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Perturbative QCD analysis of the photon structure function

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General two-scale formalism of the optimized perturbation theory (OPT) is successfully applied to the second-order QCD calculation of the photon structure function. Our findings are as follows. (i) Thorough resolution of the scheme dependences can be achieved if and only if the factorization and renormalization scheme dependences are investigated simultaneously. (ii) The two-scale OPT has dealt with the large second-order corrections so that at least part of the leading large-n terms are absorbed into the coupling constant; thus the optimized perturbation expansion may show an improved convergence. (iii) The above observation (ii) agrees up to the leading large-n terms with the results obtained through the kinematical analyses by Brodsky and Lepage and by Amati *et al.*

I. INTRODUCTION

The structure function of the photon has been discussed as a clean testing ground of quantum chromodynamics (QCD) because it is free of the unknown matrix elements of local operators at least in the first few orders of the effective coupling constant $a=\bar{g}^2/4\pi^2$. The existing second-order calculations,[1,2] which did not involve any unknown quantities except the scale parameter Λ, however, suffer from two difficulties. The first one is concerned with the calculational scheme dependences, such as the renormalization-scheme dependence,[3-6] of QCD perturbation theory, and the second one is the large negative subleading-order corrections[7] at large x.

As was studied by Politzer[8] and by two of us,[9] perturbative QCD calculations of physical quantities depend essentially on two calculational schemes;[10,11] the renormalization scheme (RS) and the factorization scheme (FS).[12,8,9] Resolution of the scheme-dependent ambiguities can be successfully carried out with the use of the optimized perturbation theory (OPT), originally proposed on the basis of the principle of minimal sensitivity by Stevenson[6] in order to solve the problem of the RS dependence. Here we should note that because the FS dependence is closely connected with the RS dependence and cannot be treated by itself, both of the two scheme dependences should be analyzed simultaneously. Such generalization of OPT has already been done,[8,9] and the photon structure function is just the quantity that clarifies the necessity and the usefulness of the generalized OPT (hereafter we call such a generalization the "two-scale" formalism of OPT). In fact if we consider only the RS dependence, ignoring the presence of the FS dependence, OPT cannot be applied to the second-order result of the photon structure function,[6] simply because its leading-order result is inversely proportional to the effective coupling constant a. This difficulty can be overcome by considering the two scheme dependences simultaneously. We shall complete this analysis in the present paper and study the consequences of the two-scale formalism of OPT applied to the second-order calculation of the photon structure function.

The two-scale formalism of OPT has already been applied to the calculations of the hadronic structure functions,[8,9] and one of the interesting consequences is concerned with the convergence behavior of perturbation expansion. Application of OPT not only solves the problem of scheme dependences, but also improves the convergence behavior.[9] This fact makes us sure that OPT may afford us a prescription to handle the scheme-dependent large higher-order corrections. Roughly speaking, through the optimization, part of the large corrections are absorbed into the optimized coupling constant and the optimized perturbation coefficients become moderate; thus the convergence behavior of the perturbation series is improved.

There are other approaches to handle the large higher-order corrections. Several authors have noted that there is a purely kinematical reason to expect higher-order corrections to become large as $x\to 1$ and that it is reasonable to hope that one can improve the convergence of the perturbation expansion by including correct kinematics at each order. Such analyses were first given for hadronic structure functions by Brodsky and Lepage[13] and by Amati *et al.*,[14] who claimed with *a posteriori* justification[15] that, by resumming such perturbation series being singular as $x\to 1$ to all orders of the strong coupling constant, the argument of the effective coupling constant at each vertex of the ladder rung is rescaled as $Q^2\to Q^2(1-x)$.

We now recognize that the outcomes of OPT and of the kinematical analyses are quite similar. To what extent does this similarity hold? And if both analyses give the same result, are there any physical and/or theoretical relations between them?

Frazer and Rossi[16] studied in detail the singularities in QCD perturbation theory for $x\sim 1$, and found an important observation: The behavior of the photon structure function for $x\sim 1$ is different from that of the hadronic ones. Although the kinematical corrections have successfully accounted for the leading large corrections for $x\sim 1$

for hadronic structure functions, *they have reproduced only a portion of them even for the leading corrections for the photon case.* Taking this observation into account we can say that the photon structure function provides us with an excellent field to study the problem set up above.

Section II is devoted to a brief survey of the two-scale formalism of OPT, and to its application to the second-order calculation of the photon structure function. In Sec. III optimization is carried out both with numerical and analytical methods. The structures of the optimized photon structure function are studied with special attention to the improvement of the convergence of perturbation expansion and to the relation with the kinematical analyses. Conclusions and several discussions are given in Sec. IV.

II. PHOTON STRUCTURE FUNCTION AND OPTIMIZATION FORMULAS

A. The second-order expression for the photon structure function

The second-order QCD calculation of the photon structure function suffers from twofold ambiguities: those coming from the FS as well as RS dependences. However, the existing calculations[2] have been carried out with the use of a special (in a sense, simple) FS, where the factorization scale M and the renormalization scale μ are identified and treated as if there were only a single scale μ ($\equiv M$) from the beginning. Thus in order to apply the two-scale formalism of OPT we should at first carry out the perturbative calculation of the photon structure function to get results where parameters denoting the RS and FS dependences are clearly discriminated from each other.

The nth moment of the photon structure function

$$F_n^\gamma(Q^2)=\int_0^1 dx\, x^{n-2}F_2^\gamma(x,Q^2) \tag{1}$$

can be factorized in terms of the Wilson operator-product expansion (OPE) as

$$F_n^\gamma(Q^2)=\vartheta_n(a(M))\cdot\mathscr{C}_n(Q/\mu,M/\mu,\tilde{a}(\mu)) , \tag{2}$$

where ϑ_n is the matrix element of the local operator with spin n, $\mathscr{C}_n$ the OPE coefficient function. Hereafter we carry out the calculation valid to first order in the electromagnetic fine-structure constant $\alpha=e^2/4\pi$. Following the general formulation of OPT (Ref. 9) we calculate the OPE coefficient function in terms of the effective coupling constant $\tilde{a}(\mu)=\bar{g}^2(\mu)/4\pi^2$ renormalized at μ, and renormalize the operator matrix element at M. In (2), ϑ_n and $\mathscr{C}_n$ are row and column vectors, respectively,

$$\vartheta_n=(\vartheta_n^\psi,\vartheta_n^G,\vartheta_n^{\rm NS},\vartheta_n^\gamma)=(\hat{\vartheta}_n,\vartheta_n^\gamma) , \tag{3a}$$

$$\mathscr{C}_n={}^t(\mathscr{C}_n^\psi,\mathscr{C}_n^G,\mathscr{C}_n^{\rm NS},\mathscr{C}_n^\gamma)={}^t({}^t(\hat{\mathscr{C}}_n),\mathscr{C}_n^\gamma) . \tag{3b}$$

(NS denotes nonsinglet.) $\hat{\vartheta}_n$ and $\hat{\mathscr{C}}_n$ are again row and column vectors whose components are obvious. The operator matrix element ϑ_n satisfies the equation

$$\frac{d\vartheta_n(a(M))}{d\ln M}=\vartheta_n(a(M))\gamma_n(a(M)) , \tag{4}$$

where γ_n denotes the anomalous-dimension matrix

$$\gamma_n=\begin{bmatrix}\gamma_n^{\psi\psi} & \gamma_n^{G\psi} & 0 & 0\\ \gamma_n^{\psi G} & \gamma_n^{GG} & 0 & 0\\ 0 & 0 & \gamma_n^{\rm NS} & 0\\ K_n^\psi & K_n^G & K_n^{\rm NS} & 0\end{bmatrix}\equiv\begin{bmatrix}\hat{\gamma}_n & 0\\ \hat{K}_n & 0\end{bmatrix} , \tag{5}$$

where $\hat{\gamma}_n$ is a three-by-three matrix and $\hat{K}_n$ is a row vector. Then Eq. (4) can be decomposed into the two equations

$$\frac{d\hat{\vartheta}_n(a(M))}{d\ln M}=\hat{\vartheta}_n(a(M))\hat{\gamma}_n(a(M))+\vartheta_n^\gamma(a(M))\hat{K}_n(a(M)) , \tag{6a}$$

$$\frac{d\vartheta_n^\gamma(a(M))}{d\ln M}=0, \text{ or } \vartheta_n^\gamma \text{ is independent of } M , \tag{6b}$$

which can be easily integrated to give[17]

$$\hat{\vartheta}_n(a(M))=AT\exp\left[\int_{a_0}^{a(M)}dx\frac{\hat{\gamma}_n(x)}{\beta(x)}\right]+\vartheta_n^\gamma\int_1^{a(M)}dx\frac{\hat{K}_n(x)}{\beta(x)}\times T\exp\left[\int_x^{a(M)}dy\frac{\hat{\gamma}_n(y)}{\beta(y)}\right] , \tag{7}$$

where A is the scheme-independent constant[7] and the β functions are defined as[10,11]

$$d\tilde{a}(\mu)/d\ln\mu=\tilde{\beta}(\tilde{a}(\mu)),\quad \tilde{a}(\mu=\mu_0)=g_\mu^2/4\pi^2 , \tag{8a}$$

$$da(M)/d\ln M=\beta(a(M)),\quad a(M=M_0)=g_M^2/4\pi^2 . \tag{8b}$$

Following Bardeen and Buras[2] (BB) we shall neglect the first term in (7), keeping only the inhomogeneous point-like term that is dominant in the asymptotic limit. Thus we are not concerned with the problem of large negative corrections near $x=0$, which has been already studied extensively by Glück and co-workers.[18]

Perturbative expressions for various quantities are as follows (hereafter we shall neglect the moment index n):

$$\begin{aligned}
&\gamma^{ij}(x)=\gamma_{ij}^0x+\gamma_{ij}^1x^2+\cdots \quad (i,j=\psi,G) ,\\
&\gamma^{\rm NS}(x)=\gamma_{\rm NS}^0x+\gamma_{\rm NS}^1x^2+\cdots ,\\
&K^j(x)=e^2(K_j^0+K_j^1x+\cdots) \quad (j=\psi,{\rm NS}) ,\\
&K^G(x)=e^2(K_g^1x+\cdots) ,\\
&\beta(x)=bx^2(1+cx+c_2x^2+\cdots) ,\\
&\tilde{\beta}(x)=bx^2(1+cx+\tilde{c}_2x^2+\cdots) ,\\
&\mathscr{C}^j(x)=e^2\delta_j(1+B_jx+\cdots) \quad (j=\psi,{\rm NS}) ,\\
&\mathscr{C}^G(x)=e^2\delta_\psi(B_Gx+\cdots) ,\\
&\mathscr{C}^\gamma(x)=e^4\delta_\gamma(B_\gamma+\cdots) ,\\
&\vartheta^\gamma=1 ,
\end{aligned} \tag{9}$$

where x denotes the effective coupling constant $a(M)$ or $\tilde{a}(\mu)$. Our notations are taken from Stevenson[6] and are somewhat different from those of Bardeen and Buras.[2] A transformation table is given for convenience (Table I).

In the present approximation, namely, in truncating everywhere, e.g., in Eqs. (9), the perturbative expansion up to the next-to-leading- (second-) order terms, there are ten parameters in total which label the RS and FS dependences: the renormalization scale μ, the factorization scale M, and the eight two-loop anomalous dimensions $\gamma^1_{\rm NS}$, γ^1_{ij} $(i,j=\psi,G)$, and K^1_j $(j=\psi,G,{\rm NS})$. It should be noted that in this order two β functions $\beta(x)$ and $\tilde{\beta}(x)$ coincide with each other. The above ten parameters should be taken, in principle, as independent variables throughout the OPT procedures, which then fix these parameters to their optimal values. Unfortunately, however, it is technically difficult to carry out the optimization inside the whole ten-parameter space. To find a way out of this difficulty we consider here a simplified model for the photon structure function, which consists of only two components: the nonsinglet point-quark component and the point-photon component, i.e.,

$$F_\gamma=\vartheta_{\rm NS}\mathscr{C}_{\rm NS}+\vartheta_\gamma\mathscr{C}_\gamma\ , \tag{10a}$$

$$d\vartheta_{\rm NS}/d\ln M=\gamma_{\rm NS}\vartheta_{\rm NS}+K_{\rm NS}\vartheta_\gamma\ ,$$
$$d\vartheta_\gamma/d\ln M=0\ . \tag{10b}$$

It should be noted that this model represents the essential structure of the photon structure function in the large-n limit: Because the hadronic anomalous-dimension matrix $\hat{\gamma}_n$ in Eq. (7) becomes a diagonal matrix in the large-n limit, the photon structure function (2) with (3)–(5) reduces essentially to the above model in the limit $n\to\infty$, in which we are particularly interested.

TABLE I. Conversion between notations used in this paper and those used by Bardeen and Buras (BB) (Ref. 2). This table should be read as, e.g., $a=g^2/4\pi^2$, $\gamma^0_{\rm NS}$(this paper) $=-\frac{1}{4}\gamma^0_{\rm NS}$(BB), etc.

This paper	Bardeen and Buras	
a	$g^2/4\pi^2$	
b	$-\beta_0/2$	
c	$\beta_1/4\beta_0$	
$\gamma^0_{\rm NS}$, γ^0_{ij}	$-\gamma^0_{\rm NS}/4$, $-\gamma^0_{ij}/4$	$(i,j=\psi,G)$
$\gamma^1_{\rm NS}$, γ^1_{ij}	$-\gamma^1_{\rm NS}/16$, $-\gamma^1_{ij}/16$	
K^0_j	$K^0_j/16\pi^2$	$(j=\psi,{\rm NS})$
K^1_j	$K^1_j/64\pi^2$	$(j=\psi,G,{\rm NS})$
B_j	$B_j/4$	
B_γ	$B_\gamma/16\pi^2$	

The model considered is simple enough to be solved analytically. The OPT analysis, given in Appendix A, shows that the optimized solution actually lies within a parameter subspace where the hadronic two-loop anomalous dimension vanishes, $\gamma^1_{\rm NS}=0$.

Guided by the above observation we start our analysis in calculating the structure function in a scheme where the hadronic two-loop anomalous dimensions $\gamma^1_{\rm NS}$ and γ^1_{ij} $(i,j=\psi,G)$ vanish. Thus taking μ, M, and K^1_j $(j=\psi,G,{\rm NS})$ as independent variables we perform the OPT procedure within the five-dimensional subspace inside the original ten-dimensional parameter space.

Now, by following the standard calculations,[2] we can write down the second-order expression for the photon structure function (in a scheme explained above):

$$\begin{aligned}\hat{F}^\gamma\equiv -bF^\gamma/e^4=&\delta_\psi K^0_\psi\left[\frac{1+d_{GG}}{(1+d_+)(1+d_-)}(1+B_\psi\tilde{a})-\frac{d_{G\psi}}{(1+d_+)(1+d_-)}B_G\tilde{a}\right]\frac{1+ca}{a}\\&+\delta_\psi(K^1_\psi-cK^0_\psi)\left[\frac{d_{GG}}{d_+d_-}(1+B_\psi\tilde{a})-\frac{d_{G\psi}}{d_+d_-}B_G\tilde{a}\right]+\delta_\psi K^1_G\left[-\frac{d_{\psi G}}{d_+d_-}(1+B_\psi\tilde{a})+\frac{d_{\psi\psi}}{d_+d_-}B_G\tilde{a}\right]\\&+\delta_{\rm NS}K^0_{\rm NS}\frac{1}{1+d_{\rm NS}}(1+B_{\rm NS}\tilde{a})\frac{1+ca}{a}+\delta_{\rm NS}(K^1_{\rm NS}-cK^0_{\rm NS})\frac{1}{d_{\rm NS}}(1+B_{\rm NS}\tilde{a})-b\delta_\gamma B_\gamma\ ,\end{aligned} \tag{11}$$

where $a\equiv a(M)$, $\tilde{a}\equiv\tilde{a}(\mu)$, and the d's are essentially the hadronic one-loop anomalous dimensions, i.e.,

$$d_{\rm NS}\equiv\gamma^0_{\rm NS}/b,\quad d_{ij}\equiv\gamma^0_{ij}/b\quad(i,j=\psi,G)\ ,$$
$$d_\pm=\{\gamma^0_{\psi\psi}+\gamma^0_{GG}\pm[(\gamma^0_{\psi\psi}-\gamma^0_{GG})^2+4\gamma^0_{\psi G}\gamma^0_{G\psi}]^{1/2}\}/2b\ . \tag{12}$$

The one-loop anomalous dimensions γ^0's and K^0's are scheme invariants and have already been calculated.[2,19]

B. Optimization formulas

In order to apply OPT to the second-order result of $\hat{F}^\gamma$ (11), we present here the optimization formulas. For this purpose we should at first evaluate the response of $\hat{F}^\gamma$ with respect to changes of the five scheme-labeling parameters μ, M, K^1_ψ, K^1_G, and $K^1_{\rm NS}$. The quantities to be optimized are the perturbative coefficients B_ψ, B_G, $B_{\rm NS}$, and B_γ and the effective coupling constants a and $\tilde{a}$, which are functions of the above optimization variables.

Before calculating the response of $\hat{F}^\gamma$ to changes of the parameters we should take notice of the following fact: The perturbative coefficients B_ψ, B_G, and $B_{\rm NS}$ are nothing but those that appear in the calculations of the hadronic structure functions; thus their structures, except their optimal values, should be determined only through the analyses of the hadronic structure functions. Analysis of the consistency condition for the nonsinglet structure function has already been done,[9] and can be done with tedious but straightforward manipulations for the singlet case. The resulting consistency equations are

$$\partial B_i/b\,\partial\ln\mu=0\quad(i=\psi,G,{\rm NS})\ , \tag{13}$$

$$\partial B_\psi/b\,\partial\ln M=-d_{\psi\psi}\ , \tag{14a}$$

$$\partial B_G/b\,\partial \ln M=-d_{\psi G}\ , \tag{14b}$$

$$\partial B_{\rm NS}/b\,\partial \ln M=-d_{\rm NS}\ . \tag{14c}$$

Obviously B_j ($j=\psi,G,{\rm NS}$) should be independent of the anomalous dimensions K^1_ψ, K^1_G, and $K^1_{\rm NS}$ that appear only in relation to the photon structure function, i.e.,

$$\partial B_i/\partial K^1_j=0\ \ (i,j=\psi,G,{\rm NS})\ . \tag{15}$$

Now we can easily write down the desired response equations of the photon structure function (11). Their explicit expressions are complicated and lengthy, and we give them in Appendix B. According to the optimization procedure, we then impose that the response of $\hat{F}^\gamma$ to changes of parameters should be of order a. This requirement imposed on the response equations (B1)—(B5) gives us the consistency equations:

$$\delta_\psi K^0_\psi\left[\frac{1+d_{GG}}{(1+d_+)(1+d_-)}\frac{\partial B_\psi}{b\,\partial\ln M}-\frac{d_{G\psi}}{(1+d_+)(1+d_-)}\frac{\partial B_G}{b\,\partial\ln M}\right]+\delta_{\rm NS}K^0_{\rm NS}\frac{1}{1+d_{\rm NS}}\frac{\partial B_{\rm NS}}{b\,\partial\ln M}$$
$$-b\delta_\gamma\frac{\partial B_\gamma}{b\,\partial\ln M}-\left[\delta_\psi K^0_\psi\frac{1+d_{GG}}{(1+d_+)(1+d_-)}+\delta_{\rm NS}K^0_{\rm NS}\frac{1}{1+d_{\rm NS}}\right]=0\ , \tag{16a}$$

$$\frac{\partial B_\gamma}{b\,\partial\ln\mu}=0\ , \tag{16b}$$

$$\frac{\partial B_\gamma}{\partial K^1_\psi}=\frac{\delta_\psi}{b\delta_\gamma}\frac{d_{GG}}{d_+d_-}\ , \tag{16c}$$

$$\frac{\partial B_\gamma}{\partial K^1_G}=-\frac{\delta_\psi}{b\delta_\gamma}\frac{d_{\psi G}}{d_+d_-}\ , \tag{16d}$$

$$\frac{\partial B_\gamma}{\partial K^1_{\rm NS}}=\frac{\delta_{\rm NS}}{b\delta_\gamma}\frac{1}{d_{\rm NS}}\ . \tag{16e}$$

These equations, simultaneously with Eqs. (13)—(15), can be integrated to give the quantities B_j ($j=\psi$, G, NS and γ) as functions of the scheme-labeling parameters and of their corresponding scheme invariants:

$$B_\psi(Q/M)=d_{\psi\psi}b\ln(Q/M)+\kappa_\psi\ , \tag{17a}$$

$$B_G(Q/M)=d_{\psi G}b\ln(Q/M)+\kappa_G\ , \tag{17b}$$

$$B_{\rm NS}(Q/M)=d_{\rm NS}b\ln(Q/M)+\kappa_{\rm NS}\ , \tag{17c}$$

$$B_\gamma(Q/M)=d_\gamma b\ln(Q/M)+[\delta_{\rm NS}K^1_{\rm NS}/d_{\rm NS}+\delta_\psi(K^1_\psi d_{GG}-K^1_G d_{\psi G})/(d_{\psi\psi}d_{GG}-d_{\psi G}d_{G\psi})]/b\delta_\gamma+\kappa_\gamma\ , \tag{17d}$$

where

$$d_\gamma\equiv(\delta_\psi K^0_\psi+\delta_{\rm NS}K^0_{\rm NS})/b\delta_\gamma\ , \tag{18}$$

and κ_ψ, κ_G, $\kappa_{\rm NS}$, and κ_γ are invariants, independent of both the FS and RS, that are calculable in some calculational schemes. Formulas to get these scheme invariants κ's are given in Appendix C.

Finally we get the optimization equations by setting the variations (B1)—(B5) to be exactly zero after the substitution of the consistency equations. These equations, obtained through straightforward manipulations, are quite complicated, and are not reproduced here. Note, however, that the numerical optimization studied in the next section is carried out by solving these exact optimization equations. In place of them we present here for later convenience the optimization equations obtained in the limit of small coupling constant ($a\approx\tilde{a}\approx 0$),

$$-(\delta_\psi K^0_\psi+\delta_{\rm NS}K^0_{\rm NS})b\ln\frac{M}{\mu}+\delta_{\rm NS}K^0_{\rm NS}\frac{B_{\rm NS}+b\ln M/\mu}{1+d_{\rm NS}}+\delta_\psi K^0_\psi\frac{(1+d_{GG})(B_\psi+b\ln M/\mu)-d_{G\psi}B_G}{(1+d_+)(1+d_-)}$$
$$+\delta_\psi K^1_\psi+\delta_{\rm NS}K^1_{\rm NS}=O(a)\ , \tag{19a}$$

$$\delta_\psi K^0_\psi\frac{(1+d_{GG})B_\psi-d_{G\psi}B_G}{(1+d_+)(1+d_-)}+\delta_{\rm NS}K^0_{\rm NS}\frac{B_{\rm NS}}{1+d_{\rm NS}}=O(a)\ , \tag{19b}$$

$$d_{GG}B_\psi-d_{G\psi}B_G=O(a)\ , \tag{19c}$$

$$-d_{\psi G}B_\psi+d_{\psi\psi}B_G=O(a)\ , \tag{19d}$$

$$B_{\rm NS}=O(a)\ . \tag{19e}$$

Equation (19a) is obtained by setting to zero the variation of $\hat{F}^\gamma$ with respect to the change of the factorization scale M, i.e., $\partial\hat{F}^\gamma/\partial \ln M=0$, Eq. (19b) by setting the variation $\partial\hat{F}^\gamma/\partial \ln\mu$ to zero, Eq. (19c) by setting $\partial\hat{F}^\gamma/\partial K_\psi^1=0$, Eq. (19d) by $\partial\hat{F}^\gamma/\partial K_G^1=0$, and Eq. (19e) by $\partial\hat{F}^\gamma/\partial K_{\rm NS}^1=0$, respectively.

III. OPTIMIZATION

A. Choice of the optimization variables

In this section we give, by solving the optimization equations, the result of the two-scale formalism of OPT applied to the second-order calculation of the photon structure function and study the consequences. In solving the optimization equations, however, a little care should be taken. By way of illustration let us consider the optimization equations in the small coupling limit, Eqs. (19). By substituting Eqs. (17) into Eqs. (19) we can easily see that each of the four equations (19b)–(19e) works as an equation that determines the optimal value of the factorization scale M, and that they are not mutually consistent: Among the four parameters μ and K_i^1 $(i=\psi,G,\text{NS})$ only one of them (or one linear combination of them) can be chosen as an independent variable and the remaining three parameters must be fixed at the outset. This is a reflection of the fact that the parameters labeling scheme dependences generally form an overcomplete set of variables in the OPT program.[8] Note also that the anomalous dimension K_G^1 cannot be chosen as an optimization variable because the optimization equations (19a)–(19d) do not contain K_G^1, thus cannot determine its optimal value.

Thus there are essentially only three choices of the optimization variables: (A) (M,μ), (B) $(M,K_{\rm NS}^1)$, and (C) (M,K_ψ^1). Parameters not listed are fixed at the beginning. Choice (C) gives essentially the same result as (B). In the following we shall mainly study the two choices (A) and (B) with the remaining parameters being fixed as

$$(\mathrm{A}_0)\quad (M,\mu),\quad K_\psi^1=K_G^1=K_{\rm NS}^1=0\ ; \tag{20a}$$

$$(\mathrm{B}_0)\quad (M,K_{\rm NS}^1),\quad \mu=M,\ K_\psi^1=K_G^1=0\ . \tag{20b}$$

It may be instructive to mention the following fact: If we set $\mu=M$ at the outset, namely, in the starting formula (11), and apply OPT, then we get a set of four optimization equations. In the small coupling limit three of them are nothing but Eqs. (19c)–(19e), and the fourth equation that comes from setting the variation $\partial\hat{F}^\gamma/\partial \ln M$ to zero is just the difference between Eqs. (19a) and (19b), i.e.,

$$\delta_\psi K_\psi^1+\delta_{\rm NS}K_{\rm NS}^1=O(a)\ . \tag{21}$$

Note that this equation does not involve B_i $(i=\psi,G,\text{NS})$. Thus even if we fix K_i^1's at the outset to be consistent with Eq. (21) and carry out the optimization with a single optimization variable $M(=\mu)$, we can determine neither the optimal values of B_i's nor that of M. This fact explains why we have been unable to apply the original single-scale OPT to the RS-dependent second-order calculation of the photon structure function,[6] which does not show any peculiarities in the present two-scale formalism of OPT.

B. Numerical optimization

First we present the results obtained by solving the exact optimization equations, which can be carried out only through the numerical computations. Hereafter we consider QCD with four quark flavors, $n_f=4$. The optimized coupling constants $\tilde{a}$ and a that are the solutions to the equations[20]

$$|b|\ln(\mu/\tilde{\Lambda})=1/\tilde{a}+c\ln[c\tilde{a}/(1+c\tilde{a})]\ , \tag{22a}$$

$$|b|\ln(M/\tilde{\Lambda})=1/a+c\ln[ca/(1+ca)]\ , \tag{22b}$$

are calculated referring to the starting calculational scheme, for which we choose the modified minimal-subtraction scheme[4] ($\overline{\rm MS}$), but are of course independent of such a scheme. The scale parameter $\tilde{\Lambda}$ defined in Eqs. (22) (Refs. 6 and 9) is related with the conventional scale-parameter[21,4] Λ by[6]

$$\tilde{\Lambda}=\Lambda(2c/|b|)^{-c/|b|}\simeq 1.118\Lambda\ , \tag{23}$$

and we use the value of the scale parameter[3] $\Lambda=\Lambda_{\overline{\rm MS}}=0.2$ GeV.

Explicit optimization is carried out for the moments of the structure function $\hat{F}_n^\gamma$, Eq. (11), where n runs over the range $3\le n\le 50$. Because we are not concerned with the problem of the large corrections for $x\sim 0$, we do not consider the second moment $\hat{F}_{n=2}^\gamma$. In Fig. 1 we have plotted the quantity $\hat{F}_n^\gamma\equiv -bF_n^\gamma/e^4$ for the leading-order calculation,[1] for the second-order calculation in the $\overline{\rm MS}$ scheme,[2] and for the optimized second-order calculation. The leading-order approximation[22] of the optimized second-order result is also plotted for comparison. Our results for the optimized structure function are, as they should be, essentially independent of the choice of the optimization variables: Two choices (A_0) and (B_0) predict the opti- mized moments that differ at worst only at the fifth significant figures, meaning that both choices give the equivalent predictions within the present accuracy level of the numerical computations. From Fig. 1 we can see that the effect of the optimization differs between the small-n and large-n moments, namely, the optimization works for the small-n moments so as to increase their absolute values while it works so as to reduce their values for the large-n moments.

In order to see such an effect of the optimization more clearly we give in Table II the optimized to original ratio of the structure function, $\hat{F}_n^\gamma(\text{OPT})/\hat{F}_n^\gamma(\text{BB})$. From Table II we recognize that the effect of the optimization really changes around $n=5$–7: The larger n becomes, the larger the effect of the optimization becomes so as to reduce the absolute values of the moments. This fact suggests that the optimization has also been able to successfully deal with the problem of the large second-order corrections for the large-n moments, i.e., the large corrections for $x\sim 1$. This point is quite interesting and is to be studied in Sec. III C in more detail in connection with the kinematical approaches.[13,14] Here we show for convenience how the structure function in the x space changes through the optimization.

We have used the method proposed by Yndurain[23] for inverting the moments to the x-space structure function

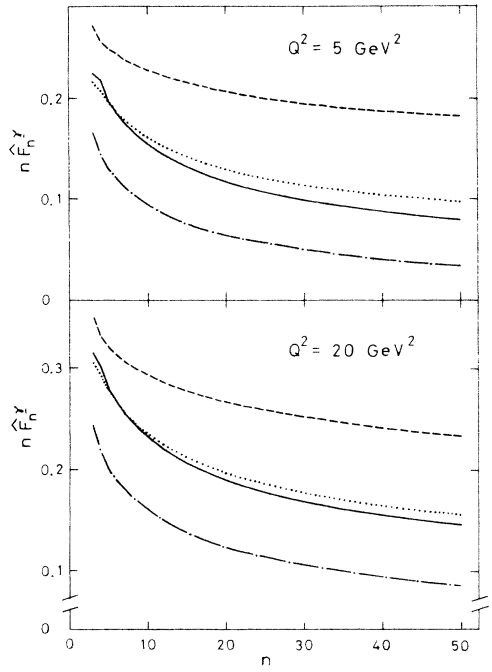

FIG. 1. Moments of the photon structure functions $\hat{F}_n^\gamma$ in units of $e^4/|b|$, predicted by the leading-order QCD calculations (Ref. 1) (dashed curve), the second-order calculations in the $\overline{\text{MS}}$ scheme (Ref. 2) (dotted curve), the optimized second-order calculations (solid curve), and the leading-order approximation of the optimized calculations, see text (dash-dotted curve). The predictions are for $\Lambda=0.2$ GeV and four flavors. We have plotted n times the structure-function moments, $n\hat{F}_n^\gamma$ for convenience.

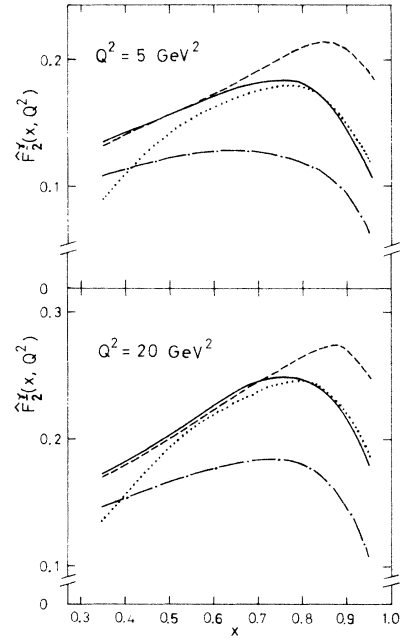

FIG. 2. Photon structure function $\hat{F}_2^\gamma(x,Q^2)$ in units of $e^4/|b|$ as predicted by QCD calculations in the leading order (dashed curve), in the second order in the $\overline{\text{MS}}$ scheme (dotted curve), in the optimized second order (solid curve), and in the leading-order approximation of the optimized second-order result (dash-dotted curve). Predictions are for $\Lambda=0.2$ GeV and four flavors.

$\hat{F}_2^\gamma(x,Q^2)$. Information of the moments over the range $3\leq n\leq 50$ was used for this purpose. In Fig. 2 the results are given, together with the results for the leading-order and for the original second-order calculations. The structure function reconstructed from the leading-order approximation[22] of the optimized second-order result is also plotted for comparison. Through the optimization the structure function has been significantly reduced for large x but has been significantly enhanced for smaller x ($x\lesssim 0.6$).

Finally we show how the optimization improves the convergence of the perturbation expansion by studying the second-order perturbation coefficients. The second-order expression of $\hat{F}_n^\gamma$ expanded in powers of the effective coupling constant has the form

$$\hat{F}_n^\gamma(Q^2)=\frac{a_n}{a(M(Q))}+b_n\ , \tag{24}$$

where the second-order coefficients b_n depend on the schemes used, whereas the leading-order ones a_n do not [the coefficients a_n defined in (24) and (27) below differ from those given in Ref. 2 by a factor $1/8\pi^2$, i.e., $a_n=a_n^{\text{BB}}/8\pi^2$]. In Table III we give the ratios b_n/a_n and $b_n/[a_n/a(M(Q))]$ for the original $\overline{\text{MS}}$ calculation and for the optimized result.[24] The optimized coupling constant $a(M_{\text{OPT}}(Q))$ is evaluated via Eq. (22b) at $Q^2=5$ and 20 GeV^2, where the coupling constant in the $\overline{\text{MS}}$ scheme takes the value 0.1234 and 0.0940, respectively. We see that the optimized second-order coefficients have been reduced drastically compared to the $\overline{\text{MS}}$ results. It seems also interesting that the optimized second-order coefficients are *always positive* for large n and thus the optimized perturbation expansion may behave as the ordinary positive term series, whereas in the original $\overline{\text{MS}}$ calculations the second-order coefficients are *always negative* and thus the perturbation series behaves as alternating series.

TABLE II. The ratio of the optimized to the original second-order calculations of the photon structure function, $\hat{F}_n^\gamma(\text{OPT})/\hat{F}_n^\gamma(\text{BB})$. The original calculations are those in the $\overline{\text{MS}}$ scheme by Bardeen and Buras (Ref. 2) with $\Lambda_{\overline{\text{MS}}}=0.2$ GeV.

n	5 GeV^2	20 GeV^2
3	1.042	1.034
5	1.001	1.006
6	0.989	1.000
10	0.959	0.986
15	0.933	0.975
20	0.911	0.967
25	0.892	0.960
30	0.875	0.954
35	0.859	0.949
40	0.845	0.944
45	0.831	0.940
50	0.818	0.936

TABLE III. The second-order to the leading-order ratios b_n/a_n and $b_n/[a_n/a(M(Q))]$ for the original $\overline{\text{MS}}$ (Ref. 2) and for the optimized calculations. The two-loop effective coupling constant in the $\overline{\text{MS}}$ scheme is calculated with $M=Q$ and $\Lambda_{\overline{\text{MS}}}=0.2$ GeV.

	b_n/a_n		$b_n/[a_n/a(M(Q))]$			
			$Q^2=5$ GeV^2		$Q^2=20$ GeV^2	
n	OPT results	BB	OPT results	BB	OPT results	BB
3	−0.688	−4.482	−0.0735	−0.553	−0.0539	−0.421
5	−0.140	−4.524	−0.0174	−0.558	−0.0123	−0.425
10	0.031	−5.403	0.0046	−0.667	0.0031	−0.508
15	0.112	−5.924	0.0187	−0.731	0.0119	−0.557
20	0.193	−6.259	0.0352	−0.772	0.0215	−0.588
25	0.272	−6.496	0.0535	−0.801	0.0315	−0.611
30	0.346	−6.675	0.0730	−0.824	0.0416	−0.628
35	0.416	−6.817	0.0933	−0.841	0.0515	−0.641
40	0.481	−6.933	0.1142	−0.855	0.0612	−0.652
45	0.542	−7.030	0.1357	−0.867	0.0707	−0.661
50	0.598	−7.113	0.1577	−0.878	0.0798	−0.669

C. Optimization in the large-n limit

In order to see how the optimization has dealt with the problem of the large second-order corrections for the large-n moments, or the large corrections for $x\sim 1$, we study here the optimization in the large-n limit.[25] In this limit the anomalous-dimension matrix $\hat{\gamma}_n$ [see Eq. (5)] becomes a diagonal matrix, and we can carry out the optimization analytically.

By keeping the *leading terms in the large-n limit* for quantities appearing in the optimization equations, we get the optimal value of the factorization scale $M_{\text{OPT}}\equiv M_{\text{OPT}}(Q)$ as

$$\ln(M_{\text{OPT}}/Q)\simeq -\tfrac{1}{4}\ln n\ , \tag{25}$$

and the optimized coupling constant $a_{\text{OPT}}\equiv a(M_{\text{OPT}})$ that is the solution to Eq. (22b) as

$$a_{\text{OPT}}=1/|b|\ln(Qn^{-1/4}/\tilde{\Lambda}) + O\left(\frac{\ln\ln(Qn^{-1/4}/\tilde{\Lambda})}{\ln^2(Qn^{-1/4}/\tilde{\Lambda})}\right)\ . \tag{26}$$

Expressing the optimized structure function $\hat{F}_n^{\gamma}(\text{OPT})$ as

$$\hat{F}_n^{\gamma}(\text{OPT})\simeq\frac{a_n}{a_{\text{OPT}}}+b_n(\text{OPT})\ , \tag{27}$$

we also get

$$b_n(\text{OPT})/a_n\simeq\tfrac{2}{3}\ln n+O(1)\ , \tag{28}$$

which should be compared with the result in the $\overline{\text{MS}}$ calculation[2,16]

$$b_n(\overline{\text{MS}})/a_n\simeq(b/4+\tfrac{2}{3})\ln n+O(1)\ . \tag{29}$$

Namely, through the optimization, *part of the second-order corrections, i.e., the first term in (29), have been absorbed into the effective coupling constant* $a_{\text{OPT}}\equiv a(M_{\text{OPT}})$, Eq. (26).

With Eqs. (26) and (27) we can also show, by following the analysis by Frazer and Rossi,[16] that the structure function in x space can be reexpressed as

$$\hat{F}_2^{\gamma}(x,Q^2)\simeq h(x)/a((1-x)^{1/2}Q^2/\tilde{\Lambda}^2)+\tilde{H}(x)\ , \tag{30}$$

where $\hat{F}_0^{\gamma}=h(x)/a(Q^2/\tilde{\Lambda}^2)$ is the large-x form of the leading-order result. Note that Eqs. (28) and (30) are the same results as were obtained through the analysis of phase-space boundary effects.[14,16] We can perform the same analyses for the hadronic structure functions, where we get the same results, namely, the optimized coupling constant eats the leading large corrections for $x\sim 1$ coming from the phase-space boundary effects.

Thus we can conclude that through the optimization based on the principle of minimal sensitivity (PMS) we can successfully deal with the large corrections that come from the kinematical boundary effects. This observation may justify the validity of OPT and shed some light on the question of what the PMS optimization does, not in the context of a mathematical prescription but as a physical effect.

IV. CONCLUSIONS AND DISCUSSION

In this paper we applied the optimized perturbation theory (OPT) to the second-order QCD calculation of the photon structure function. The result shows that the two-scale formalism of OPT is able to resolve the scheme-dependent ambiguities inherent in the perturbative QCD calculation of the photon structure function, thus giving a unique QCD prediction. In this respect we call attention to the fact[6] that the original formalism of OPT, which has taken into account only the RS dependence, cannot optimize the second-order calculation of the photon structure function. The reason why such a problem occurred can be understood in terms of the two-scale formalism of OPT. The second-order calculation of the photon structure function has, thus, served as a testing ground to prove the necessity as well as the sufficiency of

the simultaneous investigation of the RS and FS dependences in order to resolve the scheme-dependent ambiguities.

There is another important observation found through the present analysis. The OPT based on the principle of minimal sensitivity (PMS) has successfully dealt with the large second-order corrections in the large-n moments (i.e., the large corrections for $x \sim 1$) precisely the same way as the kinematical approaches have done,[14,16] *at least in the leading terms in the large-n limit.* This is also true for the hadronic structure functions. Thus *for the photon case only a portion of the leading second-order corrections in the large-n limit has been absorbed into the optimized effective coupling constant,* which "runs" in terms of the rescaled mass parameter $M_n^2(\mathrm{OPT}) \simeq Q^2 n^{-1/2}$ in the moment-space, or $M^2(\mathrm{OPT}) \simeq Q^2(1-x)^{1/2}$ in the x-space representation. The same rescaled mass parameter also appears in the optimized hadronic structure functions, and is of course consistent with the "prescription" in the kinematical approaches where the argument of the effective coupling constant *at each vertex of the ladder rung* is replaced as $Q^2 \to Q^2(1-y)$.

At present we have no idea why such "equivalence" between the optimization and the kinematical analyses has held.[26] As noted above, this equivalence has been proved only for the leading correction terms in the large-n limit. The only thing we can say now is that the OPT based on PMS may have correctly taken into account the kinematical boundary effects and that the optimized perturbation expansion shows an improved convergence behavior. This fact may justify the validity of the optimization based on PMS, giving it another physical insight.

Finally we comment on the possible breakdown of the perturbation expansion that may occur for the large-n moments. We have shown that through the optimization the dominant correction terms in the large-n limit have been absorbed into the coupling constant and thus the optimized second-order corrections become moderate. Then what happens when the $\overline{\mathrm{MS}}$ calculation gives the negative values for the structure function, as is well known to occur for large n, or large x? The answer is as follows. In OPT, though the optimized second-order coefficients become moderate, Eq. (22), which determines the optimal coupling constant, as n becomes sufficiently large comes to have no solutions, indicating that the optimal coupling constant exceeds unity, $a_{\mathrm{OPT}} \gtrsim 1$, and that the perturbation expansion breaks down. Thus, contrary to the $\overline{\mathrm{MS}}$ calculation we now have a definite criterion to decide whether or not the perturbation expansion makes sense. The region where the $\overline{\mathrm{MS}}$ calculation predicts the negative values for $\hat{F}_n^{\gamma}$ can be shown to lie outside the convergence domain of the optimized perturbation expansion. In this sense OPT is self-consistent and predicts nothing that contradicts the physical requirement

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APPENDIX A: ANALYSIS OF A SIMPLIFIED MODEL OF THE PHOTON STRUCTURE FUNCTION

Here we give an analysis of a simplified model of the photon structure function and show that the application of OPT actually chooses a scheme where the two-loop anomalous dimensions vanish. The model considered is such that the moment of the photon structure function is given as (the moment index n is suppressed)

$$F^{\gamma}(Q^2) = \vartheta(a(M)) \cdot \mathscr{C}(Q/\mu, M/\mu, \tilde{a}(\mu)) , \tag{A1}$$

where ϑ and $\mathscr{C}$ are two-component row and column vectors, respectively,

$$\vartheta(a(M)) = (\vartheta_{\mathrm{NS}}(a(M)), \vartheta_{\gamma}(a(M))) , \tag{A2}$$

$$\mathscr{C}(Q/\mu, M/\mu, \tilde{a}(\mu)) = \begin{bmatrix} \mathscr{C}_{\mathrm{NS}}(Q/\mu, M/\mu, \tilde{a}(\mu)) \\ \mathscr{C}_{\gamma}(Q/\mu, M/\mu, \tilde{a}(\mu)) \end{bmatrix} . \tag{A3}$$

Then ϑ satisfies a differential equation

$$\frac{\partial \vartheta(a(M))}{\partial \ln M} = \vartheta(a(M)) \gamma(a(M)) , \tag{A4}$$

where γ denotes a 2×2 anomalous-dimension matrix

$$\gamma = \begin{bmatrix} \gamma_{\mathrm{NS}} & 0 \\ K_{\mathrm{NS}} & 0 \end{bmatrix} . \tag{A5}$$

With the use of the renormalization-group equation

$$\frac{da(M)}{d \ln M} = \beta(a(M)), \; a(M = M_0) = g_M^2/4\pi^2 , \tag{A6}$$

Eq. (A4) can be integrated to give

$$\vartheta_{\mathrm{NS}}(a(M)) = \vartheta_{\mathrm{NS}}^0 \exp\left[\int_{a_0}^{a(M)} dx \frac{\gamma_{\mathrm{NS}}(x)}{\beta(x)} \right] + \vartheta_{\gamma}^0 \int_1^{a(M)} dx \frac{K_{\mathrm{NS}}(x)}{\beta(x)} \times \exp\left[\int_x^{a(M)} dy \frac{\gamma_{\mathrm{NS}}(y)}{\beta(y)} \right] . \tag{A7}$$

Using the perturbative expressions for various quantities [analogous to Eqs. (9) in the text], we can get the second-order expression for the photon structure function

$$\widetilde{F}_\gamma \equiv F^\gamma/e^4$$

$$=\delta_\gamma B_\gamma+\int_1^a dx\frac{K^0_{\rm NS}+K^1_{\rm NS}x}{bx^2(1+cx)}\left[\frac{a(1+cx)}{x(1+ca)}\right]^{\gamma^0_{\rm NS}/b}\left[\frac{1+ca}{1+cx}\right]^{\gamma^1_{\rm NS}/bc}\delta_{\rm NS}(1+B_{\rm NS}\tilde{a})$$

$$=\delta_\gamma B_\gamma-\frac{\delta_{\rm NS}}{b}(1+B_{\rm NS}\tilde{a})\left[\frac{ca}{1+ca}\right]^{\gamma^0_{\rm NS}/b}(1+ca)^{\gamma^1_{\rm NS}/bc}\int_{ca/(1+ca)}^{c/(1+c)}dz[cK^0_{\rm NS}z^{-\gamma^0_{\rm NS}/b-2}(1-z)^{\gamma^1_{\rm NS}/bc+1}$$

$$+K^1_{\rm NS}z^{-\gamma^0_{\rm NS}/b-1}(1-z)^{\gamma^1_{\rm NS}/bc}]\,, \tag{A8}$$

where we have neglected the hadronic contributions [i.e., the first term in Eq. (A7)], as explained in some detail in the text. In this model a set of parameters labeling the scheme dependences is $\{\mu, M, K^1_{\rm NS}$, and $\gamma^1_{\rm NS}\}$, and the quantities to be optimized are $B_{\rm NS}$, B_γ, and the effective coupling constants $a=a(M)$ and $\tilde{a}=\tilde{a}(\mu)$.

In applying OPT we take here all the above four parameters as independent variables. Now it is easy to show that the optimization really chooses such a scheme where $\gamma^1_{\rm NS}=0$. For this purpose let us evaluate the response of $\widetilde{F}_\gamma$ with respect to changes of μ and $K^1_{\rm NS}$. They become (the shorthand notations $d^0\equiv\gamma^0_{\rm NS}/b$ and $d^1\equiv\gamma^1_{\rm NS}/bc$ are used)

$$\frac{\partial\widetilde{F}_\gamma}{\partial\ln\mu}=\delta_\gamma\frac{\partial B_\gamma}{\partial\ln\mu}-\delta_{\rm NS}B_{\rm NS}\tilde{a}^2(1+c\tilde{a})\left[\frac{ca}{1+ca}\right]^{d^0}(1+ca)^{d^1}$$

$$\times\int_{ca/(1+ca)}^{c/(1+c)}dz[cK^0_{\rm NS}z^{-d^0-2}(1-z)^{d^1+1}+K^1_{\rm NS}z^{-d^0-1}(1-z)^{d^1}]\,, \tag{A9}$$

$$\frac{\partial\widetilde{F}_\gamma}{\partial K^1_{\rm NS}}=\delta_\gamma\frac{\partial B_\gamma}{\partial K^1_{\rm NS}}-\frac{\delta_{\rm NS}}{b}(1+B_{\rm NS}\tilde{a})\left[\frac{ca}{1+ca}\right]^{d^0}(1+ca)^{d^1}\int_{ca/(1+ca)}^{c/(1+c)}dz\,z^{-d^0-1}(1-z)^{d^1}\,, \tag{A10}$$

where we have made use of Eqs. (13) and (15) in the text. Imposing the responses (A9) and (A10) to be of order a, we get the consistency equations

$$\frac{\partial B_\gamma}{\partial\ln\mu}=0,\quad\frac{\partial B_\gamma}{\partial K^1_{\rm NS}}=\frac{\delta_{\rm NS}}{\delta_\gamma\gamma^0_{\rm NS}}\,. \tag{A11}$$

Next we set the responses to be exactly zero, and obtain the optimization equations

$$B_{\rm NS}\tilde{a}^2(1+c\tilde{a})\left[\frac{ca}{1+ca}\right]^{d^0}(1+ca)^{d^1}\int_{ca/(1+ca)}^{c/(1+c)}dz[cK^0_{\rm NS}z^{-d^0-2}(1-z)^{d^1+1}+K^1_{\rm NS}z^{-d^0-1}(1-z)^{d^1}]=0\,, \tag{A12}$$

$$d^0\left[\frac{ca}{1+ca}\right]^{d^0}(1+ca)^{d^1}\int_{ca/(1+ca)}^{c/(1+c)}dz\,z^{-d^0-1}(1-z)^{d^1}=1\,. \tag{A13}$$

The solutions in the physically acceptable range to these equations are

$$B_{\rm NS}=0\,, \tag{A14}$$

which hold exactly, and

$$d^1\equiv\gamma^1_{\rm NS}/bc=0\,, \tag{A15}$$

which is exact up to the hadronic contributions.

In the present model we can carry out, with the straightforward manipulations, the complete optimization explicitly, but the results are not necessary for the present purpose and thus not reproduced.

APPENDIX B: THE RESPONSE OF STRUCTURE FUNCTION OF THE PHOTON TO CHANGES OF THE SCHEME-LABELING PARAMETERS

In this appendix we write down the response equations of the photon structure function:

$$\frac{\partial \hat{F}^{\gamma}}{b\,\partial \ln M}=\left[\delta_{\psi}K_{\psi}^{0}\left[\frac{1+d_{GG}}{(1+d_{+})(1+d_{-})}\frac{\partial B_{\psi}}{b\,\partial \ln M}-\frac{d_{G\psi}}{(1+d_{+})(1+d_{-})}\frac{\partial B_{G}}{b\,\partial \ln M}\right]+\delta_{\rm NS}K_{\rm NS}^{0}\frac{1}{1+d_{\rm NS}}\frac{\partial B_{\rm NS}}{b\,\partial \ln M}\right]\frac{\tilde{a}}{a}(1+ca)$$
$$-\left[\delta_{\psi}K_{\psi}^{0}\left[\frac{1+d_{GG}}{(1+d_{+})(1+d_{-})}(1+B_{\psi}\tilde{a})-\frac{d_{G\psi}}{(1+d_{+})(1+d_{-})}B_{G}\tilde{a}\right]+\delta_{\rm NS}K_{\rm NS}^{0}\frac{1}{1+d_{\rm NS}}(1+B_{\rm NS}\tilde{a})\right](1+ca)$$
$$+\delta_{\psi}(K_{\psi}^{1}-cK_{\psi}^{0})\left[\frac{d_{GG}}{d_{+}d_{-}}\frac{\partial B_{\psi}}{b\,\partial \ln M}-\frac{d_{G\psi}}{d_{+}d_{-}}\frac{\partial B_{G}}{b\,\partial \ln M}\right]\tilde{a}+\delta_{\psi}K_{G}^{1}\left[\frac{-d_{\psi G}}{d_{+}d_{-}}\frac{\partial B_{\psi}}{b\,\partial \ln M}+\frac{d_{\psi\psi}}{d_{+}d_{-}}\frac{\partial B_{G}}{b\,\partial \ln M}\right]\tilde{a}$$
$$+\delta_{\rm NS}(K_{\rm NS}^{1}-cK_{\rm NS}^{0})\frac{1}{d_{\rm NS}}\frac{\partial B_{\rm NS}}{b\,\partial \ln M}\tilde{a}-b\delta_{\gamma}\frac{\partial B_{\gamma}}{b\,\partial \ln M}\ , \tag{B1}$$

$$\frac{\partial \hat{F}^{\gamma}}{b\,\partial \ln \mu}=\left[\delta_{\psi}K_{\psi}^{0}\frac{(1+d_{GG})B_{\psi}-d_{G\psi}B_{G}}{(1+d_{+})(1+d_{-})}+\delta_{\rm NS}K_{\rm NS}^{0}\frac{B_{\rm NS}}{1+d_{\rm NS}}\right]\tilde{a}^{2}(1+c\tilde{a})\frac{1+ca}{a}$$
$$-\delta_{\psi}(K_{\psi}^{1}-cK_{\psi}^{0})\frac{d_{GG}B_{\psi}-d_{G\psi}B_{G}}{d_{+}d_{-}}\tilde{a}^{2}(1+c\tilde{a})-\delta_{\psi}K_{G}^{1}\frac{-d_{\psi G}B_{\psi}+d_{\psi\psi}B_{G}}{d_{+}d_{-}}\tilde{a}^{2}(1+c\tilde{a})$$
$$-\delta_{\rm NS}(K_{\rm NS}^{1}-cK_{\rm NS}^{0})\frac{B_{\rm NS}}{d_{\rm NS}}\tilde{a}^{2}(1+c\tilde{a})-b\delta_{\gamma}\frac{\partial B_{\gamma}}{b\,\partial \ln \mu}\ , \tag{B2}$$

$$\frac{\partial \hat{F}^{\gamma}}{\partial K_{\psi}^{1}}=\frac{\delta_{\psi}}{d_{+}d_{-}}[d_{GG}+(d_{GG}B_{\psi}-d_{G\psi}B_{G})\tilde{a}]-b\delta_{\gamma}\frac{\partial B_{\gamma}}{\partial K_{\psi}^{1}}\ , \tag{B3}$$

$$\frac{\partial \hat{F}^{\gamma}}{\partial K_{G}^{1}}=\frac{\delta_{\psi}}{d_{+}d_{-}}[-d_{\psi G}+(-d_{\psi G}B_{\psi}+d_{\psi\psi}B_{G})\tilde{a}]-b\delta_{\gamma}\frac{\partial B_{\gamma}}{\partial K_{G}^{1}}\ , \tag{B4}$$

$$\frac{\partial \hat{F}^{\gamma}}{\partial K_{\rm NS}^{1}}=\frac{\delta_{\rm NS}}{d_{\rm NS}}(1+B_{\rm NS}\tilde{a})-b\delta_{\gamma}\frac{\partial B_{\gamma}}{\partial K_{\rm NS}^{1}}\ , \tag{B5}$$

where we have made use of Eqs. (13) and (15) in the text.

APPENDIX C: SCHEME-INVARIANT FORMULAS

We give in this appendix the formulas to calculate the scheme invariants κ_i ($i=\psi$, G, NS, and γ). These formulas can be obtained by solving simultaneously the consistency equations obtained through the analyses of the hadronic as well as the photon structure functions. Such analyses of the hadronic structure functions have been already done in Ref. 9 for the nonsinglet case, and can be done with straightforward but tedious manipulations for the singlet case with the help of the results by Bardeen *et al.*[4] The solutions to these consistency equations, namely, the formulas to calculate the scheme invariants are as follows:

$$\kappa_{\rm NS}=B_{\rm NS}-d_{\rm NS}b\ln(Q/M)+d_{\rm NS}^{1}\ , \tag{C1}$$

$$\kappa_{\psi}=B_{\psi}-d_{\psi\psi}b\ln(Q/M)-\tilde{B}_{\psi}\ , \tag{C2}$$

$$\kappa_{G}=B_{G}-d_{\psi G}b\ln(Q/M)-\tilde{B}_{G}\ , \tag{C3}$$

$$\kappa_{\gamma}=B_{\gamma}-d_{\gamma}b\ln(Q/M)-\tilde{B}_{\gamma}\ , \tag{C4}$$

where

$$\tilde{B}_{\psi}=-[1-(d_{\psi\psi}-d_{GG})^{2}-4d_{G\psi}d_{\psi G}]^{-1}$$
$$\times\{[1-(d_{\psi\psi}-d_{GG})^{2}-2d_{G\psi}d_{\psi G}]d_{\psi\psi}^{1}+[1-(d_{\psi\psi}-d_{GG})]d_{\psi G}d_{G\psi}^{1}-(1+d_{\psi\psi}-d_{GG})d_{G\psi}d_{\psi G}^{1}-2d_{G\psi}d_{\psi G}d_{GG}^{1}\}\ , \tag{C5}$$

$$\tilde{B}_{G}=[1-(d_{\psi\psi}-d_{GG})^{2}-4d_{G\psi}d_{\psi G}]^{-1}$$
$$\times[(1+d_{\psi\psi}-d_{GG})d_{\psi G}d_{\psi\psi}^{1}-(1+d_{\psi\psi}-d_{GG}-2d_{G\psi}d_{\psi G})d_{\psi G}^{1}+2(d_{\psi G})^{2}d_{G\psi}^{1}-(1+d_{\psi\psi}-d_{GG})d_{\psi G}d_{GG}^{1}]\ , \tag{C6}$$

$$\tilde{B}_\gamma = -\frac{1}{b\delta_\gamma}\left[\delta_{\rm NS}\frac{K^0_{\rm NS}d^1_{\rm NS}-K^1_{\rm NS}}{d_{\rm NS}}-\delta_\psi\frac{K^1_\psi d_{GG}-K^1_G d_{\psi G}}{d_{\psi\psi}d_{GG}-d_{\psi G}d_{G\psi}}-\frac{\delta_\psi K^0_\psi}{1+d_{\psi\psi}+d_{GG}+d_{\psi\psi}d_{GG}-d_{\psi G}d_{G\psi}}\right.$$
$$\times\left[(1+d_{GG})\tilde{B}_\psi-d_{G\psi}\tilde{B}_G-\frac{1}{d_{\psi\psi}d_{GG}-d_{\psi G}d_{G\psi}}\right.$$
$$\left.\left.\times[(1+d_{GG})(d_{GG}d^1_{\psi\psi}-d_{\psi G}d^1_{G\psi})+d_{G\psi}(-d_{GG}d^1_{\psi G}+d_{\psi G}d^1_{GG})]\right]\right]. \quad \text{(C7)}$$

In the above equations, d^1's are essentially the two-loop anomalous dimensions, i.e., $d^1_{\rm NS}\equiv\gamma^1_{\rm NS}/b$, and $d^1_{ij}\equiv\gamma^1_{ij}/b$ $(i,j=\psi,G)$. In the MS and $\overline{\rm MS}$ schemes we have already had the expressions for[4,2] $B_{\rm NS}$, B_ψ, B_G, and B_γ and also for the two-loop anomalous dimensions $d^1_{\rm NS}$, d^1_{ij} $(i,j=\psi,G)$ (Ref. 27), and K^1_i $(i={\rm NS},\psi,G)$ (Ref. 2). Remembering the usual calculations in the schemes MS and $\overline{\rm MS}$, we understand that the factorization scale in Eqs. (C1)–(C4) are taken as $M=Q$. With this information we can get the expressions for scheme invariants κ_i $(i=\psi,\ G,\ {\rm NS},\ \text{and}\ \gamma)$ from Eqs. (C1)–(C4).

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