) will just remove the undersirable contribution to Eq. (5) coming from the Thomas precession term. That is, we would like to have

$$\begin{aligned}\frac{d\tilde{\Phi}}{dt}=&\,c\widetilde{D_0\Phi}+\frac{1}{2m}\beta(\tilde{\Pi}^i\widetilde{D_i\Phi}+\widetilde{D_i\Phi}\tilde{\Pi}^i)\delta\\ &+\frac{ig}{2mc}\epsilon_{iks}\beta[\tfrac{1}{2}(\tilde{W}^s\tilde{F}_{ik}+\tilde{F}_{ik}\tilde{W}^s),\tilde{\Phi}]\delta\\ &-\frac{ig}{2m^2c^2}\epsilon_{iks}[\tfrac{1}{2}\tilde{W}^s(\tilde{F}_{0i}\tilde{\Pi}^k+\tilde{\Pi}^k\tilde{F}_{0i})\\ &\qquad+\tfrac{1}{2}(\tilde{F}_{0i}\tilde{\Pi}^k+\tilde{\Pi}^k\tilde{F}_{0i})\tilde{W}^s,\tilde{\Phi}]\delta^2\\ &-\frac{ig\hbar}{8m^2c^2}[\widetilde{D_iF_{0i}},\tilde{\Phi}]\delta^2 ,\end{aligned} \tag{10}$$

where

$$\widetilde{D_\rho\Phi}=\Omega D_\rho\hat{\Phi}\Omega^{-1}=\partial_\rho\tilde{\Phi}+\frac{ig}{\hbar}[\tilde{A}_\rho,\tilde{\Phi}],$$

and

$$\begin{aligned}\tilde{W}^i&=\hat{S}^i+\frac{1}{4m^2c^2}[\tilde{\Pi}^i(\tilde{\Pi}^k\hat{S}^k)+(\tilde{\Pi}^k\hat{S}^k)\tilde{\Pi}^i]\delta^2\\ &=\hat{W}^i+O(\delta^4) .\end{aligned}$$

The additional symmetrization in the third and fourth terms on the RHS of (10) is introduced because $\widetilde{W}^i$ does not commute with $\widetilde{F}_{\mu\nu}$ and $\widetilde{\Pi}^i$. However, the commutators of $\widetilde{W}$ with $\widetilde{F},\widetilde{\Pi}$ would give contributions to Eq. (10) of order higher than δ^2. Therefore, in the order δ^2 this additional symmetrization is superfluous.

Comparing (10) with (5) and using (9) we see that in the order δ^2 we have the relation in the Heisenberg picture

$$\left[\frac{d\Omega}{dt}\Omega^{-1},\widetilde{\Phi}\right] = -\frac{ig}{4m^2c^2}\epsilon_{iks}\tfrac{1}{2}[\widetilde{W}^s(\widetilde{E}^i\widetilde{\Pi}^k+\widetilde{\Pi}^k\widetilde{E}^i) + (\widetilde{E}^i\widetilde{\Pi}^k+\Pi^k\widetilde{E}^i)\widetilde{W}^s,\widetilde{\Phi}]\delta^2 . \quad (11)$$

Neglecting in (11) the terms of the order higher than δ^2 we can effectively put there

$$\widetilde{W}^s=\hat{S}^s,\ \ \widetilde{\Pi}^i=\hat{\Pi}^i,\ \ \widetilde{E}^i=\hat{E}^i .$$

Then (11) will be satisfied if

$$\frac{d\Omega}{dt}\Omega^{-1}=-\frac{ig}{4m^2c^2}\epsilon_{iks}\hat{S}^s(\hat{E}^i\hat{\Pi}^k+\hat{\Pi}^k\hat{E}^i)\delta^2+O(\delta^3) . \quad (12)$$

Notice that the RHS of (12) is of order δ^2, i.e., $(mc)^{-2}$.

Let us observe that the transformation (9) also changes the form of the equation (3) for $\hat{\Pi}^k$. Namely,

$$\frac{d\widetilde{\Pi}^k}{dt}=-\frac{g}{2mc}\beta(\widetilde{F}_{ki}\widetilde{\Pi}^i+\widetilde{\Pi}^i\widetilde{F}_{ki})\delta+g\widetilde{E}^k+\frac{g\hbar}{2mc}\beta(\widetilde{W}^i\widetilde{D_kB^i}+\widetilde{D_kB^i}\widetilde{W}^i)\delta + \frac{g\hbar}{4m^2c^2}\epsilon_{ils}[\widetilde{W}^s(\widetilde{D_kE^i}\widetilde{\Pi}^l+\widetilde{\Pi}^l\widetilde{D_kE^i})+(\widetilde{D_kE^i}\widetilde{\Pi}^l+\widetilde{\Pi}^l\widetilde{D_kE^i})\widetilde{W}^s]\delta^2+\frac{g\hbar^2}{8m^2c^2}D_k(\widetilde{D_iE^i})\delta^2 . \quad (13)$$

The classical counterpart of Eq. (13) will be given in Sec. IV.

The form of Ω should be calculated from (12). We do not know the explicit solution of (12) for generic external field $\hat{E}^i$, except for the rather formal expression

$$\Omega=T\exp\left[-\frac{ig}{4m^2c^2}\int_{t_0}^{t}\epsilon_{iks}\hat{S}^s(\hat{E}^i\hat{\Pi}^k+\hat{\Pi}^k\hat{E}^i)\delta^2\right] ,$$

where T stands for the ordinary time ordering. For $\hat{E}^i$ covariantly constant in time, i.e., $D_0\hat{E}^i=0$, we can take in the order δ^2

$$\Omega=\exp\left[-\frac{igt}{4m^2c^2}\epsilon_{iks}\hat{S}^s(\hat{e}^i\hat{\Pi}^k+\hat{\Pi}^k\hat{E}^i)\delta^2+O(\delta^3)\right] , \quad (14)$$

because

$$\frac{d\hat{\Pi}^k}{dt}=g\hat{E}^k+O(\delta),\ \ \frac{dS^t}{dt}=0+O(\delta) ,$$
$$\frac{d\hat{E}^i}{dt}=\frac{i}{\hbar}[H_2,\hat{E}^i]+\left[\frac{\partial\hat{E}^i}{\partial t}\right]_H=(D_0\hat{E}^i)_H+O(\delta)=O(\delta) . \quad (15)$$

The terms neglected in (15), would contribute to the term $O(\delta^3)$ in Eq. (12). When $D_0D_0\hat{E}^i=0$, we have

$$\Omega=\exp\left[-\frac{ig}{4m^2c^2}\epsilon_{iks}\hat{S}^s\left[t(\hat{E}^i\hat{\Pi}^k+\hat{\Pi}^k\hat{E}^i)-\frac{t^2}{2}(D_0\hat{E}^i\hat{\Pi}^k+\hat{\Pi}^kD_0\hat{E}^i)+\frac{t^3}{3!}g(D_0\hat{E}^i\hat{E}^k+\hat{E}^kD_0\hat{E}^i)\right]\delta^2+O(\delta^3)\right] , \quad (16)$$

where we have used the fact that

$$\frac{d\hat{E}^i}{dt}=D_0\hat{E}^i+O(\delta) .$$

Similarly, one can write Ω when $D_0{}^3\hat{E}^i=0$, etc.

It is natural to ask whether the operatorial gauge transformation (9) can be related to a Lorentz boost, in a manner analogous to that as (7b) mimics on the quantum level the Lorentz boost (7a). In Ω is to be related to a boost then one would expect that its classical counterpart obtained by the rules (6) should be equal to 1 when the velocity of the particle vanishes. The solution (16) does not have this property. This suggests that the passage from $\hat{\Phi}$ to $\widetilde{\Phi}$ is not a representation of the Lorentz boost. Therefore, $\hat{\Phi}$ and $\widetilde{\Phi}$ should not be called, respectively, the rest-frame and the laboratory-frame color operators, in contradistinction to the case of $\hat{S}^i$ and $\widetilde{W}^i$.

IV. RELATIVISTIC CLASSICAL EQUATIONS OF MOTION

As the rules for classical limit we adopt now

$$\tilde{\Pi}^i \to m\dot{x}^i,\ \ \tilde{W}^i \to w^i,\ \ \tilde{\Phi}(x) \to \Phi^a(x)K^a,$$
$$\tilde{F}_{\mu\nu} \to F^a_{\mu\nu}K^a,\ \ \widetilde{D_\mu \Phi} \to (D_\mu \Phi)^a K^a,\ \text{etc.}, \tag{17}$$
$$\beta = 1,\ \ \delta = 1,\ \ \hbar^n = 0\ \text{if}\ n > 0 .$$

Here K^a should be regarded as the expectation value of the operator $\tilde{T}^a = \Omega \hat{T}^a \Omega^{-1}$. These rules can be easily established within the framework of Ehrenfest's approach to the classical limit, presented in Ref. 4. In fact one would expect that $\tilde{\Phi} \to \tilde{\Phi}^a K^a$, $\tilde{F}_{\mu\nu} \to \tilde{F}^a_{\mu\nu} K^a$, etc., where $\tilde{\Phi}^a(x)$, $\tilde{F}^a_{\mu\nu}$ are the classical counterparts, obtained by the rules (17) of $\Omega\Phi^a(x)\Omega^{-1}$ and $\Omega F^a_{\mu\nu}\Omega^{-1}$. However, because in the Schrödinger picture $\Phi^a(x)$, $F^a_{\mu\nu}$ are number-valued functions, $\Omega\Phi^a(x)\Omega^{-1}$ differs from $\Phi^a(x)$ only by the terms such that $\hat{\Pi}^i$ present in Ω acts on $\Phi^a(x)$. These terms are of order $\hbar/m^2c^2$. The same is true for $F^a_{\mu\nu}$. Therefore, in the classical limit $\tilde{\Phi}^a(x) = \Phi^a(x)$, $\tilde{F}^a_{\mu\nu} = F^a_{\mu\nu}$, $(\widetilde{D_\mu F_{\rho\nu}})^a = (D_\mu F_{\rho\nu})^a$, etc.

The rules (17) applied to Eq. (13) give a classical equation which is the small-velocity limit of the equation

$$m\frac{d^2x^\mu}{d\tau^2} = \frac{g}{c} F^{a\mu}{}_\nu K^a \frac{dx^\nu}{d\tau}, \tag{18}$$

where τ is the proper time.

Let us remark that in Eq. (13) the terms linear in $\hbar$ give in the classical limit (17) (with the $\hbar \to 0$ prescription suspended) the contribution

$$\frac{g\hbar}{mc} w^i (D_k B^i)^a K^a + \frac{g\hbar}{2mc^2} \epsilon_{ils} w^s \dot{x}^l (D_k E^i)^a K^a ,$$

which can be regarded as the small-velocity limit of the contribution

$$-\frac{g\hbar}{2mc^2} \epsilon_{\mu\nu\rho\sigma} w^\sigma u^\rho F^{a\mu\nu} K^a$$

to the RHS of Eq. (18). This term is included in the classical equation of motion by many authors.[2,3] Our opinion is that it should be excluded from the classical equation of motion for the spin-$\frac{1}{2}$ particle. The reason is that we would like to regard the classical equations as the equations for expectation values of quantum operators in the state of the form of a wave packet. In particular, we do not accept the formal substitutions like $\hbar \to 0$, $\hat{S}^i \to \infty$, $\hbar\hat{S}^i = \text{const}$. Therefore, in our approach $\hbar$ has its true experimental value. When some term containing $\hbar$ is relatively large this means that quantum effects are important—then the classical approximation is likely to be incorrect. In fact, in the case of large terms containing a spin operator one would expect phenomena similar to those observed in the Stern-Gerlach experiment. So, the terms containing $\hbar$ have to be small if the classical approximation, which is based on the assumption that there exists a single classical trajectory, is to be the correct one. Therefore, the terms containing $\hbar$ can safely be neglected.

The rules (17) applied to (10) yield the small-velocity limit of the equation

$$\frac{d}{d\tau}(\Phi^a K^a) = (D_\mu \Phi)^a K^a u^\mu - \frac{g}{2mc^2} \epsilon_{\sigma\mu\nu\rho} f_{acb} u^\sigma w^\rho F^{a\mu\nu} \Phi^c K^b , \tag{19}$$

where $u^\mu = dx^\mu/d\tau$ is the four-velocity. Substituting into (19) $\Phi^a = \delta^{ac}$ we obtain the classical equation for the new classical color vector K^a

$$\frac{dK^c}{d\tau} = -\frac{g}{\hbar c} A^b_\mu K^a u^\mu f_{bca} + \frac{g}{2mc^2} \epsilon_{\sigma\mu\nu\rho} f_{bca} u^\sigma w^\rho F^{a\mu\nu} K^b . \tag{20}$$

The quantum equation (8) for the laboratory-frame spin operator $\tilde{W}^i$ can be written in the form (again we neglect all terms of the order higher than δ^2)

$$\frac{d\tilde{W}^t}{dt} = \frac{g}{2mc} \epsilon_{tkl} \beta (\tilde{B}^l \tilde{W}^k + \tilde{W}^k \tilde{B}^l)\delta + \frac{g}{4m^2c^2} [\tilde{W}^p (\tilde{E}^t \tilde{\Pi}^p + \tilde{\Pi}^p \tilde{E}^t) + (\tilde{E}^t \tilde{\Pi}^p + \tilde{\Pi}^p \tilde{E}^t) \tilde{W}^p] \delta^2 . \tag{21}$$

The rules (17) applied to Eq. (21) give the classical equation which is the small-velocity limit at the BMT equation

$$\frac{dw^\mu}{d\tau} = -\frac{g}{mc} w^\nu F^a{}_\nu{}^\mu K^a , \tag{22}$$

where $w^0 = c^{-1}\dot{\mathbf{x}} \cdot \mathbf{w}$, (w^μ) is the Pauli-Lubanski four-vector.

V. DISCUSSION

(a) The essence of our result is the following: The operatorial gauge transformation (9) with Ω obeying (12) leads to the new color and momentum operators. These new operators obey Heisenberg equations of motion of such a form that the formal classical limit (17) gives equations which are the small-velocity limit of the Lorentz-covariant equations of motion (19), (20), (22). The equation (12) for the operatorial gauge transformation Ω has been calculated up to the order δ^2. Calculation of the higher orders requires knowledge of the higher-order terms in the Hamiltonian (1).

(b) The statement that the classical equations of motion (18), (20), (22) are Lorentz covariant is based on the assumption that K^a is a Lorentz scalar. This assumption implies that $\hat{T}^a$ is not Lorentz scalar because Ω is not Lorentz scalar. This follows from the fact that equation (12) cannot be written in Lorentz-invariant form

$$\frac{d\Omega}{d\tau}\Omega^{-1} = \text{Lorentz scalar} .$$

A direct check of the transformation properties of $\hat{\Phi}$ and $\tilde{\Phi}$ would be possible if one knows the generators of the representation of the Poincaré group acting in the space of solutions of the Dirac equation in the presence of the external Yang-Mills field. The eigenvalues of these operators would have the meaning of the true momentum, angular momentum, etc., of the particle. Therefore, the eigenvalues should be gauge invariant. For this, the generators should be gauge covariant (under the local gauge transformations). It is well known that it is a rather nontrivial task to construct such generators, see, e.g., Ref. 11. Thus, a direct check of the transformation properties of $\hat{\Phi}$ and $\tilde{\Phi}$ is not possible at the moment.

(c) The rules (6) and (17) for the classical limit are rather formal. Nevertheless, they are the most frequently used rules, see, e.g., Refs. 1 and 3. A more refined approach to the classical limit is presented in Ref. 4—there mixing of spin and color is observed. In the present paper we use the formal rules for two reasons. First, in order to shorten reasonings. Second, the lengthy and tedious derivation of the rules of the kind (6) or (17), done in the spirit of Ref. 4, is not very interesting because the region of their validity is severely limited by the difficulties with extracting the notion of classical trajectory for a colored particle from quantum mechanics.[7] Our main goal in the present paper has been to solve the puzzle posed by the fact that the classical counterpart of Eq. (5) does not have the form of the small-velocity limit of a Lorentz-covariant equation.

(d) The obvious continuation of the present paper is to calculate the Heisenberg equations of motion and Ω in the orders higher than δ^2. Such an investigation would also provide a check of the idea that in order to obtain Lorentz-covariant equations one should utilize the operatorial gauge transformation. In the present paper we have shown that the operatorial gauge transformation works in the order δ^2. Unfortunately, the calculations in the higher orders will be rather tedious. This can be seen from the rather complicated form of the Hamiltonian in the FW representation calculated in the orders δ^3 and δ^4 (Ref. 8).

ACKNOWLEDGMENT

This paper was supported in part by the Polish Ministry of Higher Education, Science and Technology, project MR.I.7.

*Present address: Zakład Teorii Pola, Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

[1]S. K. Wong, Nuovo Cimento **LXVA**, 689 (1970).

[2]A. Barducci, R. Casalbuoni, and L. Lusanna, Nucl. Phys. **B124**, 521 (1977); L. Brink, P. diVecchia, and P. Howe, *ibid.* **B118**, 76 (1977); P. Salomonson, B.-S. Skagerstam, and J.-O. Winnberg, Phys Rev. **D 16**, 2581 (1977); C. Duval and P. A. Horvathy, Ann. Phys. (N.Y.) **142**, 10 (1982); H. J. Wospakrik, Phys. Rev. **D 26**, 523 (1982); A. P. Balachandran, G. Marmo, B.-S. Skagerstam and A. Stern, *Gauge Symmetries and Fibre Bundles* (Lecture Notes in Physics, Vol. 188) (Springer, Berlin, 1983) (in this book one can find references to many other papers devoted to classical colored particles); W. Drechsler and A. Rosenblum, Phys. Lett. **106B**, 81 (1981); S. Ragusa, Phys. Rev. **D 26**, 1979 (1981).

[3]W. Drechsler, Fortschr. Phys. **27**, 489 (1979); Nuovo Cimento **50A**, 405 (1979); U. Heinz, Phys. Lett. **144B**, 228 (1984).

[4]H. Arodź, Phys. Lett. **116B**, 251 (1982); Acta Phys. Pol. **B13**, 519 (1982).

[5]L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[6]K. M. Case, Phys. Rev. **95**, 1323 (1954).

[7]H. Arodź, Phys. Lett. **116B**, 255 (1982); Acta Phys. Pol. **B14**, 13 (1983); **B14**, 757 (1983).

[8]K. Golec, Acta Phys. Pol. B (to be published).

[9]J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Chap. 11 (there one can also find references to the original papers).

[10]A. A. Sokolov and I. M. Ternov, *The Relativistic Electron*, 2nd ed. (Nauka, Moscow, 1983), Chap. 3, Sec. 16 (there one can also find references to earlier papers).

[11]A. Peres, Phys. Rev. **167**, 1449 (1968).