Path-integral solubility of two-dimensional models

Ashok Das
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510
and Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627*

Vishnu S. Mathur
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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We apply the technique of Fujikawa to solve for simple two-dimensional models by looking at the nontrivial transformation properties of the fermion measure in the path-integral formalism. We obtain the most general solution for the massless Thirring model and point out how the one-parameter solution reduces to that of Johnson and Sommerfield in a particular limit. We present the most general solution for the massive vector model indicating how it reduces to the solutions of Brown and Sommerfield for different values of the parameter. The solution of a gradient-coupling model is also discussed.

I. INTRODUCTION

In the past, soluble two-dimensional field-theoretic models have been studied extensively, notable among them being the massless Thirring model, the Schwinger model, and the massive vector model.[1–10] The solutions of these models have been obtained by various methods, namely, by explicitly solving the equations of motion, by operator methods, by bosonization, etc. The main reason behind the solubility of these models is the fact that they contain a massless fermion in the theory which leads to classically conserved currents:

$$\partial_\mu j^\mu = 0\ ,\quad \partial_\mu j_5^\mu = \epsilon^{\mu\nu}\partial_\mu j_\nu = 0\ . \tag{1}$$

Because of quantum effects these conservation laws become anomalous[11] and it is the anomaly content of the theory that leads to the nontrivial solutions of these theories.

Recently Fujikawa has shown in a series of papers[12–15] how various anomalies arise in the path-integral formalism. The observation essentially is the fact that the fermion measure may transform nontrivially under various transformations leading to the anomalous Ward identities. It is, therefore, of interest to examine how one can obtain the already known solutions of various two-dimensional models following the method of Fujikawa. The observation here is quite simple. For a two-dimensional fermion, interacting with an external vector field, one can always redefine the fermion fields so that they decouple. Therefore, if the theory contains any nontrivial dynamics, it must essentially be contained in the transformation properties of the fermionic measure under the field redefinition.

In general, the two-dimensional models can have an Abelian or a non-Abelian symmetry. We would consider only Abelian models for the present. Further, they may possess a local gauge invariance, as is the case with the Schwinger model.[12] Roskies and Schaposnik[16] have studied this model from the path-integral point of view. Of the models without any local gauge invariance, only the derivative-coupling model was studied earlier[17,18] in the path-integral formalism. In the present paper we investigate the other nongauge models, namely, the massless Thirring model, the massive vector model, and the gradient-coupling model. Our philosophy is as follows. Since these models do not possess any local symmetry we try to find out the most general change in the fermionic measure under a field redefinition which would allow for an anomaly even in the vector current. In Sec. II we work out this general change in the measure of a fermion interacting with an external vector field. In Sec. III we use the results of the change in measure to obtain the most general solution for the massless Thirring model. It is a one-parameter family of solutions and we indicate which value of the parameter leads to the Johnson[4] and Sommerfield[5] result. In Sec. IV we study the massive vector model and discuss how different values of the one-parameter family correspond to the results of Brown[6] and Sommerfield.[5] The pseudoscalar derivative-coupling model has been studied earlier.[17,18] In Sec. V we discuss the scalar model with a gradient coupling. Finally we present some concluding remarks in Sec. VI.

II. FERMIONS IN AN EXTERNAL VECTOR FIELD

Let us now study the case of a fermion field interacting with an external vector field. We want to emphasize that our purpose is to study the most general way the fermionic measure changes in such a case and, therefore, we do not associate any dynamical local gauge invariance with the vector field. That is, we allow for the possibility that the vector-current conservation in such a case may become anomalous. The generating functional is given by

$$Z = \int D\bar\psi D\psi\, e^{iS}\ ,$$

where

$$S = \int d^2x\,[i\bar\psi\gamma^\mu(\partial_\mu - iA_\mu)\psi]\ . \tag{2}$$

Note that in $1+1$ dimensions, the vector field can be written as

$$A_\mu=\partial_\mu\sigma+\epsilon_{\mu\nu}\partial^\nu\zeta \quad (3)$$

so that formally

$$\sigma=\Box^{-1}\partial_\mu A^\mu\ ,\quad \zeta=\tfrac{1}{2}\Box^{-1}\epsilon^{\mu\nu}F_{\mu\nu}\ , \quad (4)$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\ .$$

Let us now calculate the most general change in the fermion measure if we make a simultaneous, infinitesimal gauge and chiral transformation, i.e.,

$$\begin{aligned}\psi&\to\psi'=[1-i\epsilon(x)-i\gamma_5\bar{\epsilon}(x)]\psi\ ,\\ \bar\psi&\to\bar\psi'=\bar\psi[1+i\epsilon(x)-i\gamma_5\bar{\epsilon}(x)]\ .\end{aligned} \quad (5)$$

Before considering the effect of this transformation, let us establish our notations. We work in the metric, $g^{00}=1$, $g^{11}=-1$, and in our notation γ^0 is Hermitian with $(\gamma^0)^2=1$ whereas γ^1 is anti-Hermitian with $(\gamma^1)^2=-1$. $\gamma_5=\gamma^0\gamma^1$ and it follows that it is Hermitian with square equal to unity. In continuing to Euclidean space, we treat ψ and $\bar\psi$ as independent complex quantities. Further, we let $x^0\to -ix_4$, $\partial_0\to i\partial_4$, $\gamma^0\to i\gamma_4$, $\gamma_5=\gamma^0\gamma^1\to i\gamma_1\gamma_4$. γ_5 is Hermitian in Euclidean space although the γ_μ's are anti-Hermitian and in Euclidean space we have the identity

$$\gamma_\mu\gamma_\nu=-\delta_{\mu\nu}-i\epsilon_{\mu\nu}\gamma_5\ ,\quad \epsilon_{14}=+1\ . \quad (6)$$

With these conventions, it is clear that

$$\begin{aligned}S&=\int d^2x\,[i\bar\psi\gamma^\mu(\partial_\mu-iA_\mu)\psi]\\ &=\int d^2x\,[i\bar\psi\gamma^\mu(\partial_\mu-i\partial_\mu\sigma-i\epsilon_{\mu\nu}\partial^\nu\zeta)\psi]\\ &=\int d^2x\,[i\bar\psi\gamma^\mu(\partial_\mu-i\partial_\mu\sigma-i\gamma_5\partial_\mu\zeta)\psi]\\ &\to\int(-id^2x_E)[-i\bar\psi\gamma_\mu(\partial_\mu-i\partial_\mu\sigma-i\gamma_5\partial_\mu\zeta)\psi]\\ &=i\int d^2x_E[i\bar\psi\gamma_\mu(\partial_\mu-i\partial_\mu\sigma-i\gamma_5\partial_\mu\zeta)\psi]\\ &=iS_E\end{aligned} \quad (7)$$

so that

$$Z=\int D\bar\psi D\psi\, e^{-S_E}\ . \quad (8)$$

To find the change in the measure we have to choose a set of basis states in which to expand ψ and $\bar\psi$. It is clear that if we choose the same basis states for both ψ and $\bar\psi$, then that would lead to a gauge-invariant result. And since we are interested in finding the most general change in the measure, i.e., since we are allowing the vector current to be anomalous, we must expand ψ and $\bar\psi$ in terms of basis states which are eigenstates of different operators.[19] Because of the dual nature of A_μ and A_μ^5 in Minkowski space, namely,

$$A_\mu^5=\epsilon_{\mu\nu}A^\nu\ ,$$

the covariant Dirac operator can be written as

$$\begin{aligned}\gamma^\mu D_\mu&=\gamma^\mu(\partial_\mu-iA_\mu)\\ &=\gamma^\mu(\partial_\mu-i\xi A_\mu-i\eta\gamma_5A_\mu^5)\end{aligned}$$

with

$$\eta+\xi=1\ . \quad (9)$$

In continuing to Euclidean space we let $D_0\to iD_4$, i.e., we require $A_0\to iA_4$ and $A_0^5\to iA_4^5$. It is clear, therefore, that in continuing to Euclidean space it is impossible to maintain simultaneously the reality of A_μ and A_μ^5 as well as the duality between them. We, therefore, choose to treat them as independent real quantities in Euclidean space. Once all the calculations have been done and we have rotated back to Minkowski space can the duality relation be used. With all these rules, the Dirac operator of Eq. (9) in Euclidean space becomes

$$\not{D}=\gamma_\mu D_\mu=\gamma_\mu(\partial_\mu-i\xi A_\mu-i\eta\gamma_5A_\mu^5)\ . \quad (10)$$

Here η and ξ are arbitrary parameters with the constraint in Eq. (9) and as we will show later, they essentially measure the gauge and chiral noninvariance of the functional measure. It is quite well known[15] that the Dirac operator with an axial coupling is not Hermitian in Euclidean space and in the present case it is easy to check that

$$\not{D}^\dagger=\gamma_\mu(\partial_\mu-i\xi A_\mu+i\eta\gamma_5A_\mu^5)=\gamma_\mu D_\mu(A,-A_5)\ . \quad (11)$$

In recent literature,[15] there has been a lot of controversy over the use of non-Hermitian operators. In fact, when one uses non-Hermitian operators, the orthonormality and the completeness of the basis states is not guaranteed. Therefore, to avoid all such criticism we use the two Hermitian operators

$$\Delta=\not{D}^\dagger\not{D}\ ,\quad \tilde\Delta=\not{D}\not{D}^\dagger\ , \quad (12)$$

in whose eigenstates we expand the fields ψ and $\bar\psi$. Thus, for example (see Ref. 15),

$$\Delta\phi_n=\lambda_n{}^2\phi_n\ ,\quad \tilde\Delta\chi_n=\lambda_n{}^2\chi_n\ , \quad (13)$$

and

$$\psi(x)=\sum_n a_n\phi_n\ ,\quad \bar\psi(x)=\sum_n\chi_n^\dagger b_n\ , \quad (14)$$

so that $D\bar\psi D\psi=\prod_n db_n da_n$.

The calculation of the change in measure under the combined transformations of Eq. (5) is quite standard.[12,13,15,18] However, we will include it here for completeness. Note that

$$\begin{aligned}\psi'(x)&=[1-i\epsilon(x)-i\gamma_5\bar\epsilon(x)]\psi\\ &=\sum_n a_n'\phi_n=\sum_{n,m}C_{nm}a_m\phi_n\end{aligned}$$

so that

$$\prod_n da_n'=(\det C_{nm})^{-1}\prod_n da_n\ ,$$

where

$$C_{nm}=\delta_{nm}-i\int d^2x_E\phi_n^\dagger(x)[\epsilon(x)+\gamma_5\bar\epsilon(x)]\phi_m(x)\ .$$

Thus

$$\det C_{nm} = \exp(\mathrm{Tr}\ln C_{nm}) = \exp\left[-i\sum_n \int d^2x_E \phi_n^\dagger(x)[\epsilon(x)+\gamma_5\bar{\epsilon}(x)]\phi_n(x)\right].$$

The exponent is divergent and can be regulated in the following way:

$$\begin{aligned}-i\sum_n \int d^2x_E \phi_n^\dagger(x)[\epsilon(x)+\gamma_5\bar{\epsilon}(x)]\phi_n(x) &= \lim_{M^2\to\infty} -i\sum_n \int d^2x_E \phi_n^\dagger(x)[\epsilon(x)+\gamma_5\bar{\epsilon}(x)]\phi_n(x) e^{-\lambda_n{}^2/M^2}\\ &= \lim_{M^2\to\infty} -i\sum_n \int d^2x_E \phi_n^\dagger(x)[\epsilon(x)+\gamma_5\bar{\epsilon}(x)]e^{-\Delta/M^2}\phi_n(x)\\ &= \lim_{M^2\to\infty} -i\int d^2x_E \frac{d^2k_E}{(2\pi)^2}\mathrm{Tr}[\epsilon(x)+\gamma_5\bar{\epsilon}(x)]e^{-ik_E\cdot x_E}e^{-\Delta/M^2}e^{ik_E\cdot x_E}.\end{aligned}$$

Here we have used the completeness relation of the eigenstates in the momentum space. Expanding out the operator Δ and doing the Dirac trace gives the finite result as

$$\begin{aligned}-i\sum_n \int d^2x_E \phi_n^\dagger(x)[\epsilon(x)+\gamma_5\bar{\epsilon}(x)]\phi_n(x) = -\frac{i}{2\pi}\int d^2x_E \Bigg[\epsilon(x)\left[-i\xi\partial_\mu A_\mu + \frac{\eta}{2}\epsilon_{\mu\nu}F_{\mu\nu}{}^5\right]\\ +\bar{\epsilon}(x)\left[-i\eta\partial_\mu A_\mu^5+\frac{\xi}{2}\epsilon_{\mu\nu}F_{\mu\nu}\right]\Bigg].\end{aligned}$$

Clearly then, we can write the change in the measure for ψ as

$$D\psi = J_\psi D\psi',$$

where

$$J_\psi = \exp\left\{-\frac{i}{2\pi}\int d^2x_E\left[\epsilon(x)\left[-i\xi\partial_\mu A_\mu+\frac{\eta}{2}\epsilon_{\mu\nu}F_{\mu\nu}{}^5\right]+\bar{\epsilon}(x)\left[-i\eta\partial_\mu A_\mu^5+\frac{\xi}{2}\epsilon_{\mu\nu}F_{\mu\nu}\right]\right]\right\}. \tag{15}$$

Similarly one can show that the change in measure associated with $\bar{\psi}$ under a combined gauge and chiral transformation is given by

$$D\bar{\psi} = J_{\bar{\psi}} D\bar{\psi}',$$

where

$$J_{\bar{\psi}} = \exp\left\{\frac{i}{2\pi}\int d^2x_E\left[\epsilon(x)\left[-i\xi\partial_\mu A_\mu-\frac{\eta}{2}\epsilon_{\mu\nu}F_{\mu\nu}{}^5\right]-\bar{\epsilon}(x)\left[i\eta\partial_\mu A_\mu^5+\frac{\xi}{2}\epsilon_{\mu\nu}F_{\mu\nu}\right]\right]\right\}. \tag{16}$$

Thus under a combined gauge and chiral transformation (infinitesimal), the most general change in the fermion measure is given by

$$J = J_{\bar{\psi}}J_\psi = \exp\left[-\frac{i}{2\pi}\int d^2x_E[\eta\epsilon(x)\epsilon_{\mu\nu}F_{\mu\nu}{}^5 + \xi\bar{\epsilon}(x)\epsilon_{\mu\nu}F_{\mu\nu}]\right]. \tag{17}$$

It is clear from Eq. (17) that both the vector and the axial-vector currents become anomalous as

$$\partial_\mu j^\mu = \frac{\eta}{\pi}\partial_\mu A^\mu,\quad \partial_\mu j_5^\mu = -\frac{\xi}{\pi}\partial_\mu A_5^\mu, \tag{18}$$

with

$$\eta+\xi=1.$$

Since we have omitted the tedious, but rather straightforward technical details of the calculation,[12,13,18] several comments are in order. First of all, the arbitrariness in the choice of the operators in whose basis we expand the fields simply corresponds to the arbitrariness in the choice of the regularization scheme in conventional calculations. Therefore, we must choose the basis states to respect all other symmetries the theory may have, for example, Lorentz invariance, translation invariance, etc. In addition the basis states must also be chosen such that the anomalies satisfy the consistency conditions.[20] However, in this case since the anomalies are Abelian, this does not lead to any particular restriction. However, we can make a connection between our result of Eq. (18) and the perturbative result in the following way. For example, from Lorentz invariance and Bose symmetry we can write the current correlation function as

$$\langle j_\mu j_\nu\rangle = \frac{1}{\pi}\left[\left[\frac{k_\mu k_\nu}{k^2}-g_{\mu\nu}\right]+\eta g_{\mu\nu}\right]. \tag{19}$$

Here the coefficient of the finite transverse part is unique and the arbitrariness of any regularization scheme can only reflect in the longitudinal part. This form of the correlation function leads to the vector anomaly as

$$k^\mu \langle j_\mu j_\nu \rangle = \frac{\eta}{\pi} k_\nu .$$

By using the duality relation, we can obtain the axial-vector—vector correlation function as

$$\langle j_\mu^5 j_\nu \rangle = \epsilon_\mu{}^\lambda \langle j_\lambda j_\nu \rangle = \frac{1}{\pi}\left[(-1+\eta)\epsilon_{\mu\nu} + \epsilon_{\mu\lambda}\frac{k^\lambda k_\nu}{k^2}\right] . \quad (20)$$

This determines the axial anomaly to be

$$k^\mu \langle j_\mu^5 j_\nu \rangle = \frac{1}{\pi}(-1+\eta)\epsilon_{\mu\nu} k^\mu .$$

Comparing this with the form of the axial anomaly in Eq. (18) we obtain

$$-\xi = -1+\eta$$

or

$$\eta + \xi = 1 .$$

This now completes our derivation of the most general change in the fermion measure under a combined infinitesimal gauge and chiral transformations and is given by

$$J = \exp\left[-\frac{i}{\pi}\int d^2x\,[\eta\epsilon(x)\partial_\mu A^\mu + \xi\bar{\epsilon}(x)\partial_\mu A_5^\mu]\right]$$

with

$$\eta + \xi = 1 . \quad (21)$$

At this point we would like to make contact with conventional calculations. In that case, there does exist a regularization scheme which leads to these general results. This is the point-splitting method where one defines the fermion current with a phase factor which neither preserves the gauge invariance nor the chiral invariance,[7,8] namely,

$$J^\mu(x) = \lim_{\substack{x' \to x \\ x'_0 = x_0}} \bar{\psi}(x')\gamma^\mu \times \exp\left[i\int_{x'}^{x} dx''_\mu(\xi A^\mu - \eta\gamma_5 A_5^\mu)\right]\psi(x) .$$

Let us note here that if we take the action of Eq. (7) and make a field redefinition

$$\psi(x) = e^{i(\sigma+\gamma_5\xi)}\chi(x) , \qquad \bar{\psi}(x) = \bar{\chi}(x) e^{-i(\sigma-\gamma_5\xi)} , \quad (22)$$

then the fermion fields decouple completely. Namely,

$$S_E = \int d^2x_E [i\bar{\psi}\gamma_\mu(\partial_\mu - i\partial_\mu\sigma - i\gamma_5\partial_\mu\xi)\psi] = \int d^2x_E i\bar{\chi}\gamma_\mu\partial_\mu\chi . \quad (23)$$

However, in this case, the field redefinitions correspond to a finite gauge and chiral transformation. Therefore, the measure changes nontrivially. The calculation of the change of measure under a finite transformation is slightly tricky and has been studied earlier.[16,18] In this case, it can be shown in a straightforward manner to be equal to

$$D\bar{\psi}D\psi = D\bar{\chi}D\chi \times \exp\left[-\frac{i}{2\pi}\int d^2x\, A_\mu(\eta g^{\mu\alpha}g^{\nu\beta} + \xi\epsilon^{\mu\alpha}\epsilon^{\nu\beta})\partial_\alpha\partial_\beta\Box^{-1}A_\nu\right] . \quad (24)$$

Therefore, after the field redefinition the effective action in Minkowski space becomes

$$S_{\text{eff}} = \int d^2x\,(i\bar{\chi}\gamma^\mu\partial_\mu\chi + \tfrac{1}{2}A_\mu D^{\mu\nu}A_\nu) ,$$

where

$$D^{\mu\nu} = -\frac{1}{\pi}(\eta g^{\mu\alpha}g^{\nu\beta} + \xi\epsilon^{\mu\alpha}\epsilon^{\nu\beta})\partial_\alpha\partial_\beta\Box^{-1} \quad (25)$$

with $\eta + \xi = 1$.

Since the fermion fields have now decoupled, they can be integrated out leading to an unimportant constant. Thus the generating functional of Eq. (2) becomes

$$Z = C\exp\left[\frac{i}{2}\int d^2x\, A_\mu D^{\mu\nu}A_\nu\right] . \quad (26)$$

This is the most general form of the generating functional in the absence of any local symmetries and conforms to earlier calculations by other methods.[7,8]

III. MASSLESS THIRRING MODEL

The generating functional for this theory is given by

$$Z_{\text{TH}} = \int D\bar{\psi}D\psi \exp\left[i\int d^2x\,(i\bar{\psi}\gamma^\mu\partial_\mu\psi - \tfrac{1}{2}\lambda J_\mu J^\mu)\right] ,$$

where

$$J_\mu = \bar{\psi}\gamma_\mu\psi . \quad (27)$$

If we want to compute the current correlation functions in this theory, we can add a source term for the currents and write

$$Z_{\text{TH}} = \int D\psi \bar{D}\psi \exp\left[i\int d^2x\,(i\bar{\psi}\gamma^\mu\partial_\mu\psi - \tfrac{1}{2}\lambda j_\mu j^\mu + j_\mu A^\mu)\right] = \exp\left[\frac{i\lambda}{2}\int d^2x \frac{\delta^2}{\delta A_\mu(x)\delta A^\mu(x)}\right]\int D\bar{\psi}D\psi \exp\left[i\int d^2x[i\bar{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi]\right] . \quad (28)$$

The functional integral on the right-hand side is what we have already calculated in Eq. (26). Substituting the value for that we obtain[21]

$$Z_{\mathrm{TH}}=C\exp\left[\frac{i\lambda}{2}\int d^2x\frac{\delta^2}{\delta A_\mu(x)\delta A^\mu(x)}\right]\exp\left[\frac{i}{2}\int d^2x A_\mu D^{\mu\nu}A_\nu\right]$$
$$=K\exp\left[-\frac{i}{2\pi}\int d^2x\, A_\mu\left[\frac{\eta}{1-\lambda\eta/\pi}g^{\mu\alpha}g^{\nu\beta}+\frac{\xi}{1+\lambda\xi/\pi}\epsilon^{\mu\alpha}\epsilon^{\nu\beta}\right]\partial_\alpha\partial_\beta\Box^{-1}A_\nu\right]. \tag{29}$$

This gives the most general solution of the massless Thirring model. We note that there is a one-parameter family of solutions as had been observed earlier.[7] Furthermore, the particular solution of Johnson[4] and Sommerfield[5] simply corresponds to the values of our parameters $\eta=\frac{1}{2}$ and $\xi=\frac{1}{2}$.

IV. MASSIVE VECTOR MODEL

The Lagrangian for this theory is given by

$$L_{\mathrm{MV}}=-\tfrac{1}{4}(\partial_\mu B_\nu-\partial_\nu B_\mu)(\partial^\mu B^\nu-\partial^\nu B^\mu)+\frac{\mu^2}{2}B_\mu B^\mu+i\bar\psi\gamma^\mu(\partial_\mu-igB_\mu)\psi\,. \tag{30}$$

Both the Schwinger model and the massless Thirring model can be thought of as limiting cases of this theory. For example, in the limit of $\mu\rightarrow 0$, we obtain the Schwinger model. On the other hand, in the infinite-mass limit if we let

$$\lambda=\frac{g}{\mu}=\text{fixed}\,,$$

then we recover the massless Thirring model.

The generating functional for this theory is given by

$$Z_{\mathrm{MV}}=\int DB_\mu D\bar\psi D\psi\, e^{iS}=\int DB_\mu e^{iS(B_\mu)}\int D\bar\psi D\psi \exp\left[i\int d^2x\,[i\bar\psi\gamma^\mu(\partial_\mu-igB_\mu)\psi]\right]. \tag{31}$$

Here $S(B_\mu)$ is just the action for the vector field. Furthermore, we can substitute for the functional integral on the right-hand side from Eq. (26) to obtain

$$Z_{\mathrm{MV}}=C\int DB_\mu e^{iS(B_\mu)}\exp\left[\frac{i}{2}\int d^2x\, B_\mu D^{\mu\nu}B_\nu\right]$$
$$=C\int DB_\mu e^{iS_{\mathrm{eff}}(B_\mu)}\,,$$

where

$$S_{\mathrm{eff}}(B_\mu)=\int d^2x\left[-\tfrac{1}{4}(\partial_\mu B_\nu-\partial_\nu B_\mu)(\partial^\mu B^\nu-\partial^\nu B^\mu)+\frac{\mu^2}{2}B_\mu B^\mu+\frac{g^2}{2\pi}\times B_\mu[\xi g^{\mu\nu}-(\eta+\xi)\partial^\mu\partial^\nu\Box^{-1}]B_\nu\right]$$
$$=\int d^2x\left[-\tfrac{1}{4}(\partial_\mu B_\nu-\partial_\nu B_\mu)(\partial^\mu B^\nu-\partial^\nu B^\mu)+\frac{1}{2}\left[\mu^2+\frac{\xi g^2}{\pi}\right]B_\mu B^\mu-\frac{g^2}{2\pi}B_\mu\partial^\mu\partial^\nu\Box^{-1}B_\nu\right]. \tag{32}$$

Note that here we have made use of the constraint equation $\eta+\xi=1$. Furthermore, the Euler-Lagrange equations for this theory require

$$\partial_\mu B^\mu=0$$

for consistency. If we put this condition back we note that the effective vector-boson theory is a free theory with the mass renormalized to

$$m_R{}^2=\mu^2+\frac{\xi g^2}{\pi}\,. \tag{33}$$

This is also the result obtained earlier[8] by using point splitting. In particular note that for $\xi=1$, the mass renormalization corresponds to the result of a gauge-invariant regularization and is the value obtained by Brown[6] whereas Sommerfield's result[5] corresponds to the particular value of $\xi=\frac{1}{2}$.

We also note that if we set $\mu=0$, then we obtain the Schwinger model and if we regularize in a gauge-invariant manner, i.e., if we choose $\eta=0$, $\xi=1$, then we obtain

$$m_R{}^2=\frac{g^2}{\pi}$$

which one immediately recognizes as the Schwinger solution.[2]

V. GRADIENT-COUPLING MODEL

In Ref. 18 the most general solution was worked out for a pseudoscalar field interacting with a massless fermion field through a derivative coupling. The Lagrangian is given by

$$L=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2+i\bar\psi\gamma^\mu\partial_\mu\psi+g\bar\psi\gamma_5\gamma^\mu\psi\partial_\mu\phi\,. \tag{34}$$

We only quote the result of Ref. 18 where it was shown that after a field redefinition the effective Lagrangian for the theory becomes

$$L_{\rm eff}=\frac{1}{2}\left[1-\frac{\xi g^2}{\pi}\right]\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2\ . \tag{35}$$

This can be derived in a straightforward manner from the result of Sec. II and it shows that the massive field ϕ is effectively free with a renormalized mass given by

$$m_R{}^2=\frac{m^2}{(1-\xi g^2/\pi)}\ . \tag{36}$$

Let us now consider the gradient-coupling model where we consider ϕ to be a scalar field and, therefore, the Lagrangian is given by

$$L_{\rm GC}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2+i\bar\psi\gamma^\mu\partial_\mu\psi+g\bar\psi\gamma^\mu\psi\partial_\mu\phi\ . \tag{37}$$

Note that we can rewrite this Lagrangian simply as

$$L_{\rm GC}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2+i\bar\psi\gamma^\mu(\partial_\mu-igA_\mu)\psi \tag{38}$$

with $A_\mu=\partial_\mu\phi$.

The generating functional, therefore, can be written as

$$\begin{aligned}Z&=\int DA_\mu D\bar\psi D\psi D\phi\,\delta(A_\mu-\partial_\mu\phi)e^{iS_{\rm GC}}\\&=\int DA_\mu D\phi\,\delta(A_\mu-\partial_\mu\phi)e^{iS(\phi)}\\&\quad\times\int D\bar\psi D\psi\exp\left[i\int d^2x\,[i\bar\psi\gamma^\mu(\partial_\mu-igA_\mu)\psi]\right]\ .\end{aligned} \tag{39}$$

Using the result of Eq. (26) for the functional integral on the right-hand side and then integrating over the functional δ function we obtain

$$Z_{\rm GC}=\int D\phi\, e^{iS_{\rm eff}(\phi)}\ ,$$

where

$$S_{\rm eff}=\frac{1}{2}\left[1+\frac{\eta g^2}{\pi}\right]\partial_\mu\phi\partial^\mu\phi-\frac{m^2}{2}\phi^2\ .$$

Again this shows that the effective scalar field is a free, massive field with a renormalized mass given by

$$m_R{}^2=\frac{m^2}{(1+\eta g^2/\pi)}\ . \tag{40}$$

VI. CONCLUSIONS

We have shown how one can obtain solutions of simple Abelian models like the massless Thirring model, massive vector model, and the gradient coupling model in the path-integral formalism following the methods due to Fujikawa. Roskies and Schaposnik[16] had already used these methods in connection with the Schwinger model which has a local gauge invariance. We have shown how these methods should be extended in a general manner to cases where no local symmetries exist. One general feature, in the absence of local symmetries, is that the solutions involve a generally arbitrary parameter.

Although the solutions of the derivative coupling model and the gradient coupling model are new, those for the massless Thirring model and the massive vector model have been derived before by other techniques.[7,8] We have only presented a different way of looking at them and hope that these methods will help our understanding of more complicated models possessing complicated non-Abelian structure.

Note added. After this manuscript was written, we became aware of a recent report [M. A. Rubin, Report No. Fermilab-Pub-85/76-T, 1985 (unpublished)] where the author studies the most general anomaly structure in the case of the massless Thirring model by making a Pauli-Gursey-Pursey transformation in the path-integral formalism. We would like to thank Professor K. Tanaka for bringing this to our attention. While we are not sufficiently familiar with the methods used in that paper to be able to comment on the similarities, we believe that the approach and the spirit of our calculations are quite different. G. Duerksen has also studied the gauge-invariant solution of Thirring model [Phys. Lett. **103B**, 200 (1981)].

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*Permanent address.
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