Critical behavior near deconfinement

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We determine the temperature dependence of the pressure, the energy density, and the entropy density in statistical QCD with dynamical quarks. Our results are based on Monte Carlo calculations with Wilson fermions in a fourth-order hopping-parameter expansion on an $8^3\times 3$ lattice. Using these results, we study the velocity of sound and the thermodynamic quantities relevant for the collective transverse expansion of the system and for the transverse momentum of secondaries in nuclear collisions.

I. INTRODUCTION

In recent years, many theoretical arguments were given that at sufficiently high temperature (200–300 MeV) and/or baryon-number density (a few fm^{-3}) hadronic matter should undergo one or two phase transitions, in which color is deconfined and chiral symmetry is restored.[1]

As a result of these transitions, a new state of hadronic matter, the quark-gluon plasma, is produced. In the region of temperature and density significantly above the transition region, some of the thermodynamical properties of this plasma can be deduced from perturbative QCD. The critical behavior of hadronic matter, however, is of nonperturbative nature and hence requires some new method of investigation. One such method is based on the lattice regularization of QCD and subsequent Monte Carlo simulation.

Lattice calculations prove to be quite successful and transparent for pure SU(3) gauge theory, where the existence of a first-order deconfinement phase transition at a critical temperature $T_c \sim 200$ MeV was established.[2,3] The situation for full lattice QCD with dynamical fermions is not yet quite so clear. There is considerable evidence[1] that for vanishing baryon number the system undergoes at $T_c \sim 200$ MeV a rapid transition which yields both deconfinement and chiral-symmetry restoration. The order of the transition remains unclear, however, and there still are problems concerning the extension of the theory to finite baryon-number density.[4] Nevertheless, the lattice evaluation of statistical QCD appears to be well on the way to providing us with the phase diagram for strongly interacting matter.

Present estimates of the energy densities which can be achieved in ultrarelativistic nuclear collisions, or in $\bar{p}p$ collisions with very high multiplicities, suggest values sufficiently high for the experimental formation of the quark-gluon plasma.[1] For a full understanding of the production of the plasma and its subsequent hadronization, however, a space-time description appears necessary. Thus the proper theoretical treatment of the phenomena requires an application of relativistic hydrodynamics rather than just equilibrium thermodynamics. In order to solve the hydrodynamic equations, one has to have some thermodynamic information about the system, such as its equation of state and the velocity of sound in the medium. The most reliable way to obtain this information is evidently lattice QCD, which thus can be considered as an input for relativistic hydrodynamics. Besides this, if the physical states are close to an equilibrium distribution, then statistical thermodynamics alone can already describe many features of hot hadronic matter. Hence the thermodynamics of lattice QCD not only provides useful information for hydrodynamics, but is also directly relevant for the phenomenology of the quark-gluon plasma.

Both these aspects have recently been considered: first attempts to calculate the velocity of sound for pure SU(3) gauge theory were made in Ref. 5, and some information concerning the evolution of strangeness in the quark-gluon plasma was obtained by means of lattice studies in Ref. 6.

In the present paper, we shall use Monte Carlo results obtained[7] on an $8^3\times 3$ lattice with dynamical Wilson fermions to study the behavior of some physical quantities of particular relevance for the phenomenology of the plasma evolution.

The paper is organized as follows: in Sec. II we give a thermodynamical description of the system, computing its pressure and energy and entropy densities. In Sec. III we shall then show how the calculation of the velocity of sound, the transverse expansion, and the transverse-momentum distribution can be carried out. In Sec. IV we present our numerical results.

II. THERMODYNAMICS OF LATTICE QCD

The starting point of our considerations is the Euclidean lattice form of the QCD partition function

$$Z_E(\beta,V)=\int \prod_{\text{links}} dU\, e^{-S_{\text{eff}}} , \tag{1}$$

as is obtained after integration of the quark spinor fields. Here

$$S_{\text{eff}}=S_G+S_F , \tag{2}$$

where S_G (S_F) is gluon (quark) contribution to the effective action. In the Wilson formulation of lattice QCD one has

$$S_G = \frac{6}{g_\beta^2}\left[\frac{a_\sigma}{a_\beta}\right]\sum_{P_\beta}(1-\tfrac{1}{3}\mathrm{Re\,Tr}UUUU) + \frac{6}{g_\sigma^2}\left[\frac{a_\beta}{a_\sigma}\right]\sum_{P_\sigma}(1-\tfrac{1}{3}\mathrm{Re\,Tr}UUUU) \quad (3)$$

with the sum taken over all space-time (P_β) and space-space plaquettes (P_σ); a_σ and a_β are the spatial and temporal lattice spacings, g_σ and g_β the corresponding couplings. In a fourth-order hopping-parameter expansion, the quark contribution to the effective action (2) with SU(3) gauge fields is given by[7]

$$S_F = -4N_f(2\kappa)^{N_\beta}\sum_{\text{sites}}\mathrm{Re}L - 16N_f\kappa^4\sum_{P_\beta,P_\sigma}\mathrm{Re\,Tr}UUUU + O(\kappa^5)\ , \quad (4)$$

where L denotes the thermal Wilson loop, and $\kappa=\kappa(g)$ the hopping parameter. From the partition function (1)—(4) one can obtain by differentiation any thermodynamic quantity of interest.

Since the general thermodynamics of the system described by Eqs. (1)—(4) was studied in detail in Refs. 7 and 8, we present here only the final expressions for the energy density ϵ, the pressure P, and the entropy density s. With $\epsilon=\epsilon_F+\epsilon_G$, we have on an isotropic lattice ($a_\sigma=a_\beta$)

$$\epsilon_G a^4 = 18[g^{-2}(\bar P_\sigma-\bar P_\beta)+c'_\sigma(\bar P-P_\sigma)+c'_\beta(\bar P-\bar P_\beta)] \quad (5)$$

for the gluon sector and

$$\epsilon_F = N_f[3(2\kappa)^{N_\beta}\mathrm{Re}\bar L + 144\kappa^4(\bar P_\sigma-\bar P_\beta)+O(\kappa^5)] \quad (6)$$

for the fermion sector. The constants c'_σ and c'_β come from the differentiation of the couplings g_σ and g_β with respect to a_β; for color SU(3), Wilson fermions, and $N_f=2$, one has[9] $c'_\sigma=0.193\,66, c'_\beta=-0.132\,463$. With $\bar P_\sigma, \bar P_\beta$ we denote the space-space and space-time plaquette averages on the $N_\sigma{}^3\times N_\beta$ lattice; $\bar P$ is the plaquette average on a large ($N_\sigma{}^4$) symmetric lattice. In Eq. (6), $\bar L$ denotes the average of the thermal Wilson loop.

Similarly, the pressure $P=(\partial/\partial V)(\ln Z/\beta)$ becomes

$$3Pa^4 \simeq \epsilon a^4 + a\frac{\partial g^{-2}}{\partial a}\left[2^{N_\beta+2}N_f\kappa^{N_\beta-1}\frac{\partial\kappa}{\partial g^{-2}}\mathrm{Re}\bar L - 18(\bar P_\sigma+\bar P_\beta-2\bar P)\right]\ , \quad (7)$$

with the parameters defined above; terms beyond order κ^4 are neglected. Finally, we want to consider the entropy density of our system,

$$sa^3 = N_\beta(\epsilon a^4 - fa^4)\ , \quad (8)$$

where f is the free-energy density $f=-(1/\beta V)\ln Z_E$. Since in the Monte Carlo simulation one can only compute the lattice averages of U-dependent observables, using the weight $e^{-S_{\text{eff}}}$, the free energy is not a suitable quantity for this method. We can, however, calculate $\partial\ln Z/\partial g^{-2}$ and then integrate over g^{-2} to find

$$fa^4 - f^0a_0{}^4 \simeq 3\int_{\beta_0}^{\beta}d\beta(P_\sigma+P_\beta-2\bar P) - \tfrac{1}{6}N_f2^{N_\beta+2}\int_{\beta_0}^{\beta}d\beta\left[\kappa^{N_\beta-1}\frac{\partial\kappa}{\partial g^{-2}}\mathrm{Re}\bar L\right]\ ; \quad (9)$$

here $\beta\equiv 6/g^2$, and $f^0a_0{}^4$ is the integration constant. In Eq. (9), we have neglected the contribution from κ^4 term in the fermion action (3), since it was found to be small as compared to the remaining terms.

To study the behavior of pressure and entropy density as function of the temperature, we still have to know the relation between hopping parameter κ and coupling constant g^2, as well as that between g^2 and the lattice spacing a. Following Ref. 7, we use for κ the weak-coupling relation

$$\kappa(g^2) = \tfrac{1}{8}[1+0.11g^2+O(g^4)] \quad (10)$$

with possible deviations of about 10—20% in the region $6\geq\beta\geq 5$. For $a(g^2)$, we use the renormalization-group relation

$$a\Lambda_L = \exp\left[-\frac{4\pi^2}{33-2N_f}\beta + \frac{459-57N_f}{(33-2N_f)^2}\ln\left[\frac{8\pi^2}{33-2N_f}\beta\right]\right]\ , \quad (11)$$

which is obtained from the weak-coupling expansion of the β function. The approach to the continuum limit has not really been studied for QCD with dynamical fermions. Extrapolating the behavior found in pure gauge theory,[1] we expect here as well deviations of around 20% from the continuum value. This is in accord with the results of Ref. 7 for $N_\beta=3$, 4, and 5, which agree among each other to within this accuracy.

The Monte Carlo results[7] for the energy density as well as for the average value of the thermal Wilson loop show a deconfinement transition at $\beta_c\sim 5.3$. Since there is a rapid but smooth variation of both $\bar L$ and ϵ around T_c, it has been suggested that it is a continuous phase transition. Similar results were obtained for staggered fermions in other evaluation schemes.[10]

III. VELOCITY OF SOUND AND TRANSVERSE MOTION

Having established the general thermodynamics of hadronic matter, we now want to determine the properties of some quantities particularly relevant in the study of the evolution of the quark-gluon plasma.

Let us consider first the velocity of sound in the plasma system; it is defined as

$$v^2 = \left[\frac{\partial P}{\partial\epsilon}\right]_S\ . \quad (12)$$

Since v^2 is defined as the inverse of the isentropic compressibility, it is of interest to ask here what kind of thermodynamic variation one in fact studies on the lattice. When the coupling g^2 and hence the lattice spacing $a(g^2)$ are varied for a lattice of fixed N_β and N_σ, both the temperature $T=1/(N_\beta a)$ and the volume $V=(N_\sigma a)^3$ are changed. However, $VT^3=(N_\sigma/N_\beta)^3$ remains constant, and in the ideal-gas regime at least, this quantity is just the entropy.

As already mentioned, v^2 has already been considered on the lattice for pure SU(3) gauge theory.[5] Since this theory exhibits a first-order phase transition, which implies that $P(\epsilon)$ is constant in the transition region, the velocity of sound should become zero at $T=T_c$. For $T>T_c$, it must eventually approach the asymptotic value $v_\infty{}^2=\frac{1}{3}$. In the confinement region, below T_c, the dependence of v^2 on the temperature is determined by the equation of state of a gluonium gas. The results of Ref. 5 agree qualitatively with these expectations. [The actual calculation of v^2 for pure SU(3) gauge theory becomes difficult directly in the critical region because of hysteresis behavior. This may be the origin of discrepancies in the energy density calculated in Ref. 5 as compared to the results of Ref. 2.]

For the full theory with dynamical fermions, we expect a somewhat different behavior of the velocity of sound. First of all, in the deconfinement region we now deal with hadrons, rather than with gluonium states, as constituents. In this situation v^2 should increase with temperature more rapidly than for a gluonium gas. Thus for a given temperature below T_c, the value of v^2 should be larger than the corresponding value for pure SU(3) theory.

When we cross the transition point in the case of full QCD, we expect as before that v^2 should vanish: v^2 is inversely proportional to the specific heat c_v, and this presumably diverges at T_c—at least for an infinite system. In the lattice formulation, however, we deal with finite systems, and hence a finite specific heat. Due to finite volume effects, the velocity of sound will thus most likely not completely vanish at $T-T_c$.

Next, we consider the ratio $P/(P+\epsilon)$, which determines the collective transverse expansion of the system:[10] the average transverse rapidity $\langle y\rangle_T$ of a volume of expanding matter is given by

$$\langle y\rangle_T \sim \text{const}\times \int \frac{P}{\epsilon+P}\frac{d\tau}{r_t}\ , \tag{13}$$

where r_t is the dimension of the volume in the transverse direction, and τ the time in the local rest frame.

Finally, we want to consider the transverse momentum $\langle p_t\rangle$ of inclusively produced secondaries in $\bar{p}p$ or heavy-ion collisions. It has been suggested by Van Hove[11] that an anomalous behavior of $\langle p_t\rangle$ as function the multiplicity could be a signal for the occurrence of a phase transition in hadronic matter. His conjecture is based on the idea that the $\langle p_t\rangle$ distribution of secondaries reflects the temperature of the system[12] and its evolution in the transverse direction, while the multiplicity per unit rapidity provides a measure of the entropy.

If the system undergoes a first-order phase transition, both temperature and pressure remain constant as hadronic matter is converted from hadron gas to quark-gluon plasma; the energy and entropy density, however, change discontinuously. This leads to a flattening of $\langle p_t\rangle$ over a finite range of multiplicities. For a continuous transition there would be a similar though less dramatic effect.[13]

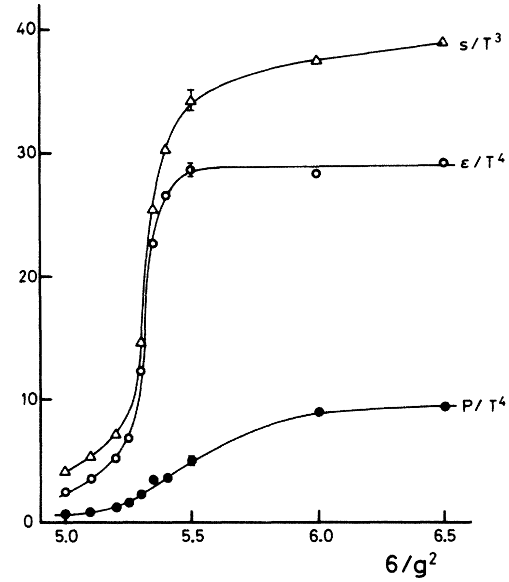

FIG. 1. Energy density ϵ, pressure P, and entropy density s, calculated on an $8^3\times 3$ lattice in a fourth-order hopping-parameter expansion.

To make these considerations more quantitative, we note that the initial energy density in the rest system of a head-on collision has been argued to be[14]

$$\epsilon \sim \langle p_t\rangle \left[\frac{dN}{dy}\right]_0 \Big/ V_A\ . \tag{14}$$

Here $(dN/dy)_0$ is the observed multiplicity of secondaries per unit rapidity interval in the c.m. system of a A-A collision, while V_A denotes the volume into which the energy is deposited. Similarly, the initial entropy density is by hydrodynamic considerations[15] expected to be

$$s \sim \left[\frac{dN}{dy}\right]_0 \Big/ V_A\ . \tag{15}$$

TABLE I. Lattice results for the basic thermodynamic variables.

$6/g^2$	T (Λ_L)	P/T^4	ϵ/T^4	s/T^3
6.5	714.2	9.39	29.26	39.0
6.0	373.69	8.81	27.90	37.4
5.5	196.07	5.08	28.64	34.15
5.4	172.41	3.81	26.41	30.18
5.35	161.68	3.50	22.68	25.48
5.3	151.62	2.37	12.28	14.66
5.25	142.2	1.64	6.86	9.0
5.2	133.36	1.33	5.21	7.18
5.1	117.32	0.92	3.52	5.35
5.0	103.22	0.66	2.48	4.23

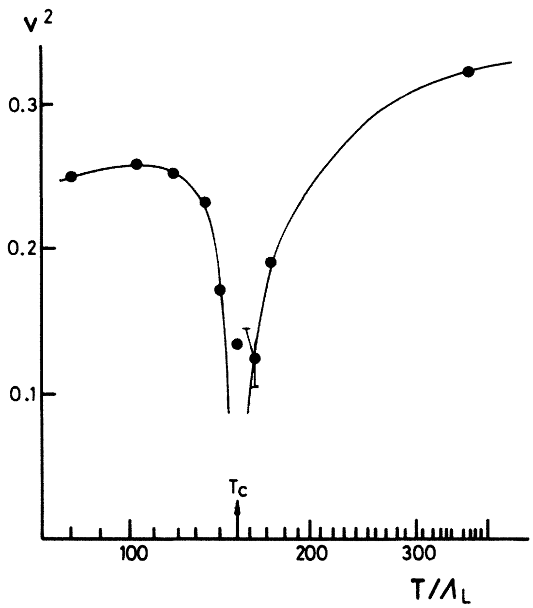

FIG. 2. Velocity of sound, $v^2 = dP/d\epsilon$, vs temperature in lattice units.

As a result, the energy density ϵ per entropy density s,

$$\epsilon/s \sim \langle p_t \rangle \ , \tag{16}$$

determines the mean transverse momentum of the secondaries. If the central volume V_A is constant in the collision, then s is related directly,

$$s \sim (dN/dy)_0 \ , \tag{17}$$

to the multiplicity. Accepting these relations between experimentally measured observables and thermodynamical quantities, then we can study the properties of transverse-momentum behavior within the thermodynamics of lattice QCD.

IV. NUMERICAL RESULTS

Our results are obtained on an $8^3 \times 3$ lattice and $N_f = 2$, using the Monte Carlo data for $\bar{P}_\sigma, \bar{P}_\beta, \bar{L}$ presented in Ref. 7.

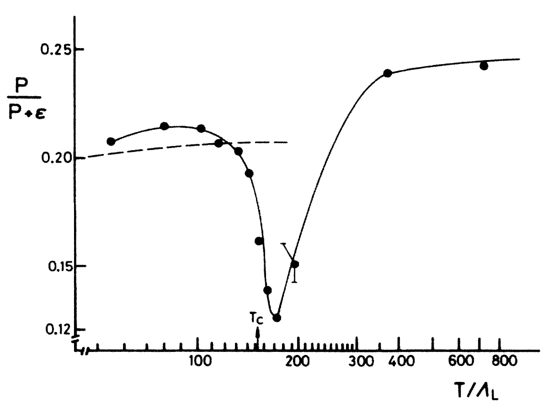

FIG. 3. $P/(P+\epsilon)$ vs temperature in lattice units; also shown are the results for an ideal gas of π, ρ, and ω mesons (dashed line).

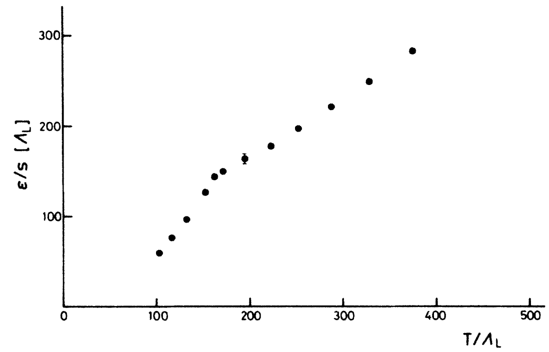

FIG. 4. Energy per degree of freedom, ϵ/s, as transverse-momentum measure, vs temperature, in lattice units.

In Fig. 1, we show the behavior of the pressure, the energy density, and the entropy density as a function of $\beta \equiv 6/g^2$; the results are listed in Table I. To compute the entropy density from Eqs. (8) and (9), we have assumed that for $\beta = 6.5$ the system behaves like a noninteracting quark-gluon gas, so that $f_0 a_0^{\,4} \mid_{\beta=6.5} \simeq -\frac{1}{3}\epsilon$.

In contrast with pure SU(3) gauge theory, the deconfinement transition now results in a rapid but smooth change of both energy and entropy densities around the critical point $6/g_c^{\,2} \simeq 5.3$, corresponding to a critical temperature $T_c \simeq 150\Lambda_L$.

The velocity of sound versus the temperature in lattice units Λ_L is shown in Fig. 2. To evaluate $dP/d\epsilon$, we have followed Ref. 5 and first expressed P and ϵ in physical units, using the renormalization-group equation (10); then the derivative in Eq. (11) was approximately calculated by taking finite differences. The temperature dependence of v^2 shows the expected from: at the critical point $T_c \simeq 150\Lambda_L$, there is a pronounced dip as evidence for the phase transition in the system. As far as we can tell, the velocity of sound at $T = T_c$ is different from zero; we expect that this is mainly due to finite volume effects, which could be checked by seeing if for sufficiently large lattices $v^2(T_c)$ converges to zero. Figure 2 also shows that in the deconfinement region $v^2(T)$ approaches rather quickly the ideal gas value $v^2(T) = \frac{1}{3}$.

In Fig. 3 the ratio $P/(P+\epsilon)$, which determines the transverse expansion rapidity $\langle y_t \rangle$ through Eq. (14), is plotted versus temperature. Also here the most striking feature is the dip in the transition region, showing that the transverse expansion of the system is suppressed around

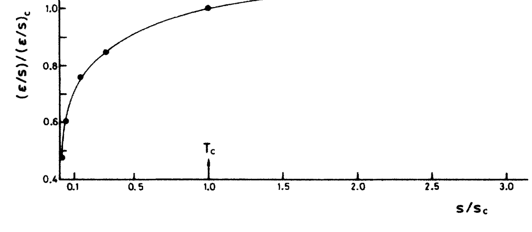

FIG. 5. Energy per degree of freedom, ϵ/s, normalized to its critical value, vs entropy density s, normalized the same way.

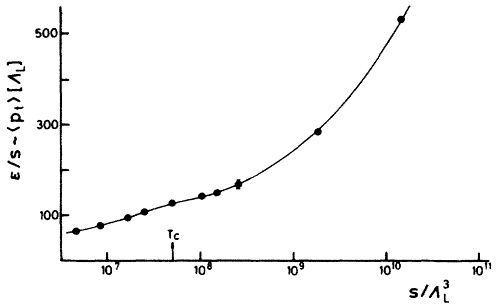

FIG. 6. Energy per degree of freedom, ϵ/s, as transverse-momentum measure, vs entropy density s, in lattice units.

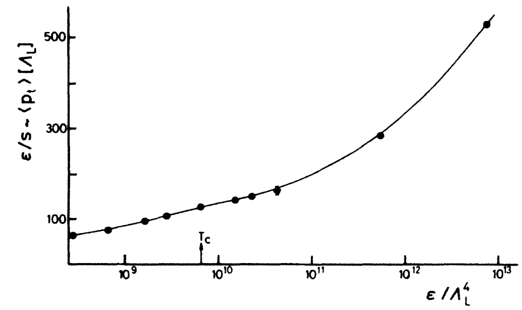

FIG. 7. Energy per degree of freedom, ϵ/s, as transverse-momentum measure, vs entropy density ϵ, in lattice units.

deconfinement.

In Figs. 4—7, we study the relevant quantities for the transverse-momentum behavior. We first note in Fig. 4 that $\langle p_t \rangle$, given according to Eq. (16) by ϵ/s, indeed provides a measure of the temperature; it is approximately linear in T, with an abrupt charge in slope at the deconfinement point. Above T_c, in the deconfined system, the available energy is distributed among a larger number of constituents, resulting in a slower growth of $\langle p_t \rangle$ with T.

Figure 5 shows ϵ/s vs s; according to Eqs. (16) and (17), this should mirror the multiplicity dependence of $\langle p_t \rangle$. We see that in the critical region there is a noticeable but smooth flattening. This behavior is qualitatively in agreement with the results from the CERN $p\bar{p}$ collider;[16] the multiplicity increase observed there appears somewhat weaker, however. Beyond the transition region, $\langle p_t \rangle$ increases again more rapidly with $s \sim (dN/dy)_0$, as shown in Fig. 6. A similar, slightly weaker increase occurs as function of ϵ (Fig. 7). This behavior is in qualitative agreement with recent cosmic-ray data.[17]

To give at least some indication of the uncertainty involved in our calculations, we have included in each figure error bars for one typical point. These are based on the naive statistical error of the lattice evaluation only. In addition, the truncation of the hopping parameter expansion, the use of the weak-coupling forms for $\kappa(g^2)$ and $a(g^2)$ in the range of g^2 here considered, and the finite lattice size all provide additional sources of error. Hopefully these can be brought to a point of estimate by future evaluations using and comparing different possible methods on considerably larger lattices.

V. CONCLUSIONS

We have determined the temperature dependence of the pressure and the energy and the entropy densities in the lattice formulation of statistical QCD with dynamical fermions. Using these results, we have then calculated the velocity of sound, and the thermodynamic quantities expected to govern the collective transverse rapidity and the mean transverse momentum of secondaries in nuclear collisions.

In the region of the deconfinement transition, we observe a strong suppression of the sound propagation in the system and of its transverse collective expansion.

Relating the energy per entropy to $\langle p_t \rangle$ and the entropy density to the central multiplicity of secondaries $(dN/dy)_0$ we further observe a noticeable flattening of the multiplicity dependence of $\langle p_t \rangle$ in the critical region. This behavior, first proposed by Van Hove,[11] is in qualitative accord with $p\bar{p}$ and cosmic-ray data.

Our study is based on extensive lattice calculations using Wilson fermions in a hopping-parameter expansion.[7] It should be emphasized, however, that other fermion formulations and other evaluation schemes for the fermion determinant have so far led to the same thermodynamics,[10] and are thus expected to yield similar results for the quantities considered here.

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