Lehmann spectral representation for anti–de Sitter quantum field theory

Dieter W. Düsedau and Daniel Z. Freedman
Department of Mathematics and Center for Theoretical Physics, Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts, 02139
(Received 30 July 1985)

The O(3,2) invariance properties of field theories in a background anti–de Sitter space-time are used to derive a nonperturbative Lehmann spectral representation for two-point functions of scalar operators. A suitable integral transform is defined and the transform of a two-point function is shown to satisfy a dispersion relation in a variable which is the eigenvalue of the Casimir operator of O(3,2).

Field theory in anti–de Sitter (AdS) space has emerged as a research topic of some interest because AdS space occurs as a natural space-time background in gauged extended supergravity and in Kaluza-Klein theories. It was once thought that AdS space is not a suitable arena for field theory because it is not a globally hyperbolic space-time[1] and the basic Cauchy problem is not well posed. It was then suggested[2] that this difficulty could be overcome if suitable boundary conditions were imposed at spatial infinity and these conditions were implicitly contained in an earlier group-theoretic approach.[3] The free-field problem in both scalar and supersymmetric field theories[4,5] now appears to be solved, and the boundary conditions ensure conservation of the generators of the isometry group SO(3,2) and the supergroup OSp(1,4) of the theory. One may begin to explore the properties of AdS quantum field theory with interactions, and the boundary conditions play a role in recent treatments of the effective potential[6] and the vacuum energy problem.[7]

In this paper we study a nonperturbative aspect of AdS quantum field theory and derive a Lehmann spectral representation for the two-point function of scalar operators (composite or elementary). The standard Lehmann representation follows from the Poincaré invariance of flat-space quantum field theory, and it is derived for AdS space by expanding in intermediate states and exploiting SO(3,2) invariance. The standard Lehmann representation is most useful in momentum space where it leads to a dispersion relation and to the nonperturbative definition of physical mass as a pole of the propagator. The Fourier transform is not natural in AdS space, but there is an analogous transform which is natural.[8] When this is combined with the Lehmann representation, one finds that the transform of the two-point function satisfies a dispersion relation as a function of a variable which is essentially the eigenvalue of the quadratic Casimir operator of SO(3,2). The notion of "physical mass" as a pole of the transform then follows naturally. Readers who wish to preview the key results are invited to peek at Eqs. (25), (29), (30), and (34) below.

Let us start the discussion with the fact that four-dimensional AdS space $(\mathrm{AdS})_4$ is really the hyperboloid $\eta_{AB}y^Ay^B=a^{-2}$ embedded in $\mathbf{R}^5$ with Cartesian coordinates y^A, $A=0,1,2,3,4$ and flat metric $\eta_{AB}=(+,-,-,-,+)$. In this setting infinitesimal transformations of SO(3,2) are realized by Killing vectors

$$K_{AB}=y_A\frac{\partial}{\partial y^B}-y_B\frac{\partial}{\partial y^A}\ . \tag{1}$$

After introducing intrinsic coordinates t,ρ,x^i, with $\rho^2=x^ix^i$, and the usual spherical coordinates via

$$\begin{aligned}
&y^0=a^{-1}\sin t\sec\rho\ ,\\
&y^i=(a\rho)^{-1}x^i\tan\rho\ ,\\
&y^4=-a^{-1}\cos t\sec\rho\ ,
\end{aligned} \tag{2}$$

one finds the induced line element

$$ds^2=(a\cos\rho)^{-2}\{(dt)^2-(d\rho)^2-\sin^2\rho[(d\theta)^2+\sin^2\theta(d\varphi)^2]\} \tag{3}$$

and the Killing vectors take the form $K_{AB}=K_{AB}{}^{\mu}\partial_\mu$ with

$$\begin{aligned}
&K_{04}=\frac{\partial}{\partial t}\ ,\\
&K_{i4}=-\sin t\sin\rho\hat{\mathbf{x}}^i\frac{\partial}{\partial t}+\cos t\left[\frac{\rho}{\sin\rho}\left[\frac{\partial}{\partial \mathbf{x}^i}-\hat{\mathbf{x}}^i\hat{\mathbf{x}}^j\frac{\partial}{\partial x^j}\right]+\cos\rho\hat{\mathbf{x}}^i\hat{\mathbf{x}}^j\frac{\partial}{\partial x^j}\right]\ ,\\
&K_{i0}=-\cos t\sin\rho\hat{\mathbf{x}}^i\frac{\partial}{\partial t}-\sin t\left[\frac{\rho}{\sin\rho}\left[\frac{\partial}{\partial \mathbf{x}^i}-\hat{\mathbf{x}}^i\hat{\mathbf{x}}^j\frac{\partial}{\partial x^j}\right]+\cos\rho\hat{\mathbf{x}}^i\hat{\mathbf{x}}^j\frac{\partial}{\partial x^j}\right]\ ,\\
&K_{ij}=x^j\frac{\partial}{\partial x^i}-x^i\frac{\partial}{\partial x^j}\ .
\end{aligned} \tag{4}$$

For more details, see Ref. 4.

In any field theory on the fixed AdS background, such as the scalar theory with Lagrangian

$$\mathscr{L}=\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-\tfrac{1}{2}m^2\varphi^2-\tfrac{1}{6}g\varphi^3-\tfrac{1}{24}\lambda\varphi^4 \tag{5}$$

the SO(3,2) transformations are realized by Hermitian operators

$$M_{AB}=\int d^3x\sqrt{-g}\;T^0{}_\nu K^\nu_{AB}\ , \tag{6}$$

where $T_{\mu\nu}(x)$ is the stress tensor. (Additional "improvement" terms in these generators are required in supersymmetric theories,[7] but this is irrelevant for the present purpose which is to fix conventions.) These operators have commutators

$$[M_{AB},M_{CD}]=i(\eta_{BC}M_{AD}-\eta_{AC}M_{BD}-\eta_{BD}M_{AC}+\eta_{AD}M_{BC}) \tag{7}$$

and act on any scalar field, such as $\phi(x)$ or a composite scalar, as

$$[M_{AB},\phi(x)]=-iK_{AB}\phi(x)\ . \tag{8}$$

The derivation of the Lehmann spectral representation requires some facts[3,4,9] about the positive-energy infinite-dimensional unitary irreducible representations of (the covering group of) SO(3,2) which we now review. Each such representation is specified by the pair of numbers (λ,s) where λ is the positive lowest eigenvalue of the energy operator M_{04} and s is the total angular momentum of the unique state with energy λ. For $s=0$ or $\frac{1}{2}$ unitarity requires $\lambda\geq s+\frac{1}{2}$, while for $s\geq 1$ this condition becomes $\lambda\geq s+1$. The lowest states of the representation are the $(2s+1)$-dimensional rotational multiplet denoted by $|(\lambda,s)\lambda,s,m\rangle$ where $-s\leq m\leq s$. Other states are obtained by acting on these by repeated application of the energy boost operators $M_k^+=iM_{0k}-M_{k4}$ which increases energy by one unit and change angular momentum as any spatial vector would. The resulting normalized states are denoted by $|(\lambda,s)\omega,j,m\rangle$ where ω, $j(j+1)$, and m are the eigenvalues of, respectively, M_{04}, $J^2=\frac{1}{2}M_{ij}{}^2$, and $J_3=M_{12}$. For $s\geq 1$, there is a further degeneracy, i.e., typically more than one rotational multiplet of given j and ω. However, we do not wish to elaborate our notation further to incorporate this. All of this information is conveniently pictured in the weight diagram of a representation, given for the $(\lambda,0)$ irreducible representation (irrep) in Fig. 1.

The quadratic Casimir operator $C_2=\frac{1}{2}M_{AB}M^{AB}$ has eigenvalue $\lambda(\lambda-3)+s(s+1)$ in the irrep (λ,s). The key property of the lowest multiplet $|(\lambda,s)\lambda sm\rangle$ is that it is annihilated by the energy deboost operators $M_k^-=M_{0k}+M_{k4}$ which satisfy

$$[H,M_k^-]=-M_k^-\ . \tag{9}$$

Thus the Casimir eigenvalue is easily verified using the expression

$$\tfrac{1}{2}M_{AB}M^{AB}=M_{04}{}^2-3M_{04}+M_k^+M_k^-+J^2\ . \tag{10}$$

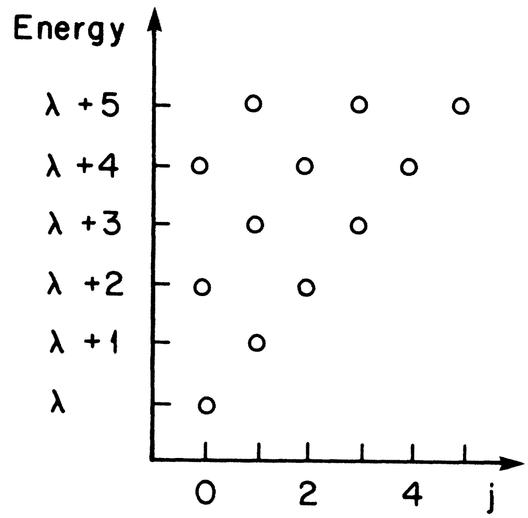

FIG. 1. Weight diagram of the $(\lambda,0)$ representation of SO(3,2). Each circle indicates the presence of a $(2j+1)$-dimensional rotational multiplet.

Let us now consider the spectrum of states of an interacting field theory such as (5). Because of SO(3,2) invariance, it is reasonable to assume that there is a unique vacuum $|0\rangle$, which satisfies $M_{AB}|0\rangle=0$, and that the remaining states can be classified in irreps (λ,s) and denoted by $|(\lambda s)\omega jm(\alpha)\rangle$. The label (α) denotes additional SO(3,2)-invariant quantum numbers which would be required to specify states uniquely. For example, the same representation (λ,s) might occur in the two- and three-particle sectors of the theory.

The free scalar theory in $(\mathrm{AdS})_4$ has been well studied.[2-4] If the Lagrangian mass parameter is m^2, then single-particle states belong to either of the two irreps $(\lambda^\pm,0)$, where $\lambda^\pm=\frac{3}{2}\pm(\frac{9}{4}+m^2/a^2)^{1/2}$. The eigenvalues $\lambda^\pm$ correspond to regular (irregular) scalar modes, respectively. Note that $m^2/a^2>-\frac{9}{4}$ leads to a stable vacuum,[4] and that in the range $-\frac{9}{4}<m^2/a^2<-\frac{5}{4}$, one can impose either set of boundary conditions and choose either the regular or irregular modes. The upper limit is determined by the unitarity condition $\lambda^->\frac{1}{2}$. If $m^2/a^2\geq-\frac{5}{4}$ then only the regular modes λ^+ are acceptable. Although the multiparticle states have not been examined in detail, it is quite clear that they are determined by direct products of the representation $(\lambda^\pm,0)$. By inspection of the weight diagram, it is clear that the two-particle sector contains the representations $(2\lambda^\pm,0)$, $(2\lambda^\pm+2,0)$, and infinitely more. Thus the allowed values of the quantum number λ are discrete, and related to the basic $\lambda^\pm$ by $\lambda=\lambda^\pm+$ (nonnegative integer). It is this picture of the spectrum of the free theory which we carry over to the interacting case.

Let us now derive the Lehmann representation by a method similar to one used in the simpler situation of the O(2,1)-invariant quantization of the Liouville theory.[10] Let $A(x)$ and $B(x)$ denote two Hermitian local scalar operators, either elementary or composite, and let us consider the Wightman function $\langle 0|A(x)B(x')|0\rangle$. We expand in a complete set of intermediate states

$$\langle 0|A(x)B(x')|0\rangle=\sum_{(\lambda,s)(\alpha)}\sum_{\omega jm}\langle 0|A(x)|(\lambda,s)\omega jm(\alpha)\rangle\langle(\lambda,s)\omega jm(\alpha)|B(x')|0\rangle\ . \tag{11}$$

We now show that only irreps which contribute in (11) have $s=0$, although $s\neq 0$ irreps may be present in the theory. From the first-order differential equations obtained from (8) it follows that the time and angular dependence of the matrix elements in (11) is

$$\langle 0|A(x)|(\lambda,s)\omega jm(\alpha)\rangle=R(\rho)e^{-i\omega t}Y_j^m(\theta,\varphi)\ , \tag{12}$$

where j is an integer and $R(\rho)$ is an unknown radial function. From (8) applied to the energy deboost M_k^-, one finds, using the fact that the lowest-energy multiplet is annihilated by M_k^-, the three differential equations

$$\begin{gathered}(iK_{0i}+K_{i4})\langle 0|A(x)|(\lambda,s)\lambda sm(\alpha)\rangle=0\ ,\\ \left[i\sin\rho\hat{\mathbf{x}}^i\frac{\partial}{\partial t}+\frac{\rho}{\sin\rho}\left[\frac{\partial}{\partial x^i}-\hat{\mathbf{x}}^i\hat{\mathbf{x}}^j\frac{\partial}{\partial x^j}\right]+\cos\rho\hat{\mathbf{x}}^i\hat{\mathbf{x}}^j\frac{\partial}{\partial x^j}\right]\langle 0|A(x)|(\lambda,s)\lambda sm(\alpha)\rangle=0\ ,\end{gathered} \tag{13}$$

where (4) has been used to obtain the last line. Since $\hat{\mathbf{x}}^j\partial/\partial x^j=\partial/\partial\rho$, the longitudinal component of the set of differential equations is just

$$\left[\lambda\sin\rho+\cos\rho\frac{\partial}{\partial\rho}\right]\langle 0|A(x)|(\lambda,s)\lambda sm(\alpha)\rangle=0 \tag{14}$$

which has the unique solution

$$R_{\lambda s}(\rho)=C(\cos\rho)^\lambda\ . \tag{15}$$

From the transverse components of (13) one finds that angular derivatives of the lowest mode matrix elements vanish. Hence only $s=m=0$ contributes.

It is no accident that (15) coincides with the known radial wave function[2–4] of the lowest mode of a free scalar field whose states transform in the irrep $(\lambda,0)$. From fairly straightforward manipulations using (8) one can deduce the action of the Casimir operator

$$\begin{aligned}\langle 0|[\tfrac{1}{2}M_{AB}M^{AB},A(x)]|(\lambda,0)\omega lm(\alpha)\rangle&=\tfrac{1}{2}K_{AB}K^{AB}\langle 0|A(x)|(\lambda,0)\omega lm(\alpha)\rangle\\ &=a^{-2}\Box\langle 0|A(x)|(\lambda,0)\omega lm(\alpha)\rangle\ .\end{aligned} \tag{16}$$

The fact that the second-order differential operator $\frac{1}{2}K_{AB}K^{AB}$ is the invariant d'Alambertian on $(\mathrm{AdS})_4$ comes as no surprise but requires a nasty computation. The equation above can be rewritten as

$$(\Box+m_\lambda{}^2)\langle 0|A(x)|(\lambda,0)\omega lm(\alpha)\rangle=0\ , \tag{17}$$

where $m_\lambda{}^2=a^2\lambda(\lambda-3)$ can be interpreted as the Lagrangian mass of a free scalar field for the irrep $(\lambda,0)$, and (17) is the corresponding wave equation.

Since (8) implies that all the operators M_{AB} have standard action on the matrix elements, it follows that these matrix elements are proportional to the known scalar mode functions[2–4]

$$\phi^\lambda_{\omega lm}(x)=\sqrt{N_{\omega lm}}e^{-i\omega t}Y_l^m(\theta,\varphi)(\sin\rho)^l(\cos\rho)^\lambda P_k^{(l+1/2,\lambda-3/2)}(\cos 2\rho)\ , \tag{18}$$

where the $P_k^{(a,b)}$ are Jacobi polynomials, $\omega=\lambda+l+2k$, and $N_{\omega lm}$ is a normalization constant determined from the standard scalar product

$$(\phi^\lambda_{\omega' l' m'},\phi^\lambda_{\omega lm})=i\int d^3x\sqrt{-g}\,g^{0\nu}(\phi'^*\overleftrightarrow{\partial}_\nu\phi)=\delta_{\omega'\omega}\delta_{l'l}\delta_{m'm}\ . \tag{19}$$

Thus we can write

$$\langle 0|A(x)|(\lambda,0)\omega lm(\alpha)\rangle=N(A,\lambda,(\alpha))\phi^\lambda_{\omega lm}(x)\ , \tag{20}$$

where $N(A,\lambda,(\alpha))$ is a constant which depends on the scalar operator A and the invariant quantum numbers.

Thus we can go back to (11) and rewrite it as

$$\langle 0|A(x)B(x')|0\rangle=\sum_\lambda\rho(\lambda,A,B)\sum_{\omega lm}\phi^\lambda_{\omega lm}(x)\phi^{\lambda *}_{\omega lm}(x')\ , \tag{21}$$

where the weight function $\rho(\lambda,A,B)$ is given by

$$\rho(\lambda,A,B)=\sum_{(\alpha)}N(A,\lambda,(\alpha))N^*(B,\lambda,(\alpha))\ . \tag{22}$$

The sum over mode functions in (21) actually gives a biscalar which is an invariant function of a free scalar field in $(\mathrm{AdS})_4$, specifically the Wightman function. It is more useful to examine the time-ordered function[11]

$$\langle 0|\, TA(x)A(x')\,|0\rangle=\sum_{\lambda}\rho(\lambda,A,A)\sum_{\omega lm}[\theta(x,x')\phi^{\lambda}_{\omega lm}(x)\phi^{\lambda *}_{\omega lm}(x')+\theta(x',x)\phi^{\lambda}_{\omega lm}(x')\phi^{\lambda *}_{\omega lm}(x)]\ . \tag{23}$$

The mode function sum in (23) is simply one way to express the scalar propagator, which has indeed been computed by doing the explicit sum[12] and by other methods.[13] The propagator is given by

$$i\Delta_F(x,x',\lambda)=\frac{a^2}{4\pi^2}\frac{d}{du}Q_{\lambda-2}(1-u)\ , \tag{24}$$

where $u=\frac{1}{2}a^2(y^A-y'^A)^2$ is the O(3,2)-invariant chordal distance variable for the points designated by $y^A(x)$ and $y'^A(x')$ as in (2), and $Q_\nu(z)$ is the Legendre function of the second kind.

Hence we can rewrite (23) as

$$\langle 0|\, TA(x)A(x')\,|0\rangle=\sum_{\lambda}\rho(\lambda,A,A)i\Delta_F(x,x',\lambda)\ , \tag{25}$$

whose weight function is clearly positive. This is the first form of the Lehmann spectral representation and it holds, under the stated spectral assumptions, as a nonperturbative result in anti–de Sitter field theory.

As one application of (25) let us take $A(x)$ to be the canonical scalar field of the Lagrangian (5). Differentiation with respect to t and the canonical commutation rules give the Lehmann sum rule

$$1=\sum_{\lambda}\rho(\lambda,\varphi,\varphi)=Z+\sum_{\lambda>\lambda_\pm}\rho(\lambda,\varphi,\varphi)\ , \tag{26}$$

where $Z=\rho(\lambda^{\pm},\phi,\phi)$ is the contribution to the weight function of the single-particle intermediate state with physical eigenvalue $\lambda^{\pm}$.

The standard Lehmann representation is most useful in momentum space, since it directly incorporates the analyticity properties of the Fourier transform of the propagator. The Fourier transform is not natural in AdS, since "plane waves" are not eigenfunctions of the wave operator. However there is a generalized Fourier transform, called the Gelfand-Graev transform, which is described in the monograph[8] of Vilenkin. The elegant geometrical basis[14] of the transform (which involves the notion of horospheres and generalized plane waves) has been discussed in connection with AdS field theory by Davis.[15]

The Gelfand-Graev transform is actually developed for functions on the hyperboloid H_4, defined as an embedded hypersurface by the equation $\bar{\eta}_{AB}\bar{y}^A\bar{y}^B=a^{-2}$ and $\bar{y}^4>0$ with $\bar{\eta}_{AB}=(-----+)$. The induced metric on H_4, with isometry group SO(4,1), can be obtained from (2,3) by the Euclidean rotation $t\to i\tau$ (or $y^0\to i\bar{y}^0$, $y^A\to\bar{y}^A A\geq 1$). Thus H_4 is a natural candidate for the Euclidean section of $(\mathrm{AdS})_4$. The analysis we now give does not rely heavily on the properties of the Euclidean field theory, and possible consequences of the global structure and the reflective boundary conditions are not considered.

A global set of coordinates for H_4 is given by

$$\begin{aligned}&\bar{y}^{\mu}=a^{-1}\sinh\theta\hat{x}^{\mu},\quad \mu=0,1,2,3\ ,\\ &\bar{y}^{4}=a^{-1}\cosh\theta,\quad \theta\geq 0\ ,\end{aligned} \tag{27}$$

where $\hat{x}^{\mu}$ is a Euclidean unit vector which can be further resolved into angular coordinates on the 3-sphere of unit radius. Details are not needed. The chordal distance variable on H_4 is

$$u=\tfrac{1}{2}a^2(\bar{y}^A-\bar{y}'^A)^2=1-\cosh\theta \tag{28}$$

if $\bar{y}'^A=(0,0,0,0,1)$. Thus u ranges over the region $u\leq 0$. For any function $F(u)$ on $(\mathrm{AdS})_4$, the analytic continuation $t\to i\tau$ takes us to the region $u\leq 0$ which is the spacelike region contiguous with the point of zero separation. See the Penrose diagram in Fig. 2. The integral transform, which we use, applies to functions $F(u)$ and can be thought of as defined in the contiguous spacelike region of $(\mathrm{AdS})_4$.

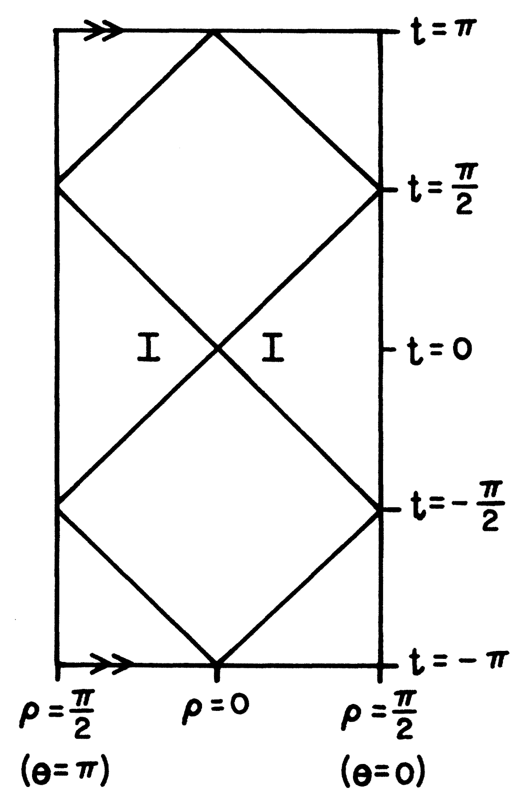

FIG. 2. The Penrose diagram of anti–de Sitter space. Light cones from the origin at the center reflect at spatial infinity $\rho=\pi/2$. The region marked I is the spacelike region contiguous with the origin in which the integral transform of the text is defined.

The transform discussed by Vilenkin takes a simple form for square integrable functions on H_4 which depend only on the single hyperbolic angle θ in (27). We set $z=\cosh\theta$. A function $f(z)$ can then be represented as

$$f(z)=\frac{i}{2}\int_{b-i\infty}^{b+i\infty} d\sigma(\sigma^2-1)\sigma\cot\pi\sigma\hat{f}(\sigma) \times P_\sigma{}^{-1}(z)(z^2-1)^{-1/2}\,, \tag{29}$$

where the transform $\hat{f}(\sigma)$ is given by

$$\hat{f}(\sigma)=\int_1^\infty dz f(z)(z^2-1)^{1/2}P_\sigma{}^{-1}(z)\,. \tag{30}$$

The integral over the vertical contour can be placed anywhere in the range $-2<b<2$, but $b=-\frac{1}{2}$ is most convenient because of the symmetries of the associated Legendre function $P_\sigma{}^{-1}(z)$. [Formulas (29) and (30) are obtained from those of p. 541 of Ref. 8 by a simple transformation.]

The conditions of square integrability with respect to the invariant measure on H_4 implies that $f(z)=O(z^{-3/2})$ as $z\to\infty$. We will relate functions $F(u)$ on $(\mathrm{AdS})_4$ to functions $f(z)$ on H_4 by the relation $F(u)\equiv f(z)$ with $u=1-z$. The transform then applies to functions which are $O(u^{-3/2})$ or $O((\cos\rho)^{3/2})$ at spatial infinity in AdS. This requires that the lowest allowed energy eigenvalue of the SO(3,2) irreps of the field theory satisfy $\lambda>\frac{3}{2}$. This implies that only regular modes are present. We accept this as a restriction on the range of validity of the present treatment, although we expect that the analysis can be extended to include irregular modes.

The transform of the propagator (24) can be obtained either by differentiation of a standard integral representation[16] of $Q_\nu(z)$ or by direct evaluation of (30). Both methods yield the representation

$$i\Delta_F(x,x',\lambda)=\frac{ia^2}{8\pi^2}\int_{b-i\infty}^{b+i\infty} d\sigma(\sigma^2-1)\sigma\cot\pi\sigma\frac{1}{\lambda(\lambda-3)-\sigma(\sigma+1)+2}\,\frac{P_\sigma{}^{-1}(z)}{(z^2-1)^{1/2}}\,. \tag{31}$$

Thus the transform of the free propagator for fields in the $(\lambda,0)$ representation is essentially a "pure pole" located at the eigenvalue $\lambda(\lambda-3)$ of the quadratic Casimir operator of O(3,2). This is no surprise. It simply indicates that the expansion is a natural spectral representation for AdS. Indeed, the special function in (31) satisfies

$$\Box_x[(z^2-1)^{-1/2}P_\sigma{}^{-1}(z)] =-a^2[\sigma(\sigma+1)-2][(z^2-1)^{-1/2}P_\sigma{}^{-1}(z)]\,, \tag{32}$$

where we regard $z(x,x')=y^A(x)y'(x')_A$ as a function of the points x and x' in AdS.

The Lehmann representation (25) [or simply O(3,2) invariance] shows that exact two-point functions depend only on the variable u or z. Thus we can set

$$F_A(u)=f_A(z)=\langle 0\,|\,TA(x)A(x')\,|\,0\rangle\,. \tag{33}$$

Then $f_A(z)$ can be represented as a transform (29) (we assume that the necessary asymptotic behavior holds). Using (29) and (31) on the left and right sides of (25), respectively, we can equate the transforms and deduce the result

$$\hat{f}_A(\sigma)=\frac{a^2}{4\pi^2}\sum_\lambda\frac{\rho(\lambda,A,A)}{\lambda(\lambda-3)-\sigma(\sigma+1)+2}\,. \tag{34}$$

Thus the transform of the exact two-point function satisfies a dispersion relation, which is entirely analogous to the situation in flat-space field theory. According to our spectral assumptions, the lowest contributing λ value in the sum, i.e., $\lambda=\lambda^+$, corresponds to the intermediate state of a single physical particle, and thus can be associated with the lowest pole in (34). Then one can define the physical mass of the particle in terms of this pole as $a^2[\sigma(\sigma+1)-2]=a^2\lambda^+(\lambda^+-3)=m_{\mathrm{phys}}{}^2$.

The significance of the Lehmann representation (25) and the dispersion relation (34) is not entirely clear. Perhaps they are merely the expected consequences of O(3,2) invariance and causality and signify little more. Nevertheless, we are struck by the close parallel between these results in AdS and their flat-space analogues, and we would like to speculate about possible further developments.

One can formally define one-particle-irreducible (1PI) vertex functions and truncated Green's functions in AdS field theory using functional methods and combinatorics, and one should be able to define "on-shell" amplitudes by integrating products of a truncated Green's function with free-field mode functions for the correct mass. We suggest that these on-shell amplitudes can be proved to share some of the properties of ordinary scattering amplitudes, such as independence of the choice of interpolating field and gauge independence (in a gauge field theory). Perhaps they are the AdS analogue of scattering amplitudes. This line of thinking may well fail because it is difficult to conceive of scattering processes in a space-time where there are no asymptotic regions where wave packets separate. Therefore we will end these speculations, simply by noting that some progress was made with the notion of scattering amplitudes in the O(2,1)-invariant situation of Ref. 10.

This work was supported in part by NSF Grant No. 84-07109-PHY and U.S. Department of Energy Contract No. DE-AC-02-76 ERO-3068.

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