Compactification of superstrings and torsion

Itzhak Bars
Department of Physics, University of Southern California, Los Angeles, California 90089-0484
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The compactification of the 10-dimensional superstring theories down to a supersymmetric 4-dimensional vacuum is reanalyzed by allowing for the possibility of torsion in the 6-dimensional compact manifold. New conditions on the generalized complex compact manifold and metric are derived. The connection and the Riemann and Ricci curvature tensors are obtained in an SU(3) basis. There is no 4-dimensional cosmological term despite a nonvanishing Ricci tensor or scalar curvature in the compact 6-dimensional manifold. However, if the Ricci tensor is required to vanish due to other considerations then torsion must also vanish, yielding a Kähler metric. Solutions to the equations are discussed and some explicit examples of 6-dimensional metrics are provided for both vanishing and nonvanishing torsion. However, we have not yet succeeded in solving for the gauge field with non-nonzero torsion.

Recently Candelas, Horowitz, Strominger, and Witten[1] (CHSW) have presented a very attractive scenario for compactifying the 10-dimensional SO(32) or $E_8 \times E_8$ superstring theories[2] to a 4-dimensional effective field theory which might be relevant to the physical world. For $N=1$, 4-dimensional supersymmetry to be valid at the compactification scale, it is necessary that the supersymmetric variations of the fermionic fields vanish in the vacuum state. Although the form of these variations, or the field theory of the massless sector of the superstring, are not yet well determined, CHSW make the assumption that (at least in the vacuum sector) the supersymmetry transformations have the Chapline-Manton form[3] suitably modified by the Chern-Simons terms introduced by Green and Schwarz[4] via the effective superstring theory. Then, within the framework of the CHSW assumptions it follows that (i) the 4-dimensional space-time background must be Minkowski space (i.e., no cosmological term) and (ii) there must exist an SO(6) Majorana spinor η in a 6-dimensional compact manifold $K=\{x_M, M=1,\ldots,6\}$ that satisfies the equations

$$[\partial_M + \tfrac{1}{4}\Gamma_{ij}(\omega_M^{ij} - 4\beta H_M^{ij})]\eta = 0 , \quad (1)$$

$$(\Gamma_{ijk}H^{ijk})\eta = 0 , \quad (2)$$

$$(\Gamma_{ij}E^{iM}E^{jN}F_{MN}^a)\eta = 0 , \quad (3)$$

where Γ_{ij} and Γ_{ijk} are totally antisymmetric products of SO(6) Dirac matrices Γ_i in 8-dimensional spinor space, and ω^{ij} is the SO(6) spin connection with $i,j=1,\ldots,6$ defined in tangent space. The totally antisymmetric tensor H_{MNP} is the B_{MN} field strength modified by the Chern-Simons forms and is most conveniently defined[4] in the form notation (i.e., $H = H_{MNP}dx^M \wedge dx^N \wedge dx^P$, $B = B_{MN}dx^M \wedge dx^N$)

$$H = dB - \tfrac{1}{30}\omega_{3y}^0 + \omega_{3L}^0 , \quad (4)$$

$$dH = -\tfrac{1}{30}F^a \wedge F^a + R^{ij} \wedge R^{ij} , \quad (5)$$

while

$$H_M^{ij} = E^{iN}E^{jP}H_{MNP} ,$$

$$H^{ijk} = E^{iM}E^{iN}E^{kP}H_{MNP} .$$

E_M^i is the sechsbein and E_i^M its inverse, while the metric in tangent space is the identity δ_{ij}. $F^a = F_{MN}^a dx^M \wedge dx^N$ is the Yang-Mills field strength with a belonging to SO(32) or $E_8 \times E_8$, and $R^{ij} = R_{MN}^{ij}dx^M \wedge dx^N$ is the curvature tensor for the SO(6) spin connection $\omega^{ij} = \omega_M^{ij}dx^M$ which must be torsionless:

$$dE^i + \omega^{ij} \wedge E^j = 0, \quad E^j = E_M^j dx^M . \quad (6)$$

CHSW specialized to the case of vanishing H_{MNP}. Dimensional arguments in Eq. (5) suggest that $H \neq 0$ might require a radius of compactification close to the Planck length, in which case-neglected derivative terms in the effective 10-dimensional field theory might become important (see below). However such issues cannot be settled until we know more about the classical vacuum solution as well as the effective 10-dimensional theory. Since $H \neq 0$ might potentially lead to interesting features for low-energy physics it is important to study it in more detail. Thus, we will analyze Eqs. (1)–(6) generally for nonzero H_{MNP} (including the case of $H=0$) and tackle Eqs. (4) and (5) last. We will see that we will arrive at generalizations of complex spaces with SU(3) holonomy that include torsion.

First, let us define a new spin connection Ω_M^{ij},

$$\Omega_M^{ij} = \omega_M^{ij} - 4\beta H_M^{ij} . \quad (7)$$

Using Eqs. (6) and (7) we see that H may be identified as the totally antisymmetric torsion of the spin connection Ω, in the spirit of Ref. 5:

$$dE^i + \Omega^{ij} \wedge E^j = 4\beta H_{MNP}E^{iP}dx^M \wedge dx^N \equiv T^i . \quad (8)$$

Second, from the point of view of SO(6)$\approx$SU(4) the 8-component Majorana spinor η contains only 4 independent complex components arranged as $4+4^*$ or $4-4^*$. Thus, by making a local SU(4) gauge transformation the 4 can be gauged to the form $\lambda(0,0,0,1)$ with λ real. We give the general solutions of Eqs. (1)–(3) in this gauge. Separating real and imaginary parts, the most general solution to Eq. (1) is given by

$$\frac{1}{4}\Gamma_{ij}\Omega^{ij}_M dx^M = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -A^T & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \eta_\pm = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \pm 1 \end{bmatrix}, \tag{9}$$

where the one-form A is any 3×3 anti-Hermitian traceless matrix, $A^\dagger = -A$, $\mathrm{tr}A = 0$, corresponding to SU(3) holonomy, and A^T is the transpose matrix.

We give the Hermitian 8×8 Γ_i matrices appropriate to this SU(3) basis as direct products of three Pauli matrices:

$$\begin{aligned} &\Gamma_1 = -\sigma_1\times\sigma_1\times\sigma_2, \quad \Gamma_2 = -\sigma_1\times\sigma_2\times 1, \\ &\Gamma_3 = \sigma_1\times\sigma_3\times\sigma_2, \quad \Gamma_4 = -\sigma_2\times\sigma_2\times\sigma_1, \\ &\Gamma_5 = -\sigma_2\times\sigma_2\times\sigma_3, \quad \Gamma_6 = \sigma_2\times 1\times\sigma_2. \end{aligned} \tag{10}$$

Then $\Gamma_7 = i\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5\Gamma_6$ is diagonal, $\Gamma_7 = \sigma_3\times 1\times 1$. It is useful to define the SU(3)-covariant combinations

$$\gamma^\alpha = \tfrac{1}{2}(\Gamma_\alpha + i\Gamma_{\alpha+3}), \quad \bar\gamma_\alpha = \tfrac{1}{2}(\Gamma_\alpha - i\Gamma_{\alpha+3}) = (\gamma^\alpha)^\dagger \tag{11}$$

that satisfy $\{\gamma^\alpha,\gamma^\beta\} = 0 = \{\bar\gamma_\alpha,\bar\gamma_\beta\}, \{\gamma^\alpha,\bar\gamma_\beta\} = \delta^\alpha{}_\beta$, where $\alpha = 1,2,3$ and lower and upper SU(3) indices are distinguished. Similarly we define the complex one-forms

$$e^\alpha = (E^\alpha_M - iE^{\alpha+3}_M)dx^M,$$
$$\bar e_\alpha = (E^\alpha_M + iE^{\alpha+3}_M)dx^M = (e^\alpha)^*$$

so that $E\cdot\Gamma = \bar e\cdot\gamma + e\cdot\bar\gamma$ is SU(3) invariant. In this basis we write the spin connection as

$$\tfrac{1}{4}\Gamma_{ij}\Omega^{ij} = \tfrac{1}{2}\bar\Omega_{\alpha\beta}\gamma^\beta\gamma^\alpha + \tfrac{1}{2}\Omega^{\alpha\beta}\bar\gamma_\beta\bar\gamma_\alpha + \tfrac{1}{2}[\gamma^\beta,\bar\gamma_\alpha]\Omega^\alpha{}_\beta \tag{12}$$

and from Eqs. (9)–(11) identify

$$\begin{aligned} &\bar\Omega_{\alpha\beta} = 0 = \Omega^{\alpha\beta}, \\ &\Omega^\alpha{}_\beta = -A_\beta{}^\alpha = (3\times 3 \text{ Hermitian, traceless}). \end{aligned} \tag{13}$$

Using our γ matrices and η in Eq. (9), we can solve Eq. (2) generally by

$$4\beta H^{ijk}\Gamma_{ijk} = 3!\begin{bmatrix} 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 \\ -h^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

where $h_{\alpha\beta} = h_{\beta\alpha}$ is any symmetric complex 3×3 matrix, whose symmetry follows from the form of the γ matrices. Hence h is in the 6-dimensional representation of SU(3). Using the definition of T in terms of H [Eq. (8)] and the result of Eq. (14) along with our γ matrices, we find the torsion in the SU(3) basis written as a two-form,

$$T^\alpha = e^\beta e^\gamma \epsilon_{\beta\gamma\delta} h^{*\delta\alpha} + 2\epsilon^{\alpha\beta\delta}\bar e_\beta e^\gamma h_{\delta\gamma}, \tag{15}$$

from which one may easily extract H_{MNP}, H^{ij}_M, and H^{ijk}, if desired (see below).

Turning to Eq. (3) we may expand in terms of the γ's

$$\begin{aligned} \tfrac{1}{2}\Gamma^{ij}E_i^M E_j^N F^a_{MN} = {} & \gamma^\alpha\gamma^\beta F^a_{\beta\alpha} + \bar\gamma_\alpha\bar\gamma_\beta \bar F^{a\beta\alpha} \\ & + [\gamma^\beta,\bar\gamma_\alpha]F^{a\alpha}_\beta. \end{aligned} \tag{16}$$

Requiring anti-Hermiticity,

$$\bar F^{a\alpha\beta} = (F^a_{\alpha\beta})^*, \; F^{a\alpha}_\beta = -(F^{a\beta}_\alpha)^*,$$

and imposing Eq. (3) we obtain

$$F^a_{\alpha\beta} = 0 = F^{a\alpha\beta}, \quad \sum_{\alpha=1}^{3} F^{a\alpha}{}_\alpha = 0. \tag{17}$$

So that the general solution is provided by the 3×3 anti-Hermitian traceless matrices $F^{a\beta}_\alpha$, or as two-forms

$$F^a = 2e^\alpha\wedge\bar e_\beta F^a{}_\alpha{}^\beta. \tag{18}$$

Thus, the spinor equations (1)–(3) are solved in terms of the unspecified complex 3×3 matrices $h_{\alpha\beta}, F^{a\beta}_\alpha$ and complex one-forms e^α and $\Omega^\alpha{}_\beta$ with the conditions that h is symmetric, $F^{a\beta}{}_\alpha$ is anti-Hermitian and traceless, $\Omega^\alpha{}_\beta$ is anti-Hermitian and traceless, $\bar e_\alpha = (e^\alpha)^*$.

These functions are further constrained by Eqs. (4) and (8). In particular, using Eq. (15) the torsion condition (8) takes the form

$$de^\alpha + \Omega^\alpha{}_\beta\wedge e^\beta = e^\beta\wedge e^\gamma\epsilon_{\beta\gamma\delta}h^{*\delta\alpha} + 2\epsilon^{\alpha\beta\delta}\bar e_\beta\wedge e^\gamma h_{\gamma\delta}. \tag{19}$$

At this point it is useful to define a complex base space via

$$z^m = \frac{1}{\sqrt{2}}(x^m - ix^{m+3}), \quad \bar z^m = \frac{1}{\sqrt{2}}(x^m + ix^{m+3}), \quad m = 1,2,3$$

and write

$$\begin{aligned} &d = dz^m\frac{\partial}{\partial z^m} + d\bar z^m\frac{\partial}{\partial\bar z^m} \equiv \partial + \bar\partial, \\ &\bar e_\alpha = dz^m\bar e_{m\alpha} + d\bar z^m e_{\bar m\alpha}, \quad e^\alpha = dz^m e^\alpha_m + d\bar z^m\bar e^\alpha{}_{\bar m}, \\ &dx^M\Omega^\alpha_{M\beta} = dz^m\Omega^\alpha_{m\beta} + d\bar z^m\bar\Omega^\beta{}_{\bar m\alpha}, \end{aligned} \tag{20}$$

where the condition on the one-form $\bar\Omega^\alpha{}_{\bar m\beta} = -(\Omega^\beta{}_{m\alpha})^*$ is required for anti-Hermiticity, while $\bar e^\alpha{}_{\bar m}$ is not related to the complex conjugate of $e^\alpha{}_m$, rather $(e^\alpha{}_m)^* \equiv e_{\bar m\alpha}$. In this paper we will specialize to solutions of the form $\bar e^\alpha{}_{\bar m} = 0$, so that

$$e^\alpha = dz^m e^\alpha{}_m, \quad \bar e_\alpha = d\bar z^m e_{\bar m\alpha}. \tag{21}$$

The inverses of the 3×3 matrices $e^{\alpha}{}_{m}$ and $e_{\bar{m}\alpha}=(e^{\alpha}{}_{m})^{*}$ will be denoted by $e^{m}{}_{\alpha}$ and $e^{\alpha\bar{m}}$, respectively. We will also define the Hermitian metric and its inverse

$$g_{\bar{m}n}=e_{\bar{m}\alpha}e^{\alpha}{}_{n},\quad g^{n\bar{m}}=e^{n}{}_{\alpha}e^{\alpha\bar{m}}\ . \tag{22}$$

If Eq. (20) were not specialized to Eq. (21) there would have been the additional $g_{mn}, g_{\bar{m}\bar{n}}$ components in the metric. We have adopted Eq. (21), which is equivalent to the requirement of a Hermitian complex space, because it greatly simplifies the system of equations, but we have lost generality which we intend to investigate elsewhere. In terms of these definitions the torsion equation (19) implies two independent complex conditions proportional to $dz^{m}\wedge dz^{n}$ and $dz^{m}\wedge d\bar{z}^{n}$:

$$\begin{aligned}&(\partial_{[m}e_{n]}+\Omega_{[m}e_{n]})^{\alpha}=e^{\beta}_{[m}e^{\gamma}_{n]}\epsilon_{\beta\gamma\delta}h^{*\delta\alpha}\ ,\\ &\partial_{\bar{m}}e^{\alpha}{}_{n}+\bar{\Omega}^{\alpha}{}_{\bar{m}\beta}e^{\beta}_{n}=2\epsilon^{\alpha\beta\delta}h_{\delta\gamma}e_{\bar{m}\beta}e^{\gamma}_{n}\ ,\end{aligned} \tag{23}$$

The solution to these equations is unique:

$$\begin{aligned}&\Omega^{\alpha}_{m\beta}=e^{\alpha\bar{n}}\partial_{m}e_{\bar{n}\beta}-e^{\alpha\bar{n}}(\partial_{m}g_{\bar{n}p}-\partial_{p}g_{\bar{n}m})e^{p}{}_{\beta}(\det\bar{e})^{-1}\ ,\\ &\bar{\Omega}^{\alpha}{}_{\bar{m}\beta}=-(\Omega^{\beta}_{m\alpha})^{*}\ ,\\ &h_{\alpha\beta}=-\tfrac{1}{2}e^{p}_{\alpha}e^{q}_{\beta}g_{\bar{m}p}\partial_{\bar{s}}g_{\bar{n}q}\epsilon^{\bar{m}\bar{s}\bar{n}}(\det\bar{e})^{-1}=h_{\beta\alpha}\ ,\end{aligned} \tag{24}$$

and the metric must satisfy

$$\det g=|\det e|^{2}=1,\quad \partial_{m}g^{m\bar{n}}=0=\partial_{\bar{m}}g^{n\bar{m}}\ . \tag{25a}$$

The determinant and divergence conditions on the metric guarantee the tracelessness of $\Omega^{\alpha}_{m\beta}$ [SU(3) holonomy] and the symmetry of $h_{\alpha\beta}$. More generally it is sufficient to require $\partial_{m}(g^{m\bar{n}}\det g)=0$, and $\det e=f(z)$, or $\det g=|f(z)|^{2}$, with $f(z)=$ analytic. But since $\det(g)$ can be mapped to identity by an analytic general coordinate transformation Eq. (25) is a general solution to Eq. (23). Note that the noncovariant looking divergence $\partial_{n}g^{n\bar{m}}$ can be rewritten covariantly in terms of a curl by using $0=\partial_{p}\ln(\det g)=g^{n\bar{q}}\partial_{p}g_{\bar{q}n}$ as

$$\begin{aligned}0=\partial_{n}g^{n\bar{m}}&=-g^{n\bar{q}}\partial_{n}g_{\bar{q}p}g^{p\bar{m}}\\ &=-g^{n\bar{q}}(\partial_{n}g_{\bar{q}p}-\partial_{p}g_{\bar{q}n})g^{p\bar{m}}\ .\end{aligned} \tag{25b}$$

We can now identify the torsion 3-form H of Eqs. (4) and (5) via Eqs. (8), (15), and (24)

$$\begin{aligned}&4\beta H=T^{\alpha}\bar{e}_{\alpha}+\bar{T}_{\alpha}e^{\alpha}=3(\partial-\bar{\partial})J\ ,\\ &4\beta dH=6\bar{\partial}\partial J\ ,\end{aligned} \tag{26}$$

where J is the metric two-form $J=g_{\bar{m}n}d\bar{z}^{m}\wedge dz^{n}$.

The curvature tensor for Ω (that includes torsion) has the components $R^{\alpha}{}_{mn\beta}$, $\bar{R}^{\alpha}{}_{\bar{m}\bar{n}\beta}$, and $R^{\alpha}{}_{m\bar{n}\beta}$ which are expressed as two-forms $(\partial\Omega+\Omega^{2})^{\alpha}{}_{\beta}$, $(\bar{\partial}\bar{\Omega}+\bar{\Omega}^{2})^{\alpha}{}_{\beta}$, and $(\partial\bar{\Omega}+\bar{\partial}\Omega+\Omega\bar{\Omega}+\bar{\Omega}\Omega)^{\alpha}{}_{\beta}$, respectively, and they are messy. The Ricci tensors obtained from the first two components vanish while the third component yields

$$\begin{aligned}R_{\bar{n}r}&\equiv e^{m}{}_{\alpha}e^{\beta}{}_{r}R^{\alpha}{}_{m\bar{n}\beta}\\ &=-g^{m\bar{s}}(\partial_{m}\partial_{[\bar{s}}g_{\bar{n}]r}-\partial_{r}\partial_{[\bar{s}}g_{\bar{n}]m})\ .\end{aligned} \tag{27}$$

Thus, the Ricci tensor and the torsion [Eq. (26)] are nonzero if the curl of the metric is nonzero. We emphasize that, despite a nonvanishing Ricci tensor in the 6-dimensional compact manifold, the cosmological term in 4 dimensions vanishes, since, by virtue of the supersymmetry properties of the vacuum assumed by CHSW, the 4-dimensional space is guaranteed to be flat Minkowski space.

Nonlinear σ models in 2 dimensions may be taken to represent the vacuum configurations of the superstring theory. According to current models of a certain type, the finiteness of the theory requires a vanishing Ricci tensor.[6] This may not be a feature of all conceivable string models, but may provide a rationale[1,7] for requiring $R_{\bar{n}r}=0$. According to our expression in Eq. (27), in 3 complex dimensions these 9 conditions are equivalent to the 9 conditions, $\epsilon^{\bar{m}\bar{p}\bar{q}}\partial_{\bar{p}}g_{\bar{q}n}=0$, that indicate a vanishing torsion. For additional arguments, see also Ref. 7. Thus, requiring a vanishing Ricci tensor would eliminate torsion and solve Eq. (25) by a unimodular Kähler metric $g_{\bar{n}m}=\partial_{\bar{n}}\partial_{m}K(z,\bar{z})$. Although this argument seems plausible, for the sake of generality and to allow for the possibility of more general finite σ models, we will not yet assume that the Ricci tensor vanishes.

The $SO(6)\approx SU(4)$ Riemann spin connection ω is solved from Eq. (6). In the SU(3) basis we have the complex nonet $\omega^{\alpha}_{m\beta}$, the triplets $\omega_{m\alpha}$, $\omega_{\bar{m}\alpha}$, and their complex conjugates, satisfying the torsion-free conditions that follow from Eqs. (6) and (21),

$$\begin{aligned}&\partial e^{\alpha}+\omega^{\alpha}{}_{\beta}\wedge e^{\beta}=0\ ,\\ &\bar{\partial}e^{\alpha}+\epsilon^{\alpha\beta\gamma}\bar{e}_{\beta}\wedge\omega_{\alpha}+\bar{\omega}^{\alpha}{}_{\beta}\wedge e^{\beta}=0\ ,\\ &\epsilon^{\alpha\beta\gamma}\bar{\omega}_{\beta}\wedge\bar{e}_{\gamma}=0\ ,\end{aligned}$$

and their complex conjugates. The unique solution is

$$\begin{aligned}&\omega_{\bar{m}\alpha}=0\ ,\\ &\omega_{m\alpha}=-\tfrac{1}{2}\epsilon^{\bar{n}\bar{p}\bar{q}}\partial_{\bar{n}}g_{\bar{p}m}e_{\bar{q}\alpha}=-\tfrac{1}{2}h_{\alpha\beta}e^{\beta}{}_{m}\ ,\\ &\omega^{\alpha}_{m\beta}=e^{\alpha\bar{n}}\partial_{m}e_{\bar{n}\beta}-\tfrac{1}{2}e^{\alpha\bar{n}}(\partial_{m}g_{\bar{n}p}-\partial_{p}g_{\bar{n}m})e^{p}{}_{\beta}\\ &\qquad=\Omega^{\alpha}_{m\beta}+\tfrac{1}{2}e^{\alpha\bar{n}}(\partial_{m}g_{\bar{n}p}-\partial_{p}g_{\bar{n}m})e^{p}{}_{\beta}\ ,\end{aligned} \tag{28}$$

and $\omega_{\bar{m}\alpha}=0$, $\omega^{\alpha}{}_{\bar{m}}=-(\omega_{m\alpha})^{*}$, $\omega_{\bar{m}}{}^{\alpha}{}_{\beta}=-(\omega_{m}{}^{\beta}{}_{\alpha})^{*}$. Note that the $\omega^{\alpha}_{m\beta}$ comes out as a traceless octet only.

It is the curvature tensor two-form of the SO(6) connection that appears in Eq. (5). Written in SU(4) notation the contribution to Eq. (5) has the form $2\,\mathrm{Tr}R^{2}$ where the traceless 4×4 matrix R is

$$R=\begin{bmatrix}R^{\alpha}{}_{\beta} & R^{\alpha}\\ R_{\beta} & R\end{bmatrix} \tag{29a}$$

with

$$R^{\alpha}{}_{\beta}=[R(2,0)+R(0,2)+R(1,1)]^{\alpha}{}_{\beta}\ ,$$

$$R_{\beta}=[R(2,0)+R(1,1)]_{\beta},\quad R^{\alpha}=-(R_{\alpha})^{*}$$

and the various (p,q) forms $R(p,q)$ given as

$$R(2,0)^\alpha{}_\beta=(\partial\omega+\omega^2)^\alpha{}_\beta,\quad R(0,2)^\alpha{}_\beta=(\overline{\partial\omega}+\overline{\omega}^2)^\alpha{}_\beta\ ,$$
$$R(1,1)^\alpha{}_\beta=(\partial\overline{\omega}+\overline{\partial}\omega+\omega\wedge\overline{\omega}+\overline{\omega}\wedge\omega)^\alpha{}_\beta+\omega_\beta\wedge\overline{\omega}^\alpha\ ,$$
$$R(2,0)_\beta=\partial\omega_\beta+\omega_\alpha\wedge\omega^\alpha{}_\beta\ , \tag{29b}$$
$$R(1,1)_\beta=\overline{\partial}\omega_\beta+\omega_\alpha\wedge\overline{\omega}^\alpha{}_\beta\ ,$$
$$R=\overline{\omega}^\alpha\wedge\omega_\alpha\ .$$

These expressions are messy when written in terms of the metric and will not be given here explicitly.

For a sechsbein of the form (21), Eq. (17) for the gauge field implies in base space

$$F^a_{mn}=0=F_{\overline{m}\,\overline{n}},\quad F^a_{m\overline{n}}g^{m\overline{n}}=0\ . \tag{30a}$$

By introducing an anti-Hermitian representation matrix t^a for SO(32) or $E_8\times E_8$ we can write the solution of Eq. (30a) for the gauge potential in the form

$$t^aA^a_m=V^{-1}\partial_mV,\quad t^aA^a_{\overline{m}}=V^\dagger\partial_{\overline{m}}V^{\dagger-1}\ , \tag{30b}$$

where V is a unimodular group element in complexified SO(32) or $E_8\times E_8$ (since A^a_m is complex). Computing $F_{m\overline{n}}$ we obtain

$$t^aF^a_{m\overline{n}}=V^{-1}[\partial_m(G\partial_{\overline{n}}G^{-1})]V\ , \tag{30c}$$

where $G=VV^\dagger$ is a unimodular Hermitian group element in SO(32) or $E_8\times E_8$. Thus, from Eq. (30) we see that G must satisfy the equation

$$g^{m\overline{n}}\partial_m(G\partial_{\overline{n}}G^{-1})=0\ . \tag{31}$$

This is analogous to the equation of motion of a nonlinear σ model in a curved background. Topologically nontrivial solutions (e.g., analogous to the Skyrmion) may be expected.

We have shown that the torsion constraint with the forms (21) have allowed us to determine uniquely the forms ω (or Ω), H, F^a in terms of the complex unimodular e^α_m, and the Hermitian unimodular G. In turn these quantities are required to satisfy Eqs. (25) and (31). Of course from the Hermitian metric $g_{\overline{m}n}$ and the Hermitian matrix G we can compute e^α_m and V by taking square roots, up to gauge transformations U in SU(3) and S in SO(32) or $E_8\times E_8$, $e^\alpha{}_m=U^{\alpha\overline{n}}(\sqrt{g}\,)_{\overline{n}m}$, $V=\sqrt{G}S$. Thus, Eqs. (1), (2), (3), (6), or (8) boil down to Eqs. (25) and (31). There remains Eq. (4) or (5).

Let us first consider in our formalisms the no torsion case $H=0$, for which a set of solutions was given by CHSW. We see from Eq. (26) that the curl of the metric must vanish and from Eq. (25) that the divergence of the inverse metric is automatically zero. Thus we must have Kähler metric $g_{m\overline{n}}=\partial_m\partial_{\overline{n}}K_{(z,\overline{z})}$. Then Eqs. (24) and (28) simplify to the one-form $\omega^\alpha_\beta=\Omega^\alpha{}_\beta=e^{\alpha\overline{s}}\partial e_{\beta\overline{s}}+e^\alpha{}_s\overline{\partial}e^s{}_\beta$, and the curvature tensor takes a form similar to the gauge field in (30c):

$$R^\alpha_{m\overline{n}\beta}=e^{\alpha\overline{s}}\partial_m(g_{\overline{s}p}\partial_{\overline{n}}g^{p\overline{q}})e_{\overline{q}\beta}\ , \tag{32}$$

and is Ricci flat by virtue of $\det g=1$ and the vanishing curl. Furthermore the metric now satisfies

$$g^{m\overline{n}}\partial_m(g_{\overline{s}p}\partial_{\overline{n}}g^{p\overline{q}})=0 \tag{33}$$

which by comparison to Eq. (31) allows one to find a solution for the gauge group element V or G by embedding the space-time manifold as a 3×3 matrix in $E_8\times E_8$ and taking $V=e_{\overline{s}\alpha}$ or

$$G=g_{\overline{n}m}=\partial_m\partial_{\overline{n}}K(z,\overline{z})\ , \tag{34}$$

thus leaving an unbroken gauge group $E_6\times E_8$. Since we now have $R=F$ and $H=0$, clearly Eqs. (4) or (5) are also automatically satisfied. There remains the determinant condition

$$\det(g_{m\overline{n}})=\det[\partial_m\partial_{\overline{n}}K(z,\overline{z})]=1\ . \tag{35}$$

which is called the Monge-Ampère equation. Calabi and Yau have provided the mathematical tools for characterizing compact 6-dimensional manifolds that solve this equation, as discussed by CHSW and Strominger and Witten.[8] Unfortunately, these methods do not provide an explicit Kähler potential $K(z,\overline{z})$ but allow one to find metrics which are in the same topological class without actually satisfying $\det g=1$. These correspond to the Calabi-Yau manifolds. The number of families of massless chiral fermions are then determined by $\frac{1}{2}$ times the Euler number of the manifold.

It is possible that Eq. (31) for the gauge field has solutions other than Eq. (34) while still maintaining a Ricci-flat Kahler metric. Then we may wonder what determines the number of families. To examine this question suppose that the Euler number of the compact manifold is zero. Then the 6-dimensional gauge-covariant Dirac equation can still have chiral zero modes if the SO(32) or $E_8\times E_8$ gauge field has certain topologically nontrivial Chern numbers. For analogies recall instanton gauge fields in a flat 4-dimensional manifold which have chiral Dirac zero modes by virtue of a nontrival second Chern number. In fact, by the use of the Atiyah-Singer theorem it can be shown[9] that it is a particular combination of Chern numbers of the gauge group element G that determines the number of families, *for any value of the Euler number of the manifold.* The result is the same even if torsion is nonzero and the metric is not Kähler. For the solution in Eq. (34) the Euler number and the third Chern number are related, but this may not be the case for other possible solutions.

Eventually the details of low-energy physics, such as the mass matrix will depend on the explicit details of the Kähler potential. Thus we are motivated to make efforts toward this goal. A general theorem states that[10] compact manifolds with SU(3) holonomy satisfying Eq. (35) cannot have symmetries globally. However, looking for solutions symmetric under certain transformations which are later broken by identifying points in the manifold may be a method of exploration. Furthermore, even though the symmetric solutions cannot correspond to a global metric they may be useful for the description of the metric in certain regions of the manifold.

The simplest case of a symmetry is obtained by taking an SU(3) invariant of the form $K(R^2)$ with

$$R^2=z_1^*z_1+z_2^*z_2+z_3^*z_3\ .$$

Then Eq. (35) becomes a simple differential equation in the variable R^2 and is solved by $\partial K/\partial R^2 = (\pm 1 + R^{-6})^{1/3}$, leading to the metrics

$$g_{m\bar{n}} = \pm\delta_{m\bar{n}}(\pm 1 + R^{-6})^{1/3} - \bar{z}_m z_n R^{-8}(\pm 1 + R^{-6})^{-2/3} . \quad (36)$$

The $(+)$ case gives positive eigenvalues for all R^2. The apparent singularity at the origin is removed by restricting the space to submanifolds satisfying the identification of points $z_n \simeq e^{i2\pi/3} z_n$. This is possible since the solution has a phase symmetry (and more). This solution (which we rediscovered with D. Caldi) was suggested in a different context by Freedman and Gibbons[11] and it is the generalization of the Eguchi-Hanson metric.[12] Unfortunately, since R can range all the way to infinity the volume of this space is infinite (recall $\det g = 1$) and therefore not useful for Kaluza-Klein compactification. However, this metric may be used, in construction of a more satisfactory solution, as a patch at singularities similar to the construction of the analogous K_3-metric in 2 complex dimensions. The $(-)$ case gives positive eigenvalues for $R^2 > 1$ and has similar properties to the $(+)$ case.

We have found other explicit solutions to Eq. (35), each having some unsatisfactory feature from the point of view of physical applications. We have also found classes of explicit solutions, involving arbitrary functions, to the closely related equation $\det(\partial_n \partial_{\bar{m}} K) = 0$ in any dimension. Our effort is still continuing and the results will be reported elsewhere.

We now return to nonvanishing torsion. It is useful to establish that Eqs. (25), which determine the metric $g_{\bar{m}n}$, have solutions that do not correspond to vanishing torsion and thus generalize the Kähler metric with SU(N) holonomy. We provide here some explicit solutions. The first solution is diagonal:

$$g_{\bar{m}n} = \begin{bmatrix} |F_2/F_3|^2 & 0 & 0 \\ 0 & |F_3/F_1|^2 & 0 \\ 0 & 0 & |F_1/F_2|^2 \end{bmatrix} , \quad (37)$$

where $F_1 = F_1(z_1, \bar{z}_1)$, $F_2 = F_2(z_2, \bar{z}_2)$, $F_3 = F_3(z_3, \bar{z}_3)$ are three arbitrary but nonvanishing and finite functions of the respective variables. The determinant is unity and the eigenvalues are positive by virtue of the absolute values. The variables $(z_i, \bar{z}_i)$ or equivalently the real and imaginary parts (x_i, y_i), are taken in any compact manifold K. For example, if we put the 6-dimensional manifold on a lattice and impose periodic boundary conditions then (x_i, y_i) may be taken within a 6-dimensional cube. This would require the functions F_i to be periodic in the variables (x_i, y_i). This manifold has a zero Euler number since it is the 6-dimensional torus. However, as pointed out above the Euler number has nothing to do with the number of families which are actually determined[9] by the Chern numbers of the gauge group element G that solves Eq. (31). Similarly, solution (37) may be defined on other compact manifolds. The functions F_i must be defined so that singularities in the metric are avoided.

A second class of solutions is obtained by taking $e_m^\alpha(z)$ analytic in z_1, z_2, z_3. In the absence of torsion this would yield a trivial Kähler metric $e_m^\alpha(z) = \partial_m f^\alpha(z)$. However, with nonzero torsion Eq. (25) reduces to $\partial_n[(\det e) e_\alpha^n] = 0$ whose solution is

$$(\det e) e^n{}_\alpha(z) = \epsilon^{npq} \partial_p A_{q\alpha}(z) ,$$
$$(\det e)^2 = \det(\epsilon^{npq} \partial_p A_{q\alpha}) , \quad (38)$$

where $A_{q\alpha}(z)$ is arbitary. The determinant of the metric now has the form $\det g = |f(z)|^2$ which is a general solution as pointed out in the discussion following Eq. (25a). The metric has positive eigenvalues since $g_{\bar{m}n} = e_{\bar{m}\alpha} e^\alpha{}_n$ is the product of Hermitian conjugate matrices.

Given a metric we must still look for solutions for the gauge group element G that satisfies Eqs. (31) and (4) or (5). Here we must consider the size L of the compact manifold relative to the Planck length l. Derivatives in the compact manifold are expected to be of order $1/L$. To be able to consider reliably an effective 10-dimensional field theory that represents the physics of the superstring (in the manner of Chapline and Manton) we must take $L \gg l$. However, the left and right sides of Eqs. (4) or (5) scale differently and the compensating scale factor is the Planck length l

$$l^{-2} dH = R^2 - \tfrac{1}{30} F^2 . \quad (39)$$

This is more clearly seen from our Eqs. (26), (29) and (30) where dH, R, and F are expressed in terms of the dimensionless metric and G. Naively, it would appear that the left-hand side is of order $l^{-2} L^{-2}$ while the right-hand side is of order L^{-4} so that $L \approx l$ is expected. If $L \approx l$, Eqs. (1)—(5) may need modification from higher derivative terms in the effective 10-dimensional field theory, so that the entire analysis becomes suspect for such a vacuum solution. Of course, if $H = 0$ this difficulty does not arise and we may concentrate on compact manifolds with $L \gg l$.[13] Another way of avoiding the problem is to look for solutions in which dH is smaller than expected on naive grounds. That is, there may be vacuum metrics for which the particular combination of curls symbolized by dH is of order l^2/L^4 instead of $1/L^2$, while a typical derivative is still of order $1/L$. It is this type of manifold that can be reliably used for the physics of compactified effective string theory if torsion is not zero. We have checked, for simpler solvable nonlinear partial differential equations, that solutions of this type do exist. The more difficult question of whether such a solution, or for that matter $H = 0$ solution, is preferred by the theory as an absolute minimum solution cannot be answered with the considerations presented so far. Our work is still in progress and further results will be reported elsewhere.

Our present conclusions are summarized in the abstract.

Note added. After submittal of this paper for publication we became aware of a revised version of Ref. 1 in which nonzero values of H are discussed without arriving at a definite solution.

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