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Masslessness and light-cone propagation in 3+2 de Sitter and 2 + 1 Minkowski spaces

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The problem of determining the most natural criteria for defining masslessness in 3+2 de Sitter space is reexamined. There is no real difficulty, for all the expected features of masslessness that make sense in de Sitter space are mutually compatible. In this paper we point out that light-cone propagation, in particular, is perfectly compatible with other criteria. The situation is very different in three-dimensional space-time, even in the flat case. Here the conventional, Maxwellian form of electrodynamics fails in many respects: it is not conformally invariant, it does not have light-cone propagation, and it has catastrophic infrared behavior as can be inferred from the fact that the "Coulomb" potential is proportional to $\ln r$. It is pointed out that there exists an alternative form of three-dimensional electrodynamics that is conformally invariant, propagates on the light cone, and has a static potential proportional to $1/r$. Finally, we point out that some so-called massive, three-dimensional gauge theories have none but the most superficial features that characterize genuine gauge theories.

I. 3 + 2 DE SITTER SPACE

It is now quite universally believed that massless, Poincaré-covariant field theories are fundamental to the formulation of physical theories. Such theories need to be regularized to make sense, especially in the infrared region, and no completely satisfactory procedure is yet available. The formulation of massive gauge fields is unavailable as an expedient in non-Abelian gauge theories, and tricky even in the Abelian case. Finite volume cutoffs destroy covariance, something that (for good reasons) has not been tolerated in the ultraviolet regime for a long time. Regularization by the introduction of a small but nonzero constant curvature in space-time therefore becomes a very attractive possibility. The infrared region of 3 + 2 de Sitter-covariant field theories is not only strongly regularized compared to the flat case, but very interesting from several different points of view. The question of how to define masslessness in 3 + 2 de Sitter space is therefore far from being purely academic. Applications to supergravity provide strong, additional motivation.

A study of this question,[1] a few years ago, examined several criteria for masslessness that seemed especially important, without inquiring whether or not propagation is confined to the light cone. Here we shall see that light-cone propagation is, in fact, a feature of massless, de Sitter field theories with any spin. We begin by listing the criteria examined in Ref. 1. (a) The index by which to recognize masslessness that was proposed first, is the emergence of gauge structure that accompanies the breakdown of the Fierz-Pauli program.[2] This led, subsequently, to the construction of a family of gauge theories with spins $s \geq 1$ (Ref. 3). The physical, one-particle states of these theories carry the representations $D(s+1,s)$ of the 3 + 2 de Sitter group. (b) Conformal extensions were mentioned in Ref. 2, but fully investigated in Ref. 1. Massless, discrete helicity representations of the Poincaré group have (unique) extensions to unitary representations of the conformal group.[4,5] The only representations of the de Sitter group that have this property are $D(s+1,s)$ for $s \geq \frac{1}{2}$ and, in addition, $D(1,0)\oplus D(2,0)$. From now on, we refer to these as the massless representations. Note that they include all the physical representations of (conventional) de Sitter gauge theories, but that this does not imply that these gauge theories are conformally invariant. (It is not even automatic in the case $s=1$, electrodynamics.[2,6]) (c) It is certainly reasonable to expect that the massless representations of the de Sitter group contract smoothly to those of the Poincaré group in the limit of vanishing curvature. This is intimately related to the property of conformal extendability, as was shown in Ref. 1. Other aspects of this contraction were examined subsequently.[6] (d) Physical representations are unitary, and in the case of the semisimple groups they are minimal weight representations. The minimal weight is bounded by the requirement of unitarity, and gauge theories are associated precisely with the lower bound. For $s \geq 1$ the bound on $D(E_0,s)$ is $E_0 \geq s+1$, and this leads to the conventional gauge theories associated with massless particles. For $s=0,\frac{1}{2}$ the lower bound is $E_0 \geq s+\frac{1}{2}$. This explains why *massless* particles with spins 0 and $\frac{1}{2}$ are not gauge theories, and points to the extraordinary gauge theories of Dirac singletons: $\mathrm{Rac}=D(1/2,0)$ (Ref. 7), and $\mathrm{Di}=D(1,\frac{1}{2})$ (Ref. 8). These subparticles are the constituents of massless particles.[9,10] We have thus returned to gauge theories. We insist that the main feature of interest in massless field theories is the fact that they involve a nontrivial gauge structure that *necessitates* indefinite metric, Gupta-Bleuler quantization.

Deser and Nepomechie[11] have recently examined propagation in de Sitter field theories, some of them massless, by means of a transformation that appears to reduce the problem to a more familiar one in Minkowski space. This method led them to conclude that the connection between null-cone propagation and gauge invariance for spins higher than 1 is specific to flat space and that it is not maintained in a 3 + 2 de Sitter universe.

We now investigate the question of whether or not massless fields in de Sitter space propagate on the light cone, by direct examination of the propagators in de Sitter space. To be systematic, it is convenient to dispose, first of all, of the case of low spin, $s=0$ and $\frac{1}{2}$. We use the customary description of de Sitter space as a covering space of the hyperboloid $y_0{}^2-y_1{}^2-y_2{}^2-y_3{}^2+y_5{}^2=1/\rho$ in R^5, ρ being the positive curvature constant. The propagator is determined in terms of boundary values of an analytic function K of the variable $z=\rho y\cdot y'$, the two-point function. The locus of points y that are lightlike relative to y' is given by[12,13] $z=\pm 1$. In the case of spin zero the field is a scalar field ϕ, and K is the vacuum expectation value:

$$K(z)=\langle 0\,|\,\phi(y)\phi(y')^*\,|\,0\rangle,\quad z=\rho y\cdot y' .$$

This function is analytic in z except for singularities at $z=\pm 1$ and at infinity. Branch cuts at $z=\pm 1$ imply "reverberations;" that is, propagation in the interior of the light cone. Propagation is confined to the light cone if and only if K is meromorphic, with poles at $z=\pm 1$. In this case the vacuum expectation value of the commutator vanishes except on the light cone. The two-point function associated with the irreducible representation $D(E_0,0)$ was calculated long ago:[13]

$$K_{E_0}(z)\propto(z^2-1)^{-1/2}Q^1_{E_0-2}(z)\ , \tag{1.1}$$

where Q_l^m is a Legendre function of the second kind. For most values of E_0, this function has a logarithmic singularity at $z=1$. [Equation 3.9(5) of Ref. 14 does not apply when μ is integral.] The discontinuity across the cut that extends along the real axis from $+1$ to $-\infty$, in the region $-1<z<+1$, is [see Ref. 14, Eq. 3.4(8)]

$$(z^2-1)^{-1/2}P^1_{E_0-2}(z)=(\partial/\partial z)P_{E_0-2}(z)\ . \tag{1.2}$$

Propagation is confined to the light cone if and only if this function vanishes identically; that is, only in the special cases $E_0=1$ and $E_0=2$. We have

$$K_1(z)=\frac{z}{z^2-1}\ ,$$

$$K_2(z)=\frac{1}{z^2-1}\ .$$

The propagator for spin-$\frac{1}{2}$ fields in the representation $D(E_0,\frac{1}{2})$ is related to the spinless propagator for $D(E_0+\frac{1}{2},0)$ by differentiation.[13] Light-cone propagation therefore occurs if $E_0=\frac{1}{2}$ and if $E_0=\frac{3}{2}$. Of the two associated representations, only the massless representation $D(\frac{3}{2},\frac{1}{2})$ is unitary.

Turning now to the case of massless fields with spin $s\geq 1$, we must recognize that we are dealing with gauge theories, and therefore first point out the obvious: the question of whether propagation into the interior of the light cone occurs depends on the choice of gauge. However, the only matter of direct physical significance is the transmission of signals between two localized, gauge-invariant (conserved) sources. We claim that such signals are transmitted along the light cone only. The easiest way to prove this is to show that there exists a choice of gauge in which all propagation (even of the physically irrelevant gauge modes) is confined to the light cone. Indeed, such a choice does exist, as we shall show.

Propagators for spin-1, de Sitter electrodynamics, have been discussed in great detail already.[6] The field is a transverse vector field, $y\cdot\mathscr{A}(y)=0$ (other possibilities exist, see Ref. 6, Appendix). Maxwell's equations, with gauge fixing, are

$$\operatorname{tr}\operatorname{pr}\{[\partial^2-\rho(\hat{N}+1)(\hat{N}+2)]\mathscr{A}_a-c\partial_a\partial\cdot\mathscr{A}\}=\mathscr{J}_a\ .$$

(Here tr pr means transverse projection and $\hat{N}\equiv y\cdot\partial$.) This is gauge-invariant if $c=1$, while $c=0$ is the "Feynman gauge." The simplest choice of c is not zero but $\frac{2}{3}$, in which case the two-point function is one of two possibilities. The first one is

$$\langle 0\,|\,\mathscr{A}_a(y)\mathscr{A}_b(y')\,|\,0\rangle=\operatorname{tr}\operatorname{pr}[\delta_{ab}K_1(z)-\rho^{-1}\partial_a\partial'_bK_2(z)]\ , \tag{1.3}$$

and the other is found by exchanging K_1 and K_2. The questions posed by the existence of two options were discussed in detail in Ref. 6 (also in Ref. 2), and will be passed over here. Evidently only poles appear and *all* propagation is on the light cone in this gauge. However, with any other choice of gauge[15] ($c\neq\frac{2}{3}$ and $c\neq 1$) logarithms appear in the two-point functions and the discontinuity across the real axis in the complex z plane extends from 1 to $-\infty$; hence, propagation extends to the interior of the light cone. Since physics is gauge independent, only gauge modes contribute to this phenomenon. (The gauge modes are not themselves logarithmic, but their canonical conjugates, the scalar modes, are. The logarithms thus occur with gauge mode factors in the modal expansion of the propagators, and therefore do not contribute to physical matrix elements.)

Just as the spin-$\frac{1}{2}$ propagator can be expressed in terms of first-order derivatives of the spin-0 propagator, one expects that the spin-$\frac{3}{2}$ propagator can be written in terms of first-order derivatives of the spin-1 propagator. If this is correct, then in the gauge that is favored ($c=\frac{2}{3}$ for spin 1), spin-$\frac{3}{2}$ propagation will be confined to the light cone. We have not actually verified this in detail. Instead we have investigated the case of arbitrary, integer spin. It turns out that there is a favored choice of gauge, in which all modes propagate on the cone. In this gauge the propagator for spin s can be obtained by differentiation of the two spin-0 propagators K_1 and K_2. Detailed calculations and explicit expressions for the propagators have been relegated to the Appendix; here we just sketch the method. The spin-s wave equation has the form[3]

$$(Q-\langle Q\rangle)k-c_s\Sigma_1[\partial y^2+(s-\hat{N}-2)y]\xi=0\ . \tag{1.4}$$

Here Q is the so(3,2) Casimir operator, and $\langle Q\rangle=2s^2-2$ is its value in $D(s+1,s)$; k is a symmetric, double-traceless and transverse tensor field of rank s, and

$$\xi=\tfrac{1}{2}\Sigma_1[\partial y^2+(s-\hat{N}-5)y](1/y^2)k'-\partial\cdot k$$

is a symmetric, traceless and transverse tensor field of rank $s-1$; k' is the trace of k and the summation symmetries. This equation is gauge invariant if $c_s=1$. We now solve this equation recursively, expressing the solutions for spin s in terms of the solutions of the field equation for spin $(s-1)$. Supposing that the latter have no logarithms, we find that the former also have no logarithms, *provided* that $c_s=2/(2s+1)$ (see Ref. 16). Logarithms do appear for every other choice of c_s. With this unique choice of gauge, we can express the propagators for spin s as derivatives of lower spin propagators, and thus avoid propagation inside the light cone for all integer spins.

Conclusions. All known, reasonable criteria for masslessness in de Sitter space are mutually compatible. In particular, this includes the intuitive idea that the physical manifestations of massless fields ought to behave like light signals and propagate only on the light cone. The lowest spins, $s=0$ and $\frac{1}{2}$, are exceptional only in that they are not described by gauge theories. In this sense their role is taken over by the singletons. Conformal extendibility holds for the physical sector for all spins. This fact is compatible with, and at least suggests, the existence of conformally invariant wave equations, but it does not imply that massless wave equations are automatically conformally invariant, nor does it imply that all conformally invariant wave equations describe massless particles.

II. 2 + 1 MINKOWSKI SPACE

Our interest in three-dimensional, massless field theories is not purely academic, since the singletons have a very interesting realization as field theories on the three-dimensional boundary of 3 + 2 de Sitter space at spatial infinity. In fact, they are the only massless particles in 2 + 1 dimensions—if by masslessness we mean that the invariant $P^\mu P_\mu$ of the (2 + 1)-dimensional Poincaré group vanishes. (For a more accurate statement, see Ref. 17.) In ordinary 3 + 1 Minkowski space it is of course a truism that $P^\mu P_\mu=0$ for massless particles. But the experience with de Sitter space, where this is not even true, has taught us to find more abstract, and therefore more physical and more meaningful, characterizations of a class of physically dominating field theories that should perhaps have been referred to by a term other than "massless." (This is a good example of the instructional value of doing physics in de Sitter space, which was in fact a motivation for embarking on the project 20 years ago.[18]) Now we want to approach three-dimensional field theories with an open mind, so we shall place the word "massless" between quotation marks from now on, in order to avoid getting trapped by the superficial generalization that it would be to define masslessness by requiring the vanishing of $P^\mu P_\mu$ from the start. An interesting approach would be to apply our experience in 3 + 2 de Sitter space to 2 + 2 de Sitter space, to discover the natural definition of "masslessness," and then to pass to the limit of flat, three-dimensional Minkowski space. The results would be the same as those obtained directly.

Let us digress for a moment to recall some conventional wisdom. It is well known that the Coulomb potential in three-dimensional space-time has the form $\ln r$ and not $1/r$. This is directly related to (i) a catastrophic infrared behavior (which we shall *not* regard as a virtue) and (ii) the fact that the scalar two-point function has the form

$$\langle 0\,|\,\phi(x)\phi(x')^*\,|\,0\rangle=[(x-x')^2]^{-1/2}\,. \tag{2.1}$$

The square root introduces a cut, and signals propagate inside the light cone. This is a well-known phenomenon sometimes called reverberation.[19] This equation is compatible with conformal invariance, with conformal degree $\frac{1}{2}$ for ϕ, which is the canonical degree in three dimensions. A conformally invariant Lagrangian for ϕ is

$$\int d^3x\,\phi^*\Box\phi\,. \tag{2.2}$$

This is the theory of free, scalar singletons (the racs). The wave equation is $\Box\phi=0$, so that $P^\mu P_\mu=0$ for free particles. However, as we said, it is not absolutely sure that this is the true (or most useful) definition of "masslessness."

It is also not completely obvious that light-cone propagation expresses the deepest meaning of "masslessness;" nevertheless, let us make the hypothesis that it does, and work out the implications. The two-point function must have poles only, and the simplest possibility for a scalar field is

$$\langle 0\,|\,\phi(x)\phi(x')^*\,|\,0\rangle=[(x-x')^2]^{-1}\,. \tag{2.3}$$

This is also conformally invariant, and in fact conformal invariance leads almost uniquely to this form. The Coulomb potential is now $1/r$. The infrared behavior may be better in this case, but we have not verified it. The conformal degree of ϕ is 1, which is not canonical, so there is no invariant Lagrangian in the usual sense. In fact, the above two-point function is incompatible with any differential equation for ϕ. To understand why, and in order to clarify the physical meaning of the field ϕ, we may expand (2.1) and (2.2) into Fourier series. This reveals the following. The modes of the field in Eq. (2.1) carry the singleton representation $D(\frac{1}{2},0)$ of so (3,2)—the three-dimensional conformal group. The wave equation $\Box\phi=0$ satisfied by these modes expresses the special property of the states of this, highly degenerate, representation. The modes of the field in Eq. (2.3), on the other hand, carry the representation $D(1,0)$ of so(3,2). This representation is less degenerate and the modes of the field are too numerous to be restricted by a differential equation in three-dimensional space-time. [They are just half of the states of the four-dimensional massless, scalar field, which carry $D(1,0)\oplus D(2,0)$.]

The two-point function (2.3) does not satisfy a wave equation, yet the Feynman propagator

$$D_F(x,x')=[(x-x')^2+i\epsilon]^{-1}$$

possesses an inverse $\Box$ such that

$$\int d^3x\,\Box(x,x')D_F(x',x'')\propto\delta^3(x-x'')\,.$$

A quantum field theory can therefore be constructed, with the free Lagrangian

$$\int d^3x\,\phi^*(x)\Box(x,x')\phi(x')\,,$$

for example, by path-integral methods. The free quantum field satisfying (2.3) can also be constructed without any difficulty. One can also take the point of view, common in the context of conformal field theory, that the (interacting) quantum field theory is defined by the bare propagators and the interaction Lagrangian. One is dealing, to put it simply, with a scalar conformal field theory with anomalous dimension. Therefore, since we have to choose between masslessness in the naive sense and "masslessness" in the sense of lightlike propagation characteristics, then the latter must at least be admitted as an attractive alternative. Let us pass now to the electromagnetic field itself.

The two-point function

$$\langle 0 \mid A_\mu(x) A_\nu(x') \mid 0 \rangle = \delta_{\mu\nu}[(x-x')^2]^{-n}$$

is incompatible with light-cone propagation if n has the canonical value $\frac{1}{2}$. Only $n=1$ is compatible with our intuitive perception of light propagation. In this case we have no wave equation for A_μ and no Maxwell equation, but we can still take the free Lagrangian

$$\int d^3x\, A_\mu(x) \Box(x,x') A_\mu(x')$$

and proceed as in the scalar case. The theory is not conformally invariant as it stands, but this defect can be removed. Following Dirac's method, we compactify $2+1$ de Sitter space and identify it with the projective $3+2$ cone. On this cone we introduce a five-potential ($\mathscr{A}_a$) $a=0,1,2,3,5$ with degree of homogeneity -1 and the manifestly conformal invariant two-point function

$$\langle 0 \mid \mathscr{A}_a(y) \mathscr{A}_b(y') \mid 0 \rangle = \delta_{ab}/y\cdot y' \; .$$

Returning to Minkowski notation one ends up with a vector potential (A_μ) $\mu=0,1,2$ and two scalar fields, just as in the four-dimensional case. The extra degrees of freedom are pure gauge and scalar modes and are eliminated by boundary conditions on the incoming states.

This theory is not a gauge theory. The Fourier expansion of $1/(x-x')^2$ includes integration over $p^2>0$, and the point $p^2=0$ does not contribute. Unitarity can therefore be assured, as in four-dimensional massive theories, by projecting out the modes that satisfy $\partial\cdot A=0$. Then the invariant inner product becomes positive definite, not semidefinite. There is no radical to divide out, no uncomplemented invariant subspaces, and hence this is not a gauge theory. The conformal extension is, nevertheless, a gauge theory, with exactly the same so(3,2) modules as de Sitter electrodynamics. In fact, $1/y\cdot y'$ is the limit, as $y^2 \to 0$, of the propagator (1.3).

A nontrivial, three-dimensional theory of gravitation could be developed along similar lines.

There is a property of four-dimensional electromagnetism that may express its essence better than any other: it is a zero-center module. Since this concept has recently been explained at length,[20] we limit ourselves to a few remarks. Both de Sitter and conformal QED have the property that all Casimir operators are nilpotent on the field module, and that the trivial representation (a vacuum mode) appears in the physical sector. In conformally invariant, three-dimensional de Sitter electrodynamics this would imply that the physical states carry the representation $D(2,1)$ of so(3,2). Conformal gravity is associated with $D(3,2)$, which is not a zero-center module. However, both these representations are included in a zero-center representation of three-dimensional conformal supersymmetry, if sufficiently extended. In fact, the extension to osp($4/N$) is justified, and N is determined, by this requirement. The result is that $N=6$ is the only extension that includes gravity without also bringing in spins higher than 2.

Finally, we would like to add some comments about a massive, three-dimensional "gauge theory." The proposed Lagrangian is[21]

$$\int d^3x\left(\tfrac{1}{8}F^2 - mF^*\cdot A\right) ,$$

where F^* is the Hodge dual of the field strength F. The wave equation is gauge invariant, in the sense that it is solved by any potential A of the form $A_\mu(x)=\partial_\mu\Lambda(x)$, for which F vanishes identically. In addition, there are solutions of the free wave equation of the form

$$A_\mu(x) = e^{ik\cdot x}\epsilon_\mu(k), \quad k^2=m^2 ,$$

with $\epsilon_\mu(k)$ complex subject to

$$\epsilon_\mu(k)^* = \epsilon_\mu(-k), \quad \epsilon^{\mu\nu\lambda}k_\nu\epsilon_\lambda = im\epsilon^\mu \; .$$

Under the action of the three-dimensional Poincaré group these modes transform as a direct sum of a massive, irreducible representation and a very reducible representation carried by the gradient modes. There is no indecomposable representation and no need for any indefinite metric. We may introduce gauge fixing in the usual way. Thus, in the Feynman gauge, the free Lagrangian is

$$\int d^3x\left(\tfrac{1}{2}A_\mu \Box A_\mu - mF^*\cdot A\right) .$$

The free solutions are now the massive modes already described and, in addition, the massless modes

$$A_\mu(x) = e^{ik\cdot x}k_\mu, \quad k^2=0 \; .$$

Both may be quantized with positive metric, which gives rise to the two-point function

$$\langle 0 \mid A_\mu(x)A_\nu(x') \mid 0 \rangle = \int e^{ik(x-x')}[\delta_{\mu\nu}+\epsilon_{\mu\nu\lambda}(k^\lambda/m) - k_\mu k_\nu] \times \delta^+(k^2-m^2)dk - \int e^{ik(x-x')}k_\mu k_\nu \delta^+(k^2)dk \; .$$

There is no need for the current to be conserved and the theory has none of the essential characteristic features of gauge theories. It should be stressed that in real gauge theories the structure of the interaction (minimal coupling) is uniquely determined by gauge invariance, which is in turn imposed by unitarity. This is the rigidity that gives gauge theories their essential predictive power. In the theory examined here, on the other hand, there is nothing that particularly distinguishes or favors minimal coupling.

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APPENDIX

The aim of this appendix is to show that massless, integral-spin fields in 3 + 2 de Sitter space propagate on the light cone. The demonstration rests on recurrence formulas that reduce the problem to the spinless case.

The positive-energy, irreducible representations of so(3,2) are denoted $D(E_0,s)$. The infinitesimal generators are denoted

$$L_{ab}=M_{ab}+S_{ab}=-L_{ba},\quad a,b=0,1,2,3,5 ,$$

where M_{ab} is the orbital part and S_{ab} is the spin part. The minimal energy E_0 is the lowest eigenvalue of L_{50} and s is the angular momentum of the lowest-energy eigenspace.

The carrier states of $D(E_0,s)$ will be described as symmetric tensor fields ϕ of rank s satisfying

$$[Q-\langle Q(E_0,s)\rangle]\phi=0 ,$$

$$(y\cdot\partial-N)\phi=0 \text{ (homogeneity)} ,$$

$$y\cdot\phi=0 \text{ (transversality)} ,$$

$$\partial\cdot\phi=0 \text{ (divergencelessness)} .$$

Here

$$Q=\tfrac{1}{2}L_{ab}L^{ab}$$

is the Casimir operator, and

$$\langle Q(E_0,s)\rangle=E_0(E_0-3)+s(s+1)$$

is its value in $D(E_0,s)$. The number N may be fixed at our convenience. The last two conditions imply tracelessness. The carrier states can be built up by induction on s.

Proposition 1. Let ϕ be a carrier state for the representation $D(E_0,s)$. Suppose that $E_0\neq s+1$ and that $E_0\neq 2-s$. Then ϕ can be represented by the formula

$$\phi=\Sigma_1\theta\cdot z\phi_1+\Sigma_2\theta\phi_2+D_s\phi_g . \tag{A1}$$

Here θ is the transverse projector

$$\theta_{ab}=\delta_{ab}-\rho y_a y_b ,$$

Σ_1 and Σ_2 symmetrize the product of two tensors, $z=(z_a)$ is a complex polarization vector that carries spin 0 or ±1. Next, ϕ_1 is a carrier state for $D(E_0,s-1)$, while ϕ_2 is a carrier state for $D(E_0,s-2)$, given in terms of ϕ_1 by the formula

$$\phi_2=-\frac{2}{2s-1}z\cdot\phi_1 .$$

The operator D_2 is a "purified" gradient operator that makes a symmetric, transverse tensor field of rank s out of a symmetric, transverse tensor field of rank $s-1$,

$$D_s\phi_g=\rho^{-1}\Sigma_1[\bar\partial+\rho(s-1)y]\phi_g ,$$

$$\bar\partial_a=\theta_a{}^b\partial_b .$$

Finally ϕ_g is given by the revealing formula

$$\phi_g=\frac{1}{(E_0-s-1)(E_0+s-2)}\times\left[z\cdot\bar\partial\phi_1-(s+1)\rho y\cdot z\phi_1+\rho\Sigma_1 yz\cdot\phi_1-\frac{\rho}{2s-1}D_{s-1}z\cdot\phi_1\right] .$$

The reason for the exclusion of $E_0=s+1$ and $E_0=2-s$ is manifest.

Applying a finite number of times the recurrence relation (A1) within its domain of validity permits us to express the propagator $K_{E_0}^{(s)}(z)$ for $D(E_0,s)$ in terms of $K_{E_0}^{(0)}(z)=K_{E_0}(z)$. This procedure leads to the following result when E_0 is an integer.

Proposition 2. The homogeneous propagator $K_{s+p}^{(s)}(z)$ for the unitary irreducible representation $D(s+p,s)$, p integer ≥ 2, possesses poles at $z=\pm1$ for any $p\geq2$. It has logarithmic singularities at $z=\pm1$ for any $p\geq3$. Poles and other singularities are displayed by the following formulas:

$$K_{s+2}^{(s)}(z)=\frac{S_2^{(s)}(z)}{(z^2-1)^{2s+1}} , \tag{A2}$$

$$K_{s+p}^{(s)}(z)=\frac{S_p^{(s)}(z)}{(z^2-1)^{2s+1}}+\frac{1}{2}K_{3-s-p}^{(s)}(z)\ln\frac{z-1}{z+1} ,\quad p\geq3 . \tag{A3}$$

Here $S_p^{(s)}(z)$ is a tensor polynomial in the z variable and $K_{3-s-p}^{(s)}(z)$ is the propagator of the *unique* finite irreducible representation $D(3-s-p,s)$, Weyl equivalent to $D(s+p,s)$. The other representations that are Weyl equivalent to $D(s+p,s)$ are infinite, nonunitary and carry higher spin: $D(s+2,s+p-2)$ and $D(1-s,s+p-2)$.

Note that proposition 2 does not yet deal with the massless representations $D(s+1,s)$. It applies to the massive representations with integer E_0, including the limiting case $D(s+2,s)$, which is also exceptional.

The case $p=2$ is remarkable in the sense that only one representation, namely, $D(1-s,s)$ is Weyl equivalent to $D(s+2,s)$. This fact prevents the appearance of the logarithmic term in the expression of the propagator $K_{s+2}^{(s)}(z)$, a feature shared by the propagator $K_{1-s}^{(s)}(z)$ for the infinite, nonunitary representation $D(1-s,s)$.

Let us now examine more closely the singularities at $E_0=s+1$ and at $E_0=2-s$ which limit the validity of the recurrence formula (A1). They are related to the fact that these values are reduction points of the minimal weight representations. The wave equation (1.4), which in the present notation reads

$$[Q-2s^2+2]\phi+c_sD_s\partial_s\cdot\phi=0 ,$$

$$\partial_s\cdot\phi=\partial\cdot\phi-\tfrac{1}{2}\Sigma_1[\bar\partial+\rho(s-4)y]\phi',\quad \phi'=\text{trace }\phi$$

determines a nondecomposable representation, more precisely a Gupta-Bleuler triplet that is characterized by a reproducing kernel or two-point function.

Proposition 3. Let ϕ be a carrier state for the spin-s Gupta-Bleuler triplet. Then ϕ has the representation

$$\phi=\Sigma_1\theta\cdot z\phi_1+\Sigma_2\theta\phi_2+D_s(\phi_g{}^1+\phi_g{}^2)\ . \tag{A4}$$

Here ϕ_1 is a carrier state for the unitary representation $D(s+1,s-1)$ or the nonunitary representation $D(2-s,s-1)$. For these particular representations proposition 2 tells us that the two-point functions have (multiple) poles at $z=\pm 1$ only and no logarithmic singularity. Next, ϕ_2 is given in terms of ϕ_1 by

$$\phi_2=-\frac{2}{2s-1}z\cdot\phi_1$$

and

$$\phi_g{}^1=(2s-1)^{-2}[z\cdot\bar{\partial}\phi_1+\rho(s-2)y\cdot z\phi_1+\rho\Sigma_1 yz\cdot\phi_1 -2(2s-1)^{-1}D_{s-1}z\cdot\phi_1]\ ,$$
$$\phi_g{}^2=\Lambda+(1-c_s)^{-1}[c_s(2s+1)-2]\Lambda'\ . \tag{A5}$$

The heart of the problem is the nature of the tensor fields Λ and Λ'.

We shall leave the full details of the specifications of Λ and Λ' to a subsequent publication[22] (see also Ref. 16).

The crux of the matter is that ϕ is determined by ϕ_1 and by Λ and Λ'. If ϕ_1 is free of logarithms, then the source of logarithms in ϕ might lie in the gaugelike parts: $D_s\Lambda$ and $D_s\Lambda'$.

The tensor field Λ satisfies the inhomogeneous equation

$$(Q-2s^2+2)\Lambda=\eta_f\ ,$$

where η_f is an arbitrary carrier state for the finite irreducible representation $D(1-s,s-1)$. The general solution has the form

$$\Lambda\sim\eta_f\ln\sqrt{\rho}y_\pm[\mathrm{mod}\,D(s+2,2-1)\text{-states} \oplus D(2-s,s)\text{-states}]\ .$$

Precisely the action of the gradient D_s in Eq. (A4) cancels any carrier state for $D(1-s,s-1)$ since

$$D_s\eta_f=0\ .$$

This remarkable property prevents the appearance of logarithms in the gauge fields $D_s\Lambda$. The only source of logarithms in ϕ is $D_s\Lambda'$. The tensor field Λ' satisfies the inhomogeneous dipole equation

$$(Q-2s^2+2)^2\Lambda'=\eta_f$$

and contributes logarithms which are not canceled by the action of D_s. Therefore logarithms appear in ϕ unless the coefficient of Λ' vanishes, that is, for all values of the gauge-fixing parameter c_s except

$$c_s=\frac{2}{2s+1}\ ,$$

as is seen from Eq. (A5). Full details, including all nondecomposable representations that appear, will be published separately. We should mention that there are actually two systems of representations, and two choices of propagators, for all spins, as conjectured by Breitenlohner and Freedman.[23] (For spins 0, $\frac{1}{2}$, and 1, see Refs. 2 and 15.)

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