Non-Abelian Aharonov-Bohm effect

P. A. Horváthy*
Centre de Physique Theorique, Laboratoire propre du C.N.R.S., Luminy Case 907, 13288 Marseille Cédex 9, France
(Received 15 May 1985)

The scattering of a nucleon beam on a non-Abelian flux line (proposed by Wu and Yang for testing the existence of gauge fields) is studied in a test-particle framework. All predictions of Wu and Yang are quantitatively confirmed. In particular, protons can be turned into neutrons under suitable conditions. (Charge conservation is restored at the field-theoretical level.) Similar results hold for the scattering of π mesons. A "shadow" of the effect survives also in the classical limit.

I. INTRODUCTION

In their celebrated paper on the nonintegrable phase factor Wu and Yang[1] propose a *Gedankenexperiment* to test the existence of gauge fields. By analogy to the Aharonov-Bohm (AB) experiment,[2] they suggest scattering a nucleon beam around a cylinder which contains a "non-Abelian flux." Although the nucleons cannot penetrate into the interior of the cylinder—the only region where the field strength is different from zero—they interact with the gauge potential outside and produce thus a nontrivial interference.

Wu and Yang assume first that the flux in the cylinder is in the τ_3 direction. They predict then that (i) if one scatters a proton (or a neutron) beam on the cylinder, the interference fringes are shifted in the opposite direction; if a coherent mixture of protons and neutrons in a pure state is scattered on the cylinder, one will observe (ii) a fluctuation in the proton-neutron mixing ratio, and (iii) a fluctuation of the nucleon intensity.

They assume next that the average direction of the flux is different from that of τ_3. Then (iv) the scattering of a proton beam will produce some neutrons as well as protons.

The predictions of Wu and Yang are purely *qualitative.* The aim of this paper is to give, at least approximately, a *quantitative description.*

In our attempt to deepen their heuristic arguments, we have encountered a number of conceptual as well as technical difficulties. First, how does one produce the "non-Abelian flux line?" (See *Note added in proof.*) Even as*suming this can be done* somehow, a physically sensible model is needed to describe the process. A nucleon is naturally identified with a spin-$\frac{1}{2}$ SU(2) doublet Ψ. The complete description of the nucleon-cylinder system would require solving the coupled Yang-Mills (YM) equations, which is technically impossible without knowing in detail what is going on in the cylinder. Fortunately, this is not needed: we are interested only in describing the motion of the nucleons which, by assumption, cannot penetrate into the cylinder. On the other hand, it is reasonable, as a first approximation, to neglect the changes in the YM field due to the nucleon. This is what we mean here by treating the nucleon as a *test particle* moving in the field A_μ due to the non-Abelian flux enclosed in the cylinder. So Ψ satisfies a background Dirac equation

$$\gamma^\mu(\partial_\mu\Psi+A_\mu\Psi)=m\Psi \ . \qquad (1.1)$$

The validity of (1.1) is restricted to the exterior region M. Dropping the irrelevant z variable and assuming the cylinder very thin, we identify M with the punctured plane $R^2\setminus\{0\}$.

This model can be *solved completely* (Sec. II). First, the "Yang-Mills vacua"—YM potentials with $F_{\mu\nu}=0$ in the exterior region—are described: there exists a gauge—analogous to the U gauge in monopole theory[3]—such that, in polar coordinates, the YM potential is given by

$$A_\theta=\frac{1}{i}\begin{bmatrix}\alpha & \\ & -\alpha\end{bmatrix},\quad A_r=0 \ , \qquad (1.2)$$

where α is a real parameter defined modulo integers. In this gauge the field is Abelian, and the wave equation (1.1) splits into two *uncoupled, electromagnetic "Aharonov-Bohm" (AB) equations* with opposite magnetic fluxes. The solution is thus obtained by mere substitution of the known Abelian results.[2,4]

So far so good. But how can we *interpret* these results in terms of protons and neutrons? As emphasized by Yang and Mills in the very first paper ever written on gauge theory,[5] the distinction between protons and neutrons is possible only by measuring electric charge. The prescription of Ref. 5 is as follows: select, at each point x independently, a preferential direction $\omega(x)$ in isospace. In the *fully dynamical theory* the electric current is given by

$$J^\mu_{\rm em}=\frac{1}{i}\overline{\Psi}\gamma^\mu\omega(x)\Psi+{\rm Tr}\{[A_\nu,F^{\mu\nu}]\omega(x)\} \qquad (1.3)$$

which is ordinarily conserved, $\partial_\mu J^\mu_{\rm em}=0$.

Observe that it is only the total rather than the nucleon's charge alone which is conserved. So transitions of the type proton$\rightarrow$neutron are, *a priori,* not excluded. As a matter of fact, in Sec. IV we show that such a transition does indeed occur. It is not difficult to show that the particle's charge alone,

$$q_{\rm nucl}=e\left[\frac{1}{2}+\frac{1}{i}\int\Psi^\dagger(x)\omega(x)\Psi(x)d^3x\right] , \qquad (1.4)$$

is conserved only if the "implementation" $\omega(x)$ generates an *internal symmetry*, $D_\mu \omega = 0$ for the background field configuration.[6,7] Otherwise, it is only the total charge $q_{tot} = q_{nucl} + q_{field}$ which is conserved, where

$$q_{field} = -e \int \mathrm{Tr}\{[A_\nu(x), F^{0\nu}(x)]\omega(x)\} d^3x \ . \qquad (1.5)$$

In the non-Abelian AB setup any direction ξ is uniquely implementable (Sec. IV). In fact, $\omega(x) = \xi$ (a constant generator) in a suitable gauge.[6,8]

To be an internal symmetry is a strong condition: in our case an $\omega(x)$ not parallel to $A_\mu(x)$ generates an internal symmetry only for the trivial field.

Having settled the problem of charge conservation, we can identify protons and neutrons, and interpret the results of Sec. II. The cases (i)—(iii) correspond to $\omega(x)$ being parallel to the electromagnetic direction. The particle's charge is now conserved: protons (neutrons) remain protons (neutrons).

To describe case (iv), a subtle mixture of the fully dynamical and the test-particle frameworks is used. We still use the test-particle equation (1.1)—hoping that its solution does not differ substantially from the exact one—but interpret the result in the dynamical framework, since it is only here that electric charge conservation can be restored. Our procedure is best understood by the following analogy: consider the reflection of an elastic ball by a wall. It is natural to describe the motion of the ball alone, but this leads to an apparent violation of momentum conservation which can only be restored by allowing some dynamics to the wall. This does not affect the motion of the ball in an essential way as long as this latter is light compared to the wall.

Our results provide a full confirmation to all predictions of Wu and Yang. The same technique allows us to describe the scattering of π mesons, and leads to similar conclusions.

Interestingly, a "shadow" of the effect survives in the classical limit. We demonstrate this by solving explicitly the equations proposed by Wong.[9] Protons and neutrons are again distinguished by electric charge. If the direction of the field coincides with the electromagnetic direction, there is no classical effect. If, however, these directions are different, isospin precession is found also classically.

The crucial step in performing the generalized AB experiment seems to be the creation of a non-Abelian flux. This may well explain the failure of the recent attempt by Zeilinger, Horne, and Shull[10] who, as suggested by Wu and Yang, scattered a neutron beam on a rotating rod of ^{238}U metal. The produced flux seems to be too weak for observation. Viewed another way, the experiment of Zeilinger *et al.* sets an upper limit $\sim 10^{-15}$ for the strength of the non-Abelian AB effect versus its electromagnetic counterpart.

II. NUCLEON SCATTERING OF A NON-ABELIAN FLUX LINE

Let us first describe the "non-Abelian vacua"—static YM potentials A_j with $F_{ij} = 0$—on $M = R^2 \setminus \{0\}$. According to Wu and Yang,[1] YM fields are characterized by their nonintegrable phase factors. To demonstrate this explicitly, choose a reference point x_0 [for example, let x_0 be (1,0), in the usual polar coordinates (r,θ)]. Let γ_0 be a loop through x_0 which winds once around the cylinder. Denote by Φ the nonintegrable phase factor

$$\Phi = P\left[\exp\left[-\oint_{\gamma_0} A_j d^j x\right]\right] . \qquad (2.1)$$

Under a gauge transformation defined by $g(x)$, Φ changes as $\Phi \to g^{-1}(x_0)\Phi g(x_0)$, so the YM field determines (2.1) up to conjugacy with an element of SU(2).

Conversely, assume we are given two YM potentials A_j and A_j' whose phase factors (2.1) are conjugate. We claim A_j and A_j' are gauge related. To see this consider the simply connected space M^* obtained by cutting M along the half-infinite line $\theta = 0$. Let us choose arbitrarily a path γ_x^* in M^* from $x_0^* = (1,0)$ to x^* for all $x^* \in M$. Now

$$g(x) = P\left[\exp\left[-\int_{\gamma_x^*} A_j d^j x^*\right]\right] \qquad (2.2)$$

does not depend on the choice of γ_x^*, since M^* is simply connected and thus for any loop γ^* there exists a continuous family $\gamma_s(t)$, $0 \le s \le 1$, $0 \le t \le 1$ such that $\gamma_0(t)$ is the trivial loop, γ_1 is γ^* itself, and $\gamma_s(0) = \gamma_s(1) = x_0$ for all s. Define

$$h(s,t) = P\left[\exp\left[\int_0^t A_j \frac{\partial x^j}{\partial t} dt\right]\right] ,$$

where the integral is taken along γ_s. Put $g(s) = h(s,1)$. According to the "non-Abelian Stokes" theorem[3]

$$g^{-1}\frac{dg}{ds} = \int_0^1 h^{-1} F_{ij} h \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s} dt \ .$$

In our case $F_{ij} = 0$, so $g(s)$ is independent of s, $g(s) = g(1)$. Hence (2.2) provides us with a well-defined function on M^*. It satisfies $\partial_j g = -A_j g$. Similarly, replacing A_j by A_j', we get a function $g'(x^*)$ on M^*. Assume now that the phase factors (2.1) belonging to A_j and A_j', respectively, are conjugate, $\Phi' = h^{-1}\Phi h$ for some $h \in$ SU(2). Then

$$f(x^*) = g(x^*) h [g'(x^*)]^{-1}$$

defines a gauge transformation over M^* which carries A_j to A_j',

$$f^{-1}A_j f + f^{-1}\partial_j f = A_j' \ .$$

$f(x^*)$ projects to a function on M if and only if $f(r,2\pi) = f(r,0)$ for all r. But this happens exactly when $\Phi' = h^{-1}\Phi h$.

Φ can be diagonalized by conjugation—physically, by a global gauge transformation. In this gauge Φ reads

$$\Phi = \begin{bmatrix} \exp(2\pi i\alpha) & 0 \\ 0 & \exp(-2\pi i\alpha) \end{bmatrix} , \qquad (2.3)$$

where α is a real number defined modulo integers. α is unique if we require $\alpha \ge 0$. Plainly,

$$A_r = 0, \quad A_\theta = \frac{1}{i}\begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} = -2\alpha\tau_3 \qquad (2.4)$$

is a gauge potential producing (2.4), and what we have just proved shows that any YM potential A_j is gauge equivalent to one of this form. The gauge where (2.4) holds is analogous to the U gauge in monopole theory.[3]

Let us now study the motion of a nucleon, considered as a test particle in a background YM vacuum A_j. Bring A_j to the form (2.4) by a suitable gauge transformation. In this gauge the Dirac equation (1.1) clearly splits into two uncoupled Dirac equations, each describing an electromagnetic Aharonov-Bohm experiment with enclosed fluxes $2\pi\alpha/e$ and $(-2\pi\alpha/e)$, respectively ($\hbar=1$). So (1.1) is solved by mere substitution of the Abelian results.

Actually, we find it convenient to replace (1.1) by its nonrelativistic limit (this is justified if the nucleons move sufficiently slowly). The standard approximation procedure, see Ref. 11, Sec. 1.4, shows that the upper 4 (positive-energy) component ψ of (the eight-component) Ψ satisfies the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{2m}(\partial_j-A_j)^2\psi \ . \qquad (2.5)$$

The Pauli term $\sigma\cdot\mathbf{B}$ does not appear now since $F_{ij}=0$ in M. Both spin components of ψ satisfy hence the same equation, and (2.5) can be thus considered as a two-component equation for a spinless SU(2) doublet ψ. In the U gauge (2.4), it becomes

$$i\frac{\partial}{\partial t}\begin{bmatrix}\phi\\ \chi\end{bmatrix}=\begin{bmatrix}\hat{H}_\alpha & 0\\ 0 & \hat{H}_{-\alpha}\end{bmatrix}\begin{bmatrix}\phi\\ \chi\end{bmatrix},\ \psi=\begin{bmatrix}\phi\\ \chi\end{bmatrix}, \qquad (2.6)$$

where

$$\hat{H}_\alpha=-\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left[\frac{\partial}{\partial\theta}+i\alpha\right]^2\right] \qquad (2.7)$$

is the standard Hamiltonian for a particle with unit electric charge in the field of an electromagnetic AB solenoid, with flux[2] $(2\pi\alpha/e)$.

We are interested in the behavior of the nucleons at large distances, so we can use scattering theory.[12] Let us briefly summarize what is known in the Abelian case.[2] Consider first time-dependent scattering. In the gauge $A_r=0$, $A_\theta=-\alpha$, the angular momentum for a charged particle is $\hat{l}=-i(\partial/\partial\theta)+\alpha$ with periodic boundary conditions $\psi(r,2\pi)=\psi(r,0)$ (Ref. 13). Choosing α so that $-1<\alpha<1$, the spectrum of angular momentum is $\lambda_m=m+\alpha$, $m=0,\pm1,\pm2,\ldots$. It is convenient to work in momentum Hilbert space $H=L^2(M,d\mathbf{k})$. H splits into angular momentum subspaces,

$$H=\oplus_m H_m \ ,$$

where H_m is defined by the projection

$$(\hat{P}_m\psi)(k,\beta)=\frac{1}{2\pi}e^{im\beta}\int_0^{2\pi}d\theta\, e^{-im\theta}\psi(k,\theta)\ , \qquad (2.8)$$

$\mathbf{k}=(k,\beta)$. This has the advantage that the restriction of the S matrix to each angular momentum subspace H_m is just a *phase shift,*

$$\hat{S}\mid H_m=\exp[2i\delta_m(\alpha)]\ , \qquad (2.9)$$

where

$$2\delta_m(\alpha)=\begin{cases}-\pi\alpha & \text{for } \lambda_m\geq 0\\ \pi\alpha & \text{for } \lambda_m<0\ .\end{cases} \qquad (2.10)$$

Intuitively, the phase shift depends on which side of the solenoid the particle passed. The origin of the AB effect is that, in quantum mechanics, a particle can be split and go both sides.

In the *time-independent* approach one works instead with an incident plane wave $\psi_{\rm inc}(x)=\exp(i\mathbf{k}\cdot\mathbf{x})$. The outgoing wave has the form

$$\psi_{\rm out}(x)=\exp(i\mathbf{k}\cdot\mathbf{x})+f(k,\theta)\exp(i\mathbf{k}\cdot\mathbf{r})r^{-1/2}\ . \qquad (2.11)$$

In the AB experiment $f(k,\theta)$ is given, for $\theta\neq0$, by

$$f(k,\theta)=\frac{\sin\pi\alpha\exp(-i\theta/2+[\alpha])}{\sqrt{2\pi ki}\,\sin(\theta/2)}\ , \qquad (2.12)$$

where $[\alpha]$ is the greatest integer less then α. [This subtlety will be important in the non-Abelian case where the upper (lower) components get opposite phase shifts because the α's have different signs.]

Equation (2.6) is solved now at once: go to the U gauge, switch to the momentum representation, and decompose the incident wave as

$$\psi_{\rm inc}=\begin{bmatrix}\phi_++\phi_-\\ \chi_++\chi_-\end{bmatrix}, \qquad (2.13)$$

where the subscript $\pm$ refers to the sign of λ_m. The outgoing wave function is then, by (2.9) and (2.10),

$$\psi_{\rm out}=\begin{bmatrix}\exp(-i\pi\alpha)\phi_++\exp(i\pi\alpha)\phi_-\\ \exp(i\pi\alpha)\chi_++\exp(-i\pi\alpha)\chi_-\end{bmatrix}. \qquad (2.14)$$

Observe that (2.14) depends only on α modulo integers—i.e., on the Wu-Yang factor.

To interpret these results in terms of protons and neutrons, these objects must be first identified.

III. ELECTRIC CHARGE IN GAUGE THEORIES

Let us first consider a fully dynamical nucleon-Yang-Mills system (A_μ denotes here the total YM field, not only a background). The field equation

$$D_\mu F^{\mu\nu}=\partial_\mu F^{\mu\nu}+[A_\mu,F^{\mu\nu}]=J^\nu\ , \qquad (3.1)$$

where J^ν is the nucleon current,

$$J_a^\nu=(1/i)\bar{\psi}\gamma^\nu\tau_a\psi,\ \ a=1,2,3\ , \qquad (3.2)$$

implies that J^ν is covariantly constant, $D_\nu J^\nu=0$. Consequently,[5]

$$j_a^\mu=\frac{1}{i}\bar{\psi}\gamma^\mu\tau_a\psi+([A_\nu,F^{\mu\nu}])_a\ , \qquad (3.3)$$

is ordinarily conserved, $\partial_\mu j_a^\mu$ for all a. Hence

$$q_a=\int j_a^0 d^3x,\ \ a=1,2,3\ , \qquad (3.4)$$

provides us with an ordinarily conserved charge for each a.

Yang and Mills argue that electric charge should be introduced by choosing, at each point independently, a preferential direction $\omega(x)$ in isospace. Choosing the gauge so that this direction is along the third axis, the charge of the nucleon (field) is expressed as

$$q_{\rm nucl}=e\left[\tfrac{1}{2}+\int d^3xJ_3^0\right], \tag{3.5}$$

$$q_{\rm field}=e\int([A_\nu,F^{0\nu}]_3)d^3x\ . \tag{3.6}$$

The total charge $q_{\rm nucl}+q_{\rm field}=q_3$ is thus ordinarily conserved.

The formulas above are valid in the "preferential gauge." In an arbitrary gauge they are reformulated as follows: let us select, at each point x, a direction $\omega(x)=\omega^a(x)\tau_a$ [with $|\omega(x)|=1$] in internal space; consider

$$q_{\rm nucl}=e\left[\frac{1}{2}+\frac{1}{i}\int\Psi^\dagger(x)\omega(x)\Psi(x)d^3x\right], \tag{3.7}$$

$$q_{\rm field}=-e\int{\rm Tr}\{[A_\nu(x),F^{0\nu}(x)]\omega(x)\}d^3x\ . \tag{3.8}$$

The fundamental assumption in this procedure is that there is a gauge such that $\omega(x)$ is position independent, $\omega(x)=\xi$, for all x. This is undoubtedly correct if the underlying space is R^3. In topologically nontrivial situations however, the fields A_μ and ψ are smoothly defined in general only on contractable coordinate patches. The local expressions $A_\mu^{(\alpha)},\psi^{(\alpha)}$ $(A_\mu^{(\beta)},\psi^{(\beta)})$ given in V_α (V_β) are compatible if and only if

$$\psi^{(\alpha)}=h_{\alpha\beta}\psi^{(\beta)}\ ,$$
$$A_\mu^{(\alpha)}=h_{\alpha\beta}{}^{-1}A_\mu^{(\beta)}h_{\alpha\beta}-\partial_\mu h_{\alpha\beta}(h_{\alpha\beta})^{-1}\ , \tag{3.9}$$

where $h_{\alpha\beta}$ is the transition function between the gauges labeled by α and β, respectively. Similarly, in V_α the direction field is given by an $\omega^{(\alpha)}$. Gauge invariance requires that

$$\omega^{(\alpha)}(x)=h_{\alpha\beta}(x)^{-1}\omega^{(\beta)}(x)h(x)_{\alpha\beta} \tag{3.10}$$

in the intersection of the domains. Now, V_α is contractible for each α by assumption, so one can choose a gauge—the so-called "rigid" gauge—such that $\omega^{(\alpha)}(x)=\xi$ is constant,[6,8] Gauge invariance requires hence that

$$h_{\alpha\beta}{}^{-1}(x)\xi h_{\alpha\beta}(x)=\xi \tag{3.11}$$

must be satisfied for all x, where $h_{\alpha\beta}$ denotes now the transition functions between the *rigid* gauges over V_α and V_β. Conversely, any ξ satisfying this constraint provides us with an admissible direction field: $\omega_\xi(x)=\xi$ is a consistent definition in the rigid gauges, and thus in all gauges.

Equation (3.11) is, as explained in Refs. 6 and 8, a *topological obstruction.* Those ξ's satisfying (3.11) are called *implementable directions.*

Let us now consider a "background" solution A_μ^* to the sourceless YM equations $\partial_\mu F^{*\mu\nu}+[A_\mu^*,F^{*\mu\nu}]=0$ in M, and assume the direction field $\omega(x)$ is background-covariantly constant, $\partial_\mu\omega+[A_\mu^*,\omega]=0$ [in the terminology of Ref. 6, $\omega(x)$ generates an internal symmetry of the field configuration A_μ^*]. Let us split the total field as $A_\mu=A_\mu^*+a_\mu$. As demonstrated in Ref. 7 the dynamical YM equation (3.1) implies that

$$\iota^\mu=J^\mu+(D_\nu F^{\mu\nu})_N\ , \tag{3.12}$$

where the subscript N means all terms quadratic and higher order in a_μ, satisfies the background conservation law $\partial_\mu\iota^\mu+[A_\mu^*,\iota^\mu]=0$. Hence

$$\iota_{\rm em}^\mu(x)={\rm Tr}[\iota^\mu(x)\omega(x)] \tag{3.13}$$

is ordinarily conserved, $\partial_\mu\iota^\mu=0$. Our definition given in the Introduction for a test particle means exactly that quadratic and higher-order terms in a_μ can be neglected,[14] and (3.13) reduces then to the particle's electromagnetic current $J_{\rm em}^\mu={\rm Tr}(J^\mu\omega)$ which must be now separately conserved. Alternatively, this statement follows also directly from the Dirac equation (1.1).

Now, adopting the philosophy of Yang and Mills,[5] a nucleon is a *proton* (*neutron*) if its charge is e (0).

Alternatively, consider the electric charge operator[3]

$$Q_{\rm em}=(e/i)\omega(x)\ . \tag{3.14}$$

A nucleon Ψ is then a proton or a neutron, if it is an eigenfunction of $Q_{\rm em}$ with eigenvalue $\pm e/2$.

IV. INTERPRETATION IN TERMS OF PROTONS AND NEUTRONS

SU(2) YM fields over the punctured plane can be described without transition function, so (3.11) is automatic: any SU(2) direction is implementable. Furthermore, the implementation is unique. As a matter of fact,

$$\omega_\xi(x)=\xi \tag{4.1}$$

in a suitable "rigid" gauge. Indeed, denoted by $\omega_\xi(x)$ an implementation of ξ, so that $\omega_\xi(x_0)=\xi$. In each contractible subset of M—for example, in the region $0\le\theta<2\pi$—we can gauge $\omega_\xi(x)$ to the constant value ξ, $g^{-1}(x)\omega_\xi g(x)=\xi$. Furthermore, the G-valued function $g(x)$ can be chosen to be smooth everywhere—except that it may happen that $\lim_{\theta\to2\pi-0}g(x)=h\neq1$. Such a singularity is, however, removable. In fact, h belongs necessarily to the U(1) subgroup of $G={\rm SU(2)}$ of rotations around ξ, because $\omega_\xi(x)$ is continuous. But H is connected, so there is a smooth path $h(t)$ in H such that $h(0)=1$ and $h(2\pi)=h$. The new gauge transformation defined by $g(r,\theta)h^{-1}(\theta)$ is then smooth and brings $\omega_\xi(x)$ to ξ.

Notice, that this unique implementation of ξ can be extended to an implementation of the full su(2) in two gauge-inequivalent ways.[6]

Now we are in the position to check the predictions in Ref. 1. Assume first that the electromagnetic and the field directions coincide. This means that, in the U gauge, $\omega(x)=\tau_3$ so a proton or a neutron has wave function

$$\text{proton}=\begin{bmatrix}\phi\\0\end{bmatrix},\quad \text{neutron}=\begin{bmatrix}0\\\chi\end{bmatrix}, \tag{4.2}$$

respectively, in this gauge. So, by (2.14),

$$\psi_{\text{out}}=\begin{cases}\left[\exp(-i\pi\alpha)\phi_{+}+\exp(i\pi\alpha)\phi_{-}\right] & \text{for a proton}\\ 0 & \\ 0 & \\ \left[\exp(i\pi\alpha)\chi_{+}+\exp(-i\pi\alpha)\chi_{-}\right] & \text{for a neutron ,}\end{cases} \tag{4.3}$$

and hence protons (neutrons) get opposite shifts, proving (i) of Sec. I.

Let now the incident beam be a *coherent mixture of protons and neutrons in a pure state.* Assume for the sake of simplicity that $\phi_{\text{inc}}=\chi_{\text{inc}}=\exp(i\mathbf{k}\cdot\mathbf{x})/\sqrt{2}$. Using (2.11) and (2.12) we get (ii) the outgoing proton/neutron mixing ratio ρ shows a direction-dependent fluctuation expressed as

$$\rho^2=\frac{2\pi kr\sin^2(\theta/2)+\sin^2\pi\alpha+\sqrt{8\pi kr}\,\sin(\theta/2)\sin\pi\alpha\cos[\pi/4+\theta/2-kr(1-\cos\theta)]}{2\pi kr\sin^2(\theta/2)+\sin^2\pi\alpha-\sqrt{8\pi kr}\,\sin(\theta/2)\sin\pi\alpha\cos[\pi/4-\theta/2-kr(1-\cos\theta)]}\,. \tag{4.4}$$

See Fig. 1(a). The picture is more symmetric for the proton-neutron difference $\ln\rho$ [Fig. 1(b)].

(iii) The outgoing *nucleon intensity* ν *fluctuates,* as shown on Fig. 2, according to

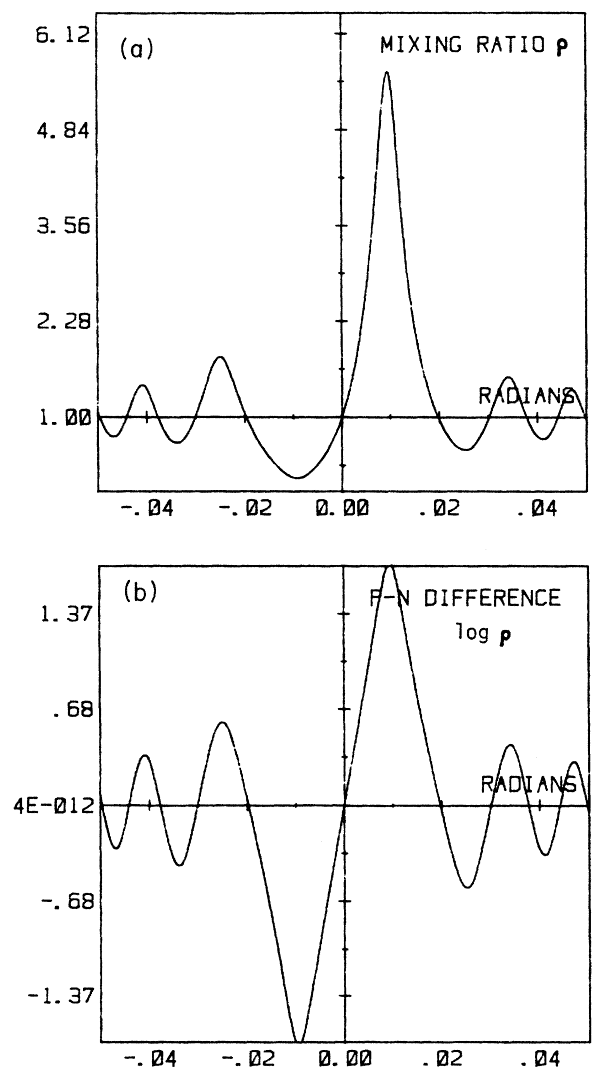

FIG. 1. Fluctuation of the proton-neutron mixing ratio calculated for $kr=10^4$ and $\alpha=0.5$. Protons and neutrons, respectively, get opposite shifts. The vertical asymmetry can be removed by considering rather the proton-neutron difference depicted in (b).

$$\nu=1+\frac{\sin^2\pi\alpha}{2\pi kr\sin^2(\theta/2)}-\left[\frac{2}{\pi kr}\right]^{1/2}\sin\pi\alpha\sin\left[\frac{\pi}{4}-kr(1-\cos\theta)\right]. \tag{4.5}$$

Observe that all results depend only on α (modulo integers)—in other words, on the nonintegrable phase factor.

Let us now assume that the field and electromagnetic directions are *different.* For simplicity, assume that $\omega(x)=\tau_1$ in the U gauge, so that a proton or a neutron has wave function

$$\text{proton}=\begin{bmatrix}\phi\\ \phi\end{bmatrix},\quad \text{neutron}=\begin{bmatrix}\chi\\ -\chi\end{bmatrix}, \tag{4.6}$$

respectively. An incoming proton is by (2.14) transformed into

$$\psi_{\text{out}}=\cos(\pi\alpha)\begin{bmatrix}\phi\\ \phi\end{bmatrix}+\frac{1}{i}\sin(\pi\alpha)\begin{bmatrix}\phi_{+}-\phi_{-}\\ -\phi_{+}+\phi_{-}\end{bmatrix}. \tag{4.7}$$

ψ_{out} is hence a flux-dependent mixture of protons and neutrons: hence *scattering a beam of protons produces some*

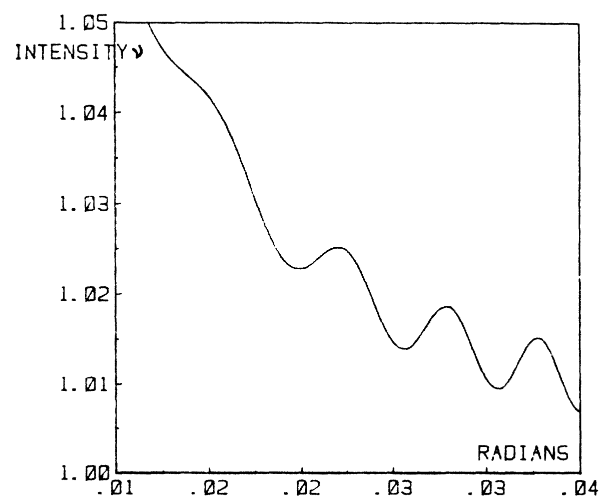

FIG. 2. Fluctuation of the nucleon intensity calculated for $kr=5\times10^4$ and $\alpha=0.5$. A symmetric picture would be obtained for negative scattering angles. The divergence at $\theta=0$ is due to the use of the Abelian formula (2.12).

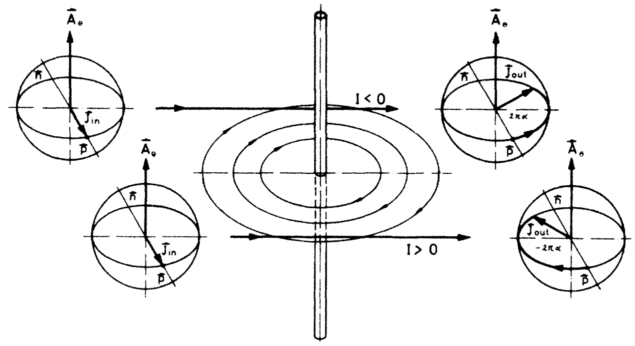

FIG. 3. Scattering of a proton on a non-Abelian flux line whose direction in isospace is perpendicular to the proton-neutron axis. Isospin is rotated by the angle $\mp 2\pi\alpha$, depending on the sign of the angular momentum. For $\alpha=\frac{1}{2}$ in particular, all protons are turned into neutrons. This effect survives in the classical limit.

neutrons as well as protons, confirming prediction (iv) of Wu and Yang. In particular for $\alpha=\frac{1}{2}$ (when the Abelian cross section is maximal), $\cos\pi\alpha=0$, and thus all protons are converted into neutrons. Similarly, an incoming neutron is changed into a proton in this case.

In the first case $\omega(x)=\tau_3$ is covariantly constant, so the particle's charge is conserved. When $\omega(x)=\tau_1$, $D_j\omega=2\alpha[\tau_3,\tau_1]=2\alpha\tau_2$ vanishes if and only if $\alpha=0$ (modulo integers), so the nucleon's charge alone is conserved only if the YM field is trivial. In other cases the interaction of the nucleons with the charged quanta of the field induces a transfer of electric charge. Our fundamental assumption is, however, that this does not change the "background" field in a substantial way and the Dirac equation (1.1) is still a good approximation for describing the real process.

When the particle goes entirely on one side of the solenoid [either $\phi_-=0$, $\chi_-=0$ or $\phi_+=0$, $\chi_+=0$ in (2.13)], the S matrix is just a *rotation* by angle $\mp 2\pi\alpha$ around τ_3:

$$\hat{S}_\pm=\hat{R}_{\mp 2\pi\alpha}=\begin{bmatrix}\exp(\mp i\pi\alpha) & 0\\ 0 & \exp(\pm i\pi\alpha)\end{bmatrix} \tag{4.8}$$

(where the sign is plus or minus depending on the sign of the angular momentum). Hence the field induces isospin precession (Fig. 3).

Notice also that the S matrix is the square root of the Wu-Yang factor, $\hat{S}_\pm=(\Phi)^{\mp 1/2}$. Intuitively, if the particle goes entirely on one side, its polar angle is changed by $\pm\pi$. So the phase shift is half of that associated to $\pm 2\pi$—which is $\Phi^{\mp 1}$.

V. MESON SCATTERING

We can, by analogy, consider the scattering of π mesons on a non-Abelian flux line. Pions are spin-0 SU(2) triplets. Let us define π^+,π^-,π^0 to be the eigenstates of Q_{em} given by (3.14) with eigenvalues e, $-e$, and 0, respectively. Identify the Lie algebra su(2) with R^3 in the standard way. The same line of thought as for nucleons leads to describing the motion of π mesons by a three-component Schrödinger equation. In the U gauge this is not diagonal; however, it can easily be diagonalized by a second gauge rotation. In this new gauge—let us call it the U' gauge—our Schrödinger equation becomes

$$i\frac{\partial}{\partial t}\begin{bmatrix}\phi\\ \chi\\ \eta\end{bmatrix}=\begin{bmatrix}\hat{H}_{2\alpha} & & \\ & \hat{H}_{-2\alpha} & \\ & & \hat{H}_0\end{bmatrix}\begin{bmatrix}\phi\\ \chi\\ \eta\end{bmatrix}. \tag{5.1}$$

Equation (5.1) splits into three uncoupled AB equations with "magnetic" fluxes $4\pi\alpha/e$, $-4\pi\alpha/e$, and 0, respectively.

It is now easy to prove the following statements, analogous to the predictions of Wu and Yang. Assume first that the preferential direction coincides with that of the field. In the U' gauge, the pion states are given by

$$\pi^+=\begin{bmatrix}\phi\\ 0\\ 0\end{bmatrix},\quad \pi^-=\begin{bmatrix}0\\ \chi\\ 0\end{bmatrix},\quad \pi^0=\begin{bmatrix}0\\ 0\\ \eta\end{bmatrix}. \tag{5.2}$$

Thus (i) π^+ (π^-) mesons get opposite phase shifts, while a π^0 beam proceeds undisturbed, and (ii) if a coherent mixture of π^+, π^0, and π^- is scattered on the cylinder, one will observe a direction-dependent fluctuation in the mixing ratios and intensities.

These statements are confirmed exactly like those valid for nucleons.

Let us now assume that $\omega(x)$ is orthogonal to the field direction, for example, $\omega(x)=\tau_1$ in the U gauge. In the U' gauge the pion states are hence represented by

$$\pi^+=\begin{bmatrix}\phi\\ -\phi\\ \phi\end{bmatrix},\quad \pi^0=\begin{bmatrix}\eta\\ \eta\\ 0\end{bmatrix},\quad \pi^-=\begin{bmatrix}\chi\\ -\chi\\ -\chi\end{bmatrix}. \tag{5.3}$$

Hence (iv) the scattering of a π^+ beam will produce also some π^- and π^0 mesons; as a matter of fact, $\psi_{\text{inc}}=\pi^+$ is scattered into

$$\psi_{\text{out}}=\cos^2\pi\alpha\underset{\pi^+}{\begin{bmatrix}\phi\\ -\phi\\ \phi\end{bmatrix}}-\sin^2\pi\alpha\underset{\pi^-}{\begin{bmatrix}\phi\\ -\phi\\ -\phi\end{bmatrix}}+\frac{1}{i}\sin 2\pi\alpha\underset{\pi^0}{\begin{bmatrix}\phi_+-\phi_-\\ \phi_+-\phi_-\\ 0\end{bmatrix}}. \tag{5.4}$$

From (5.1) one could naively deduce that meson scattering is the same for α and $\alpha+\frac{1}{2}$, because $\hat{H}_{2\alpha}$ and $\hat{H}_{2\alpha+1}$ describe the same electromagnetic AB situation. Observe however that the S matrix (2.9) and (2.10) is (-1) if α is an odd integer. In the electromagnetic situation this is unobservable, since it yields merely an overall phase. In the meson scattering, however, the upper two components in (5.1) do get this phase shift, while the lower component is unchanged, so there is a relative phase difference. Alternatively, this is understood by remembering that in case (iv) the electric charge of a test particle is conserved if and only if $\alpha=0$. So for $\alpha=\frac{1}{2}$ something must happen which causes the loss of the particle's charge. This is confirmed by the explicit solution (5.4): for $\alpha=\frac{1}{2}$ all incoming π^+'s are turned into π^-'s.

These results complete those obtained in[15] where it was shown that, for a particle which transforms according to a unitary representation U of the gauge group, two solutions of the test particle wave equation in YM vacua characterized by Wu-Yang factors Φ and Φ' are gauge equivalent if and only if $U(\Phi)=U(\Phi')$. In those papers we did not distinguish between protons and neutrons (or between π^+ and π^-), so any gauge transformation was allowed. Here, however, a preferential direction $\omega(x)$ has been chosen in order to do such a distinction. Therefore, only those gauge transformation are now allowed which preserve $\omega(x)$.

Similar results could be obtained for the scattering of "free quarks" [SU(3) triplets] or baryons [SU(3) octets] on a "chromomagnetic flux line," etc.

VI. CLASSICAL LIMIT

Generalizing the heuristic arguments of Wong,[9] Ardoź[16] has demonstrated, that the expectation value of the particle's isospin,

$$J_a=\int \psi^\dagger \tau_a \psi\, d^3x \;, \tag{6.1}$$

satisfies, with the classical trajectory $x(t)$, the equations of motion

$$\dot J=[J,A_\mu \dot x^\mu] \;, \tag{6.2}$$

$$m\ddot x_\mu=-\mathrm{Tr}(JF_{\mu\nu}\dot x^\nu) \;. \tag{6.3}$$

These equations are gauge invariant if J transforms according to $J\rightarrow g^{-1}Jg$ under a gauge transformation.

In the non-Abelian AB setup these equations are solved at once. By (6.3), the space-time motion is just that of a free particle. To solve (6.2), it is convenient to switch to the U gauge. With the initial condition $J(0)=(J_{ik})\in \mathrm{su}(2)$ the solution reads

$$J(t)=\begin{bmatrix} J_{11} & \exp\{\alpha i[\theta(t)-\theta(0)]\}J_{12} \\ \exp\{-\alpha i[\theta(t)-\theta(0)]\}J_{21} & J_{22}\end{bmatrix} . \tag{6.4}$$

So classical isospin precesses, just like its quantum counterpart, around the field direction. When the particle's polar angle changes by $\Delta\theta$, J is rotated by an angle $(-2\alpha\Delta\theta)$. In the scattering process $\Delta\theta=\pm\pi$, depending on which side of the solenoid the particle has passed. So

$$J_{\text{out}}=\begin{cases} R_{-2\pi\alpha}J_{\text{inc}} & \text{for } I\geq 0 \\ R_{+2\pi\alpha}J_{\text{inc}} & \text{for } I<0 \;, \end{cases} \tag{6.5}$$

where $\mathbf{I}=m\mathbf{x}\times\dot{\mathbf{x}}$ is the classical angular momentum. Again, the phase of (6.5) depends on the sign of the angular momentum.

To get an interpretation in terms of protons and neutrons, first these objects must be identified. Let us choose a preferential direction $\omega(x)$. According to (6.1) the electric (3.7) of the nucleon is

$$q_{\text{nucl}}=e(\tfrac{1}{2}+\omega^a J_a) \;. \tag{6.6}$$

Notice that the second term here is just the "spin-from-isospin" contribution due to the internal symmetry of the field.[6,14]

A "classical proton" (a "classical neutron") is hence one with $-\mathrm{Tr}(\omega J)=\frac{1}{2}\,(-\frac{1}{2})$.

If the field direction is parallel to the preferential direction, then, in the U gauge,

$$J=\begin{cases} \dfrac{i}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} & \text{for a proton} \\[2ex] \dfrac{i}{2}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} & \text{for a neutron}\;. \end{cases} \tag{6.7}$$

The effects (i)—(iii) above disappear in the classical limit: according to (6.4) any mixture of pure protons and neutrons remains unchanged since there are no off-diagonal elements.

On the other hand, the prediction (iv) survives even classically. Let us assume again that ω and the field direction are different, for example $\omega=\tau_1$ in the U gauge. Then

$$J=\begin{cases} \dfrac{i}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} & \text{for a proton} \\[2ex] \dfrac{i}{2}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix} & \text{for a neutron}\;. \end{cases} \tag{6.8}$$

According to (6.4), a proton is scattered hence into

$$\begin{aligned} J_{\text{out}}&=\frac{i}{2}\begin{bmatrix}0 & \exp(\mp 2\pi\alpha i)\\ \exp(\pm 2\pi\alpha i) & 0\end{bmatrix} \\ &=\cos(2\pi\alpha)\frac{1}{2}\begin{bmatrix}0 & i\\ i & 0\end{bmatrix}\pm\sin(2\pi\alpha)\frac{1}{2i}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}, \end{aligned} \tag{6.9}$$

depending on the sign of the angular momentum. This is again a rotation by angle $\mp 2\pi\alpha$, cf. (4.7) and (4.8). In particular, for $\alpha=\frac{1}{2}$ all protons are turned into neutrons.

Using (6.2), it is clear that the particle's charge alone is conserved if and only if ω is an internal symmetry for the field, $D_j\omega=0$. (The loss of electric charge is again restored in the field-theoretic framework.) This conclusion is confirmed by the explicit solution: $D_j\omega=0$ if and only if $\alpha=0$ modulo integers. But, by (6.9), then $J_{\text{out}}=J_{\text{inc}}$.

Noted added in proof. A promising candidate may be the fractionally charged self-dual solution described in P. Forgács, Z. Horváth, and L. Palla, Phys. Rev. Lett. **46**, 392 (1981).

ACKNOWLEDGMENTS

I would like to thank T. T. Wu for suggesting that I study this problem. I am indebted to Lochlainn O'Raifeartaigh and Péter Forgács for enlightening discussions and to Jose Alberty for his help in producing Figs. 1 and 2. Hospitality in Marseille and Dublin are gratefully acknowledged.

*Present address: Dublin Institute for Advanced Studies, 10 Burlington Rd., Dublin 4, Eire.

[1]T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).

[2]Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959); for details on scattering see W. C. Henneberger, Phys. Rev. A **22**, 1383 (1980); S. N. M. Ruijsenaars, Ann. Phys. (N.Y.) **146**, 1 (1983).

[3]See, e.g., P. Goddard and D. Olive, Rep. Prog. Phys. **41**, 1357 (1978).

[4]In Sec. II we find actually scattering solutions to a Schrödinger rather then to a Dirac equation. This is certainly justified for small velocities, cf. Ref. 10.

[5]C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[6]P. A. Horváthy and J. H. Rawnsley, Phys. Rev. D **32**, 968 (1985); J. Math. Phys. (to be published). The definition of symmetries is based on P. Forgács and N. Manton, Commun. Math. Phys. **72**, 15 (1980).

[7]L. F. Abbott and S. Deser, Phys. Lett. **116B**, 259 (1982).

[8]P. Nelson and A. Manohar, Phys. Rev. Lett. **50**, 943 (1983); A. P. Balachandran, G. Marmo, N. Mukunda, J. S. Nilsson, E. C. G. Sudarshan, and F. Zaccaria, *ibid.* **50**, 1553 (1983).

[9]S. K. Wong, Nuovo Cimento **65A**, 689 (1970); See also A. P. Balachandran, P. Salomonson, B. S. Skagerstam, and J. O. Winnberg, Phys. Rev. D **15**, 2308 (1977); S. Sternberg, in *Differential Geometrical Methods in Mathematical Physics, II,* proceedings of the Conference, Bonn, 1977, edited by K. Bleuler, H. R. Petri, and A. Reetz (Lecture Notes in Mathematics, Vol. 676) (Springer, Berlin, 1977), p. 1; C. Duval and P. A. Horváthy, Ann. Phys. (N.Y.) **142**, 10 (1982).

[10]A. Zeilinger, M. A. Horne, and C. G. Shull, in *Proceedings of the International Symposium on the Foundations of Quantum Mechanics, Tokyo, 1983,* edited by S. Kamefuchi (Physical Society of Japan, Tokyo, 1984), pp. 389—393.

[11]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[12]See, e.g., M. Reed and B. Simon, *Methods of Modern Mathematical Physics,* Vol. III of *Scattering Theory* (Academic, New York, 1979).

[13]There has been a recent controversy about the angular momentum operator in the field of an AB solenoid, cf. F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); R. Jackiw and A. N. Redlich, *ibid.* **50**, 555 (1983). Here we adopt the viewpoint in P. A. Horváthy, Phys. Rev. A **31**, 1151 (1985).

[14]R. Jackiw and N. Manton, Ann. Phys. (N.Y.) **127**, 259 (1980).

[15]P. A. Horváthy and J. Kollár, Class. Quant. Gravit. **1**, L61 (1984); see also in *Monopoles in Quantum Field Theory,* proceedings of the Monopole Meeting, Trieste, Italy, 1981, edited by N. S. Craigie, P. Goddard, and W. Nahm (World Scientific, Singapore, 1982), p. 277.

[16]H. Ardoź, Phys. Lett. **116B**, 251 (1982); Acta Phys. Pol. B **13**, 519 (1982). The conditions of the validity are, however, restrictive, see H. Ardoź, Phys. Lett. **116B**, 255 (1982).