Spontaneous chiral-symmetry breaking in three-dimensional QED

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A detailed analysis is given of chiral-symmetry breaking in the large-flavor (N) limit of quantum electrodynamics in $(2+1)$ dimensions. Analytical and numerical solutions of the homogeneous Dyson-Schwinger equation for the fermion self-energy combined with a computation of the effective potential for the fermion bilinear show that it is energetically preferable for the theory to dynamically generate a mass for fermions. The magnitude of the mass is roughly exponentially suppressed in N from the fundamental dimensionful scale $\alpha \equiv N e^2$ of the gauge coupling constant, but the scale at which the self-mass begins to damp rapidly appears to be of order α, so that there is no spontaneous breaking of an approximate scale invariance that the underlying theory possesses at momentum small compared to α. Higher-order $1/N$ corrections are analyzed and it is shown that the $1/N$ expansion can be used consistently to demonstrate chiral-symmetry breaking. Open issues and possible improvements of the analysis are given and some avenues for future investigation suggested.

I. INTRODUCTION

Among all the hadrons the π meson is believed to play a special role. The underlying theory of strong interactions possesses a near chiral symmetry $SU(2)_L \times SU(2)_R$ because of the approximate masslessness of the up and down quarks. This symmetry must then break spontaneously in order to explain the effective 300-MeV masses that these quarks appear to possess as the constituents of hadrons. The spontaneous breaking of any continuous symmetry necessarily leads to the existence of massless Goldstone bosons, and in the case of chiral symmetry, the π mesons approximately play this role.

Although this picture of meson dynamics is widely believed and supported by much circumstantial evidence, no complete derivation of the Goldstone nature of the π meson has ever been produced starting with the underlying gauge theory, QCD. What one would like to see is a direct analysis of the problem along the lines of, say, the original discussion of dynamical spontaneous symmetry breaking in quantum field theory by Nambu and Jona-Lasinio.[1] This discussion of a four-fermion interaction theory and the corresponding study in two space-time dimensions[2] provides prototypes for the analysis of realistic gauge field theories. With strong-coupling effects necessarily playing an important role in QCD, however, a treatment along these lines is not yet possible. In the face of these difficulties, people have turned to a variety of approximation devices[3–6] and indirect arguments[7,8] to buttress the picture of the π meson as a Goldstone boson.

In this paper, we will describe a gauge field theory in which the problem of dynamical chiral-symmetry breaking can be systematically analyzed. The model is quantum electrodynamics in 2 + 1 dimensions (QED_3) treated in a $1/N$ expansion.[9,10] Although this model is not QCD and not even four dimensional, it is a genuine gauge field theory. It is in fact the first quantum field theory we know of, above two space-time dimensions, that permits a systematic treatment of chiral-symmetry breaking.

The model, furthermore, has properties reminiscent of four-dimensional theories. It has an intrinsic dimensional parameter, the gauge coupling constant, that plays a role similar to the renormalization scale in four dimensions. The relative size of this parameter and a possible chiral-symmetry-breaking scale, a fermion mass, is then a question of considerable interest. Within the framework of a $1/N$ expansion, it will be seen that the dynamically generated fermion mass is much less than the coupling constant. There is, of course, no evidence for any large hierarchy in QCD_4 and indeed none would be expected in a theory without a small dimensionless parameter.[11] In four-dimensional theories other than QCD, however, hierarchies might well exist with interesting experimental consequences. In particular, some dynamical theories of electroweak symmetry breaking contain small parameters that can lead to hierarchies.[11,12] The hierarchy in these theories, however, is inverse to the one appearing in QED_3, leading instead to a relatively large fermion mass.

In any theory with a hierarchy between the intrinsic scale and a spontaneously generated fermion mass, an important question arises. If the intrinsic scale approximately drops out of the problem at the energy scale where the mass arises, the underlying theory will be approximately scale invariant. The fermion mass would then spontaneously break scale invariance, leading to the existence of a dilaton. The dilaton would in turn presumably pick up a small mass once the effects of the intrinsic scale are included. This question will be investigated in QED_3 (Ref. 13). We shall argue that even though the dynamical fermion mass is small compared to the coupling strength, the energy scale at which the mass turns on and off is on the order of the coupling. Therefore there is no approximate scale invariance to be spontaneously broken and there is no approximate dilaton.

In Sec. II the features of QED_3 will be presented. Its continuous symmetries with and without a fermion mass will be described. It will be shown that the massless theory is both ultraviolet and infrared finite order by order in the $1/N$ expansion. The infrared finiteness is associated with an effective scale invariance in the infrared which arises because of the existence of an infrared-stable fixed point. It is noted that spontaneous chiral-symmetry breaking will not occur to any finite order in $1/N$.

In Sec. III the possibility of spontaneous chiral-symmetry breaking will be investigated by solving the Dyson-Schwinger gap equation. The gap equation, which in effect sums dominant contributions in the $1/N$ expansion, will be solved both analytically and numerically. The different mass scales in the problem will be explained and the question of spontaneous breaking of scale invariance will be addressed.

The effective potential will be constructed in the same approximation in Sec. IV. Functional differentiation of this potential will produce the gap equation. The extremum value of the potential (its value at a solution to the gap equation) will then be computed. It will be argued that the spontaneous breaking of chiral symmetry is energetically preferred.

In Sec. V higher-order corrections in the $1/N$ expansion will be estimated. It will be argued that these corrections are indeed down in $1/N$, that is, that the leading terms in the expansion have been correctly summed in our solution to the gap equation. Section VI will contain a summary of our results and a list of open questions.

II. THE MODEL AND SOME OF ITS PROPERTIES

In this section we will define the model to be studied, discuss its symmetries, its ultraviolet and infrared behavior, and the use of the $1/N$ expansion in analyzing chiral-symmetry breaking.

The Lagrangian for massless quantum electrodynamics in three space-time dimensions (QED_3) is

$$L=-\tfrac{1}{4}F_{\mu\nu}{}^2+\bar{\psi}i\not{D}\psi \;, \tag{2.1}$$

where

$$D_\mu=\partial_\mu-ieA_\mu \;. \tag{2.2}$$

A spinorial representation of the Lorentz group SO(2,1) in three dimensions is provided by two-component spinors, with the corresponding 2×2 representation of the Dirac algebra being given by the Pauli matrices

$$\gamma^0=\sigma_2,\;\; \gamma^1=i\sigma_3,\;\; \gamma^2=i\sigma_1 \;. \tag{2.3}$$

There is, however, no other 2×2 matrix that anticommutes with all these γ_μ. There is, therefore, nothing to generate a chiral symmetry that would be broken by a mass term $m\bar{\psi}\psi$, whether it be explicit or dynamically generated. The massless theory has no more symmetry than the massive theory.

Consider therefore the basic fermion field to be a four-component spinor. The theory will be taken to contain N such fermions. The three 4×4 γ matrices can be taken to be

$$\gamma^0=\begin{bmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{bmatrix},\;\; \gamma^1=\begin{bmatrix}i\sigma_1 & 0\\ 0 & -i\sigma_1\end{bmatrix},\;\; \gamma^2=\begin{bmatrix}i\sigma_2 & 0\\ 0 & -i\sigma_2\end{bmatrix}. \tag{2.4}$$

There will then be two 4×4 matrices,

$$\gamma^3=i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\;\; \gamma^5=i\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{2.5}$$

that anticommute with γ^0, γ^1, and γ^2. The massless theory will therefore be invariant under the "chiral" transformations

$$\psi\to e^{i\alpha\gamma^3}\psi,\;\; \psi\to e^{i\beta\gamma^5}\psi \;. \tag{2.6}$$

For each four-component spinor, there will be a global U(2) symmetry with generators

$$1,\; \gamma^3,\gamma^5,\text{ and } [\gamma^3,\gamma^5] \;, \tag{2.7}$$

and the full symmetry is then U(2N). A mass term $m\bar{\psi}\psi$ would break this symmetry to the subgroup

$$SU(N)\times SU(N)\times U(1)\times U(1) \;. \tag{2.8}$$

To elucidate the difference between the two-component and the four-component spinor theories it is helpful to recall the discrete symmetries of these theories. In 2 + 1 dimensions, parity corresponds to inverting one axis, since inversion of both axes could be undone by a π rotation. Thus under the parity transformation P, $(x,y)\to(x,y)_P=(-x,y)$ say. The corresponding operation on the two-component spinors is

$$P\psi(t,\mathbf{x})P^{-1}=\sigma_1\psi(t,\mathbf{x}_P) \;, \tag{2.9}$$

and on the gauge field

$$\begin{aligned} PA^0(t,\mathbf{x})P^{-1}&=A^0(t,\mathbf{x}_P) \;,\\ PA^1(t,\mathbf{x})P^{-1}&=-A^1(t,\mathbf{x}_P) \;,\\ PA^2(t,\mathbf{x})P^{-1}&=A^2(t,\mathbf{x}_P) \;. \end{aligned} \tag{2.10}$$

Time reversal is given by

$$\begin{aligned} \tau\psi(t,\mathbf{x})\tau^{-1}&=\sigma_2\psi(-t,\mathbf{x}) \;,\\ \tau A^0(t,\mathbf{x})\tau^{-1}&=A^0(-t,\mathbf{x}) \;,\\ \tau\mathbf{A}(t,\mathbf{x})\tau^{-1}&=-\mathbf{A}(-t,\mathbf{x}) \;. \end{aligned} \tag{2.11}$$

A two-component mass $m\bar{\psi}\psi$ is odd under both P and τ. Writing a four-component spinor as

$$\psi=\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$$

the chiral-symmetry-breaking, four-component mass term $m\bar{\psi}\psi$ becomes

$$m\bar{\psi}\psi=m\psi_1^\dagger\sigma_3\psi_1-m\psi_2^\dagger\sigma_3\psi_2 \;. \tag{2.12}$$

The parity transformation becomes

$$\psi_1\to\sigma_1\psi_2,\;\; \psi_2\to\sigma_1\psi_1 \;, \tag{2.13}$$

and therefore $m\bar{\psi}\psi$ is parity conserving.

This point is being stressed because there is an alternative possibility in three dimensions. Another acceptable candidate for a mass term is

$$m\bar{\psi}\tfrac{1}{2}[\gamma^3,\gamma^5]\psi=m\psi_1^\dagger\sigma_3\psi_1+m\psi_2^\dagger\sigma_3\psi_2 \;. \tag{2.14}$$

This term is invariant under the chiral transformations

(2.6) but clearly not invariant under the parity transformation (2.12). Such a parity-violating mass is in fact the only possibility in the two-component formalism. It is known that it will induce a Chern-Simons mass term for the gauge field via one-loop vacuum polarization.[14] That such a fermion mass and the corresponding Chern-Simons mass could arise spontaneously, leading to the spontaneous violation of parity in QED_3, is an interesting and important possibility.[15] In this paper, however, attention will be restricted to the possible spontaneous appearance of a parity-conserving, chiral-symmetry-violating mass.

We describe next the perturbative properties of QED_3 and then introduce the $1/N$ expansion. Since the square of the gauge coupling constant has dimensions of mass, the theory is super-renormalizable. It is not difficult to see in fact that once gauge invariance is taken into account, the Green's functions of the theory are completely ultraviolet finite.

In the infrared, however, severe divergences force the breakdown of the loop expansion for the massless theory.[16,17] Already at two loops, infrared divergences appear in Euclidean Green's functions. Although they are presumably an artifact of the loop expansion, their consequences and the way in which they would be removed in a nonperturbative treatment are not yet clear in general. One possibility that suggests itself is that chiral symmetry will be forced to break spontaneously giving the fermion a mass.[18] This is what happens in the two-dimensional Gross-Neveu model.[2] It has been argued, however, that the infrared divergences in massless three-dimensional theories can disappear without mass generation.[16,17] In the one nonperturbative scheme that is known for this model—the $1/N$ expansion—it can be shown that the theory does indeed stay massless while becoming infrared finite.[16] Whether this finite theory in turn exhibits spontaneous chiral-symmetry breaking is the central problem of this paper to which we shall turn after describing the $1/N$ expansion.

With N four-component spinors, the Lagrangian is

$$L=\sum_{i=1}^{N}\bar{\psi}_i(i\partial\!\!\!/-eA\!\!\!/)\psi_i-\tfrac{1}{4}F_{\mu\nu}{}^2 \ . \tag{2.15}$$

In an expansion in $1/N$ with $\alpha\equiv e^2N$ fixed, only the graphs shown in Fig. 1 contribute to leading, zeroth order. It is simplest to proceed in Landau gauge and we shall do so throughout the paper.

The gauge-boson propagator $D_{\mu\nu}(p)$ is given by

$$D_{\mu\nu}(p)=\frac{g_{\mu\nu}-p_\mu p_\nu/p^2}{p^2[1+\Pi(p)]} \ , \tag{2.16}$$

where, to leading order in $1/N$, and for a constant fermion mass m,

$$\Pi(p)=\frac{\alpha}{4\pi p^2}\left[2m+\frac{p^2-4m^2}{p}\times\arcsin\left[\frac{p}{(p^2+4m^2)^{1/2}}\right]\right] . \tag{2.17}$$

In the zero-mass theory being considered so far,

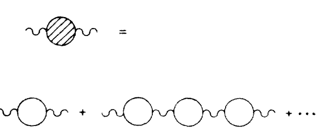

FIG. 1. The summed gauge-boson propagator to zeroth order in the $1/N$ expansion.

$$\Pi(p)=\frac{\alpha}{8p} \ . \tag{2.18}$$

The momentum p is Euclidean. It follows that in the infrared limit $p\ll\alpha$, the propagator behaves like $1/\alpha p$ rather than $1/p^2$. A simple power-counting exercise then shows that the Euclidean Green's functions are completely infrared finite.[16]

The behavior of the theory for all momenta is best described in terms of the dimensionless running coupling constant:

$$\bar{\alpha}(p)\equiv\alpha/\{8p[1+\Pi(p)]\} \ . \tag{2.19}$$

This object, formed by multiplying the natural dimensionless coupling $\alpha/8p$ by the zeroth-order (in $1/N$) gauge propagator renormalization factor $Z_3(p)\equiv[1+\Pi(p)]^{-1}$, characterizes the interaction strength of the theory. It corresponds to the renormalization-group β function

$$\beta=\bar{\alpha}(\bar{\alpha}-1) \ . \tag{2.20}$$

In the ultraviolet $\bar{\alpha}(p)$ approaches zero rapidly, as expected for a super-renormalizable theory. Zero is an ultraviolet stable fixed point. In the infrared, the $1/N$ expansion leads to the existence of an infrared stable fixed point $\bar{\alpha}=1$. The infrared well being of the theory can be traced to an effective scale-invariant behavior associated with this fixed point. The scale invariance is evident since in this limit, equivalent to taking α to infinity at fixed p, α completely drops out of all computations.

Because questions of scale invariance arise naturally in connection with chiral-symmetry breaking, it is important to discuss this effective symmetry in a bit more detail. In the limit $p\ll\alpha$, each graph in the $1/N$ expansion looks like a three-dimensional Feynman graph with a gauge propagator behaving like $1/p$ and a dimensionless coupling constant $1/\sqrt{N}$. The power counting is very similar to four-dimensional field theory and one might worry that the effective scale invariance is broken by ultraviolet divergences just as it is in four dimensions. Here, of course, the divergences would get cut off at α since the complete theory is ultraviolet finite. Still, this would bring α back in at the quantum level, destroying the naive effective low-energy scale invariance.

A simple power-counting exercise for the infrared effective theory gives the degree of divergence to be

$$d=3-B-F \ ,$$

where B (F) is the number of external boson (fermion) lines. The possible "divergences" (α dependences) are as follows.

$B=2$, $F=0$, ($d=1$): The gauge propagator. It is reduced to $d=-1$ by gauge invariance.

$B=1$, $F=2$, ($d=0$): The vertex. Its divergence ($\ln\alpha$)

is canceled by the fermion wave-function renormalization, just as in four dimensions, because of the Ward identity.

$B=0$, $F=2$, $(d=1)$: The fermion self-energy. It is reduced to $d=0$ in the usual way. The wave-function divergence ($\ln\alpha$) is canceled by the vertex. In the massless theory, there is no mass renormalization to any finite order in N— chiral symmetry forbids it.

One concludes therefore that the effective low-energy scale invariance of the massless theory holds to any order in the $1/N$ expansion. If spontaneous chiral-symmetry breaking takes place at momentum scales much less than α, there will clearly be an approximate spontaneous breaking of scale invariance.

III. CHIRAL-SYMMETRY BREAKING

The properties of QED_3 make it clear that spontaneous chiral-symmetry breakdown will not take place to any finite order in the $1/N$ expansion. To go beyond finite orders, we utilize the homogeneous Dyson-Schwinger equation for the fermion propagator. We investigate whether this equation, which effectively sums over all orders in the $1/N$ expansion, admits a nonzero fermion mass as a solution. To make the equation tractable some approximations must be made and the resulting equation will correspond to a selective resummation of terms in the $1/N$ expansion. The reliability of these approximations will be discussed in Sec. IV.

The Dyson-Schwinger gap equation will now be set up and its analytic and numerical solutions will be discussed in the various momentum ranges. The inverse Euclidean fermion propagator is written in the form

$$S_F^{-1}=-\not p[1+A(p)]+\Sigma(p)\,, \tag{3.1}$$

where $\Sigma(p)$ is the dynamically generated effective mass. The Dyson-Schwinger equation, using the lowest-order (in $1/N$) vertex and gauge propagator, is

$$-\not p A(p)+\Sigma(p)=\frac{\alpha}{N}\int\frac{d^3k}{(2\pi)^3}\frac{D_{\mu\nu}(p-k)\gamma^\mu\{\not k[1+A(k)]+\Sigma(k)\}\gamma^\nu}{\{k^2[1+A(k)]^2+\Sigma^2(k)\}}\,, \tag{3.2}$$

with $D_{\mu\nu}(p-k)$ given by Eqs. (2.16) and (2.18). $\Sigma(p)$ must be determined self-consistently by this equation.

The right-hand side of the Dyson-Schwinger equation contains an explicit factor of $1/N$. A solution $\Sigma(p)$ will clearly have to exhibit some N dependence to cancel this factor in the integral equation. It will then have to be checked, and that is the business of Sec. V, that the higher-order terms (in $1/N$), that would be added to the right-hand side of the gap equation, really are down in the $1/N$ expansion.

The wave-function renormalization $A(p)$ will be generated perturbatively in $1/N$. It is therefore expected to be suppressed in the large-N limit by $O(1/N)$ and we drop it to leading order. Consistency, in fact, demands that wave-function renormalization be dropped in Eq. (3.2). The full vertex Γ_ν has already been replaced by the lowest-order vertex γ_ν and the wave-function renormalization and vertex are connected by the Ward identity. These higher-order corrections, along with similar corrections to the gauge propagator, will be analyzed in Sec. V. The lowest-order equation is pictured in Fig. 2.

In general, $D_{\mu\nu}(p-k)$ in (3.2) will depend on $\Sigma(p)$ itself through the vacuum-polarization graph (Fig. 1). One is then faced with a complicated set of coupled integral equations. (It would presumably be possible to solve these equations numerically.) For momentum scales large compared to $\Sigma(p)$ itself, however, $\Sigma(p)$ can be ignored in $D_{\mu\nu}(p)$ so that $\Pi(p)$ is approximately given by Eq. (2.18). After angular integration, the gap equation then becomes

$$\Sigma(p)=\frac{\alpha}{2\pi^2Np}\int_0^\infty dk\frac{k\Sigma(k)}{k^2+\Sigma^2(k)}\ln\frac{k+p+\alpha/8}{|k-p|+\alpha/8}\,. \tag{3.3}$$

For momentum k of order $\Sigma(k)$ in this integral, where the nonlinearity becomes relevant, Eq. (3.3) is not reliable in detail. We nevertheless retain $\Sigma(k)$ in the denominator as an infrared cutoff and a qualitative measure of the nonlinearity at low momentum.

To study Eq. (3.3) analytically it is convenient to break the momentum integration into two regions and expand the logarithm appropriately for each region:

$$\Sigma(p)=\frac{\alpha}{\pi^2Np}\int_0^p dk\frac{k\Sigma(k)}{k^2+\Sigma^2(k)}\times\left[\frac{k}{p+\alpha/8}+O\left[\frac{k}{p+\alpha/8}\right]^3+\cdots\right]+\frac{\alpha}{\pi^2Np}\int_p^\infty dk\frac{k\Sigma(k)}{k^2+\Sigma^2(k)}\times\left[\frac{p}{k+\alpha/8}+O\left[\frac{p}{k+\alpha/8}\right]^3+\cdots\right]. \tag{3.4}$$

For both large p and small p (relative to $\alpha/8$), asymptotic forms may be found for $\Sigma(p)$ by retaining only the first term in the perturbative expansion of the logarithm. In this approximation, the integral equation may be converted to a more manageable second-order nonlinear differential equation:

$$\frac{d}{dp}\left[\frac{d\Sigma(p)}{dp}\frac{p^2(p+\alpha/8)^2}{2p+\alpha/8}\right]=-\frac{\alpha}{\pi^2N}\frac{p^2\Sigma(p)}{p^2+\Sigma^2(p)}\,. \tag{3.5}$$

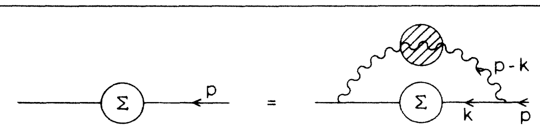

FIG. 2. The lowest-order approximation to the full homogeneous Dyson-Schwinger equation for the fermion self-mass $\Sigma(p)$.

In the limit $p \ll \alpha/8$, this equation simplifies to the form

$$\frac{d}{dp}\left[p^2\frac{d\Sigma(p)}{dp}\right]=-\frac{8}{\pi^2 N}\frac{p^2\Sigma(p)}{p^2+\Sigma^2(p)} . \tag{3.6}$$

If it is further assumed that $p \gg \Sigma(p)$, the equation may be linearized to the form

$$\frac{d}{dp}\left[\frac{p^2 d\Sigma(p)}{dp}\right]=-\frac{8}{\pi^2 N}\Sigma(p) . \tag{3.7}$$

It must be checked later that the use of this linear equation is self-consistent. Strictly speaking, only if the result $\Sigma(p) \ll \alpha/8$ emerges from the full nonlinear equation will this linear equation be relevant in any regime at all. This question will be addressed shortly using a numerical analysis of the nonlinear equation. Equation (3.7) has solutions of the form

$$\Sigma(p)=Ap^a, \text{ where } a=-\frac{1}{2}\pm\frac{1}{2}\left[1-\frac{32}{\pi^2 N}\right]^{1/2} . \tag{3.8}$$

For large N the two solutions are

$$\Sigma_1(p)\sim\frac{1}{p^{8/\pi^2 N}}, \quad \Sigma_2(p)\sim\frac{1}{p^{1-8/\pi^2 N}} . \tag{3.9}$$

Σ_1 barely falls asymptotically while $\Sigma_2(p)$ is roughly of order $1/p$. For $N<32/\pi^2$ the solutions fall like $1/\sqrt{p}$ times a function that oscillates in $\ln(p)$. Whether these oscillations are seen in the solutions depends on the range available between $\Sigma(p)$ and α. We will return to this after discussing the numerical solutions. For these values of N, of course, the use of the $1/N$ expansion becomes less than reliable. It is easily seen that both of the above solutions are consistent with the truncation of the logarithm expansion in Eq. (3.4). Substituting them back, the higher-order terms are suppressed by powers of p/α. In other words, they both solve the integral equation as well as the differential equation in the infinite-α limit.

The solution $\Sigma_1(p)$ has the form of a hard (bare) mass in the sense that it is not rapidly damped at high momentum. Instead, it is nearly constant, varying only with a small anomalous dimension. It can be equivalently generated by insertion of a bare mass into the theory and summation of the logarithmic mass renormalizations in each order in $1/N$. The solution $\Sigma_2(p)$, damping rapidly ($\sim 1/p$) as p increases, corresponds to spontaneous chiral-symmetry breaking at the momentum scales ($p \ll \alpha$) being considered here.

The question of which of these two solutions is chosen by the nonlinear equation (3.6) is very important. If it is the latter, then spontaneous chiral-symmetry breaking will take place at momentum scales on the order of Σ itself—far less than α since $\Sigma \ll \alpha$. In this range, as we discussed in Sec. II, the theory is approximately scale invariant to any order in $1/N$. Therefore the $\Sigma_2(p)$ solution would correspond to an approximate spontaneous breaking of scale invariance. The Σ_1 solution, on the other hand, looks like a bare mass, explicitly breaking both chiral and scale invariance.[19] Even this solution could, of course, evolve into a rapidly damped effective mass once momentum scales on the order of α are reached. If this happens, spontaneous chiral-symmetry breaking takes place, but at momentum scales on the order of α. In this case, the dimensionful parameter α plays a central role in the dynamics and there is clearly no approximate scale invariance to be spontaneously broken.

We now argue that the Σ_1 solution is the only possibility for $p \ll \alpha$. A solution $\Sigma(p)$ of the nonlinear Eq. (3.6) must be finite at $p=0$ if it is to correspond to mass generation. On physical grounds, it should in fact by analytic in p^2 about $p^2=0$. A power-series solution to Eq. (3.6), about $p=0$, takes the form

$$\Sigma(p)=\Sigma_0\left[1-\frac{4p^2}{3\pi^2 N\Sigma_0{}^2}+\cdots\right] , \tag{3.10}$$

where $\Sigma_0 \equiv \Sigma(p=0)$. The boundary conditions at $p=0$, expressed by Eq. (3.10), are the same whether the Σ_1 or Σ_2 solution emerges in the linearized regime $p \gg \Sigma(p)$. These conditions determine the solution for all p and there is nothing to rule out the Σ_1 solution. This solution will therefore necessarily be the asymptotic ($p \gg \Sigma$) form of the solution to the nonlinear Eq. (3.6) (Ref. 20). This conclusion has also been verified by a numerical analysis of Eq. (3.6). One concludes that at momenta much less than α, the effective mass looks very much like a bare mass. It varies very slowly with p.

A possible flaw in the above argument is that Eq. (3.3) [and therefore Eq. (3.6)] is not really reliable when $p \le \Sigma(p)$. A set of coupled integral equations must then be solved as noted above Eq. (3.3). The coupled set might then be analyzed in the same way we have just discussed the approximate equation.

It is interesting to compare the above discussion, leading to the solution $\Sigma_1(p)$, to the analysis of Ref. 9. The approach there is to assume that $\Sigma(p)$ takes on a constant value $m \ll \alpha$ for all values of p less than or on the order of α. This is simply our Σ_1 solution in the infinite-N limit. It, even more than our true solution, looks like a bare mass at these momentum scales. For $p>\alpha$, the integrand damps rapidly so that the integral in Eq. (3.3) can be cut off at $p \simeq \alpha$. The infrared cutoff is provided by m itself, and the integral can be evaluated at $p \ll \alpha$ to give

$$m \simeq \Sigma(0) \simeq m\frac{8}{\pi^2 N}\ln(\alpha/m) ,$$

and, therefore,

$$m \simeq \alpha e^{-\pi^2 N/8} . \tag{3.11}$$

One sees clearly the nonperturbative nature of the solution (3.11)—it falls with N more rapidly than any polynomial in $1/N$. The explicit factor $1/N$ in the gap equation has been compensated by the exponential hierarchy between α and m.

We turn now to an analysis of Eq. (3.5) in the asymptotic regime $p \gg \alpha/8$. There are two possible asymptotic solutions:

$$\Sigma_A(p)=\frac{A}{p^2}\left[1+a\frac{\alpha}{p}+\cdots\right] , \tag{3.12}$$

$$\Sigma_B(p)=B\left[1+b\frac{\alpha}{p}+\cdots\right]. \tag{3.13}$$

It is easy to see, however, that the $\Sigma_B(p)$ solution, corresponding to a bare mass, is not compatible with the homogeneous integral equation (3.3). A bare mass has simply been banished from the theory by not including an inhomogeneous term in this equation.

The first solution $\Sigma_A(p)$ corresponds to dynamical chiral-symmetry breaking setting in at a momentum scale on the order of α. The coefficient a is given by

$$a=-\frac{1}{8}-\frac{2}{3N\pi^2}. \tag{3.14}$$

A function that solves the truncated lower-integral equation in Eq. (3.4) is

$$\Sigma'_A(p)=\frac{D}{p^2+\frac{p\alpha}{8}}\left[\frac{p}{p+\frac{\alpha}{8}}\right]^{8/\pi^2N}. \tag{3.15}$$

Its contribution to the upper integral is suppressed by a factor α/Np relative to $\Sigma'_A(p)$ itself. It agrees with the first term in the asymptotic series for $p \gg \alpha/8$ and with the second term for large N. When p approaches $\alpha/8$, however, we do not expect it will be reliable, since the other integral becomes important in this regime. While the asymptotic power-law behavior $\Sigma_A(p)\sim 1/p^2$ has been reliably determined, even the coefficient of this leading term is sensitive to small momentum scales (on the order of α). This behavior can be further elucidated by rederiving it using the language of the operator-product expansion.[21]

The operator-product expansion for the fermion propagator has the form

$$C_1(p^2)\langle 0|1|0\rangle + C_2(p^2)\langle 0|\bar\psi\psi|0\rangle + \cdots .$$

To the leading order, $C_1(p^2)=1/p$, $C_2(p^2)$ is of dimension -3 and the dynamically generated mass Σ is proportional to $p^2C_2(p^2)\langle 0|\bar\psi\psi|0\rangle$. The diagrammatic expansion for C_2 and dimensional analysis show that for $p\gg\alpha$, C_2 behaves like α/p^4. Therefore Σ falls like $1/p^2$ for $p\gg\alpha$.

In the regime $p\gg\alpha/8$, the dynamical versus hard mass solutions are clearly distinguished for all N, in contrast with the infrared regime $p\ll\alpha/8$. This is reasonable since the $1/N$ expansion does not play a central role in this regime. It enters only in the corrections to the dominant asymptotic behavior.

Now we turn to the numerical study of the Dyson-Schwinger equation (3.3). In our discussions of analytic solutions we have made the assumption that the dynamically generated mass is much smaller than $\alpha\equiv Ne^2$. Yet the explicit solutions that we have obtained so far come from the linearized equations, and their magnitude is not determined until the nonlinearities are taken into account. A numerical analysis of the full nonlinear equation was performed for this purpose.

The Dyson-Schwinger equation has been solved using a self-consistent iterative procedure. Nonzero solutions for $\Sigma(p)$ have been found. They have the expected qualitative behavior discussed above. We see that for fixed N, $\Sigma(p)$ falls sharply to zero once p goes past $\alpha/8$, as required for a dynamically generated mass. Our study has so far been limited to values of N from 1 to 3, including intermediate nonintegral values. The main reason for this is limitation of computing time. To achieve sensitivity to the shape of the integrand, the integration grid has to be chosen to be smaller than the maximum size of $\Sigma(p)$. But the overall size of $\Sigma(p)$, represented by $\Sigma(0)$, falls sharply with increasing N. Pisarski[9] anticipated this falloff to be exponential with N. For the values of N that we have used, the falloff is even faster than e^{-N}. Therefore to track solutions for large N we have to reduce the integration grid size appropriately. This leads to a significant increase in computer time and we decided to stop our analysis at $N=3$. Given sufficient computing capacity, there appears to be no reason why we should not continue to find solutions for arbitrarily large N.

As a function of p, $\Sigma(p)$ starts at a finite value $\Sigma(0)$ and begins to fall monotonically once $p\gg\Sigma(p)$. We have not so far seen any qualitative change in the rate of falloff as p passes through $\alpha/8$. The method of solution made use of an ultraviolet cutoff $\Lambda\gg\alpha/8$ on the integral, and solutions are quite insensitive to Λ. A plot of $\Sigma(p)$ vs p for $N=2.6$ is shown in Fig. 3.

The ratio $\Sigma(0)/(\alpha/8)$ is a measure of the hierarchy between the chiral-symmetry-breaking scale and the fundamental scale of the theory. The numerical results for $-\ln[\Sigma(0)/(\alpha/8)]$ for different values of N are shown in Table I. For low N the falloff is of the form e^{-N}, and for $N>2$ it is even faster. Since we have not gone to very large N, however, we decided not to attempt a fit. The large range between $\Sigma(0)$ and α suggests that the solution $\Sigma_1(p)$ to the linearized, infinite-α equation should indeed play a role in the nonlinear, finite-α theory. This question must be studied further, both numerically and analytically, for larger values of N.

Earlier in this section we found that for $N<3.2$, $\Sigma(p)$ has oscillatory behavior. The gap between $\Sigma(0)$ and $\alpha/8$ for this range of N is not so large that the oscillatory solutions found will obviously play a role in the region $\Sigma(p)\ll p\ll\alpha/8$. We have found no numerical evidence for oscillations in our analysis.

IV. THE COMPOSITE OPERATOR EFFECTIVE POTENTIAL

In this section we describe the composite operator effective potential formalism that naturally accompanies the Dyson-Schwinger gap equation. Such a potential as a function of $\langle\bar\psi(x)\psi(y)\rangle$ [i.e., as a function of $\Sigma(p)$ and $A(p)$] can be constructed following Cornwall, Jackiw, and Tomboulis. Dyson-Schwinger equations are then obtained be extremizing the potential with respect to $\Sigma(p)$ and $A(p)$. Therefore a solution of the Dyson-Schwinger equation represents a stationary point of the effective potential. Furthermore the value of the potential at an extremum is the energy density of the corresponding field configuration. Hence the effective potential computation enables us to determine whether the symmetry-breaking solution of the gap equation that we have found is energetically preferred to the symmetric one.

First we review the composite operator effective potential formalism. The quantity $\langle 0|\bar\psi(x)\psi(y)|0\rangle$ can be computed by extremizing the Euclidean effective action Γ defined as follows. Let

$$Z=\int dA\, d\bar\psi\, d\psi \exp\left[-\int d^3x\left[\bar\psi \not{D}\psi-\frac{F^2}{4}+\bar\psi(x)K(x,y)\psi(y)\right]\right], \tag{4.1}$$

where $K(x,y)$ is an external source that induces explicit chiral-symmetry breaking (CSB).

Let $W(K)=\ln Z$ and

$$\phi_c(x,y)=\frac{\delta W(K)}{\delta K}=\langle\bar\psi(x)\psi(y)\rangle_K . \tag{4.2}$$

ϕ_c at $K=0$ is the required expectation value one needs to compute. Let

$$\Gamma(\phi_c)=W-\int \phi_c(x,y)K(x,y) . \tag{4.3}$$

Then

$$\frac{\delta\Gamma}{\delta\phi_c}=-K(x,y) \tag{4.4}$$

and the solution for ϕ_c of the equation

$$\frac{\delta\Gamma}{\delta\phi_c}=0 \tag{4.5}$$

gives the required expectation value. As shown by Cornwall, Jackiw, and Tomboulis,[22] and also Peskin,[23]

$$\Gamma=\mathrm{Tr}(S^{-1}-\not\partial)S-\mathrm{Tr}\ln S^{-1}-2\mathrm{PI\ graphs} , \tag{4.6}$$

where 2PI graphs is the sum of 2-particle-irreducible graphs shown in Fig. 4(b). Here S is the full fermion propagator $S^{-1}=\not k(1+A)+\Sigma(k)$. The gauge field contribution to the potential comes from the 2PI graphs. We shall see that this contribution makes the symmetric vacuum unstable and leads to CSB.

Integral equations for A and Σ can be obtained by letting

$$\frac{\delta\Gamma}{\delta\Sigma}=0=\frac{\delta\Gamma}{\delta A} .$$

They are equivalent to the equations for A and Σ derived from (3.2). We restricted our attention to the leading $1/N$ behavior of vertices and propagators in setting up the Dyson-Schwinger equation. For example we neglected A altogether because its effects were down in $1/N$. Therefore we must do the effective potential computation to the corresponding order, to make energetic comparisons of our symmetry breaking solution with the symmetric one.

To discuss the solutions to the gap equation (3.2), it suffices to carry the computation of the potential to second order in the $1/N$ expansion. Continuing to neglect wave-function renormalization $A(p)$, the zeroth-order potential [Fig. 4(a)] is

$$V_0=\mathrm{Tr}(S^{-1}-\not\partial)S-\mathrm{Tr}\ln S^{-1}$$
$$=\frac{N}{\pi^2}\int_0^\infty p^2dp\left[\frac{2\Sigma^2(p)}{p^2+\Sigma^2(p)}-\ln\left[1+\frac{\Sigma^2(p)}{p^2}\right]\right], \tag{4.7}$$

where a subtraction has been performed to normalize V_0 to be zero at $\Sigma(p)=0$. The next-order term corresponds to the emission and reabsorption of a photon from the fermion loop [Fig. 4(b)]. After angular integration it takes the form

$$V_1=-\frac{\alpha}{2\pi^4}\int_0^\infty p\, dp\frac{\Sigma(p)}{p^2+\Sigma^2(p)}$$
$$\times k\, dk\frac{\Sigma(k)}{k^2+\Sigma^2(k)}\ln\left[\frac{k+p+\frac{\alpha}{8}}{|k-p|+\frac{\alpha}{8}}\right]. \tag{4.8}$$

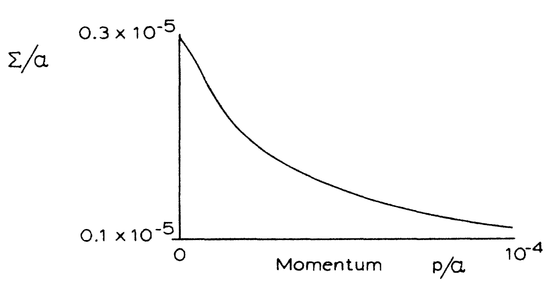

FIG. 3. Numerical solution showing the momentum dependence of the dynamically generated fermion mass for $N=2.6$ fermion flavors.

TABLE I. The fermion self-energy at zero momentum $\Sigma(0)$ as a function of the number of flavors N.

N	$-\ln\left[\frac{\Sigma(0)}{\alpha/8}\right]$
1	2.3
1.2	2.9
1.4	3.6
1.6	4.3
1.8	5.1
2.0	6.1
2.2	7.2
2.4	8.6
2.6	10.7
2.8	13.8
3.0	19.5

FIG. 4. The two leading contributions to the effective potential involving (a) a pure fermion loop and (b) a fermion loop with a gauge-boson exchange.

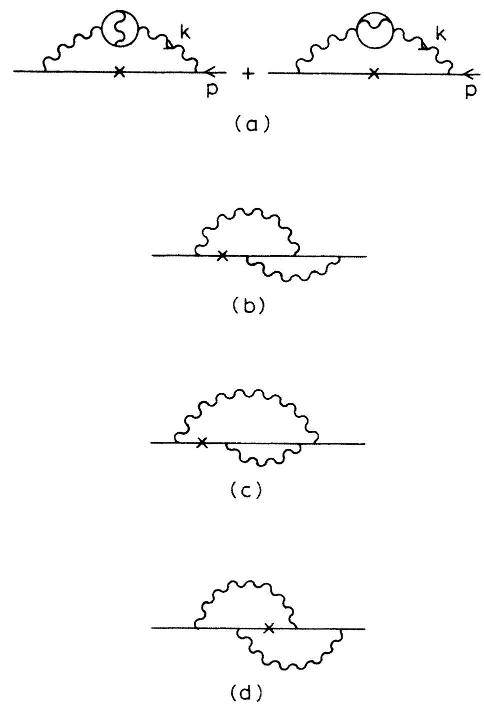

FIG. 5. The leading corrections to the right-hand side of the Dyson-Schwinger equation. In this figure, the wavy line represents the full propagator computed to leading order in the $1/N$ expansion.

[We have omitted an extra term in (4.8) which is part of wave-function renormalization and is down in $1/N$.] Note that V_1 also vanishes at $\Sigma(p)=0$.

If the potential $V=V_0+V_1$ is extremized by setting its functional derivative with respect to $\Sigma(p)$ equal to zero, the Dyson-Schwinger gap equation (3.3) is obtained. The value of the potential at this extremum is

$$V_{\text{ext}}=\frac{N}{\pi^2}\int_0^\infty p^2dp\left[\frac{\Sigma^2}{p^2+\Sigma^2}-\ln\left[1+\frac{\Sigma^2}{p^2}\right]\right]. \quad (4.9)$$

It can easily be seen that this expression is less than or equal to zero by noting that $x/(1+x)-\ln(1+x)$ is negative for all positive x. Therefore if a symmetry-breaking solution can be found it will always be preferred to the symmetric one.

Note that we did not have to make any large-p/α or small-p/α approximation in the previous argument. It can only be vitiated by large-N corrections. In the next section we shall argue that this is not the case.

V. HIGHER-ORDER $1/N$ CORRECTIONS

The analysis described so far has concluded that chiral symmetry is spontaneously broken in a $1/N$ treatment of QED_3. It was emphasized that this does not happen to any finite order in $1/N$. The homogeneous Dyson-Schwinger equation (3.3) contains an explicit factor of $1/N$ on the right-hand side and, in effect, describes a selective resummation of the $1/N$ expansion. The factor of $1/N$ is compensated dynamically by the N dependence of $\Sigma(k)$.

We now look more carefully at higher-order corrections to the right-hand side of the Dyson-Schwinger equation. These will contain explicit factors of $1/N^2$, $1/N^3$, etc., and might naturally be expected to be small corrections. What must be checked is that these terms are not dynamically enhanced, in the way that the leading term is, to bring them up to $O(1)$ in the expansion. In this section, it will be shown that this enhancement does not take place for momenta $p \ll \alpha$. A brief discussion of the region $p \geq \alpha$ will conclude the section.

The leading corrections are shown in Fig. 5. The gauge propagator represents the leading-order expression (2.16) in the $1/N$ expansion. Each graph contains a cross representing the insertion of a single mass factor $\Sigma(k)$ in the fermion propagator. In the limit $k \gg \Sigma(k)$, the Dyson-Schwinger equation can be linearized and there is only this one insertion to be made. In general, however, this is not the case. For external momenta p on the order of $\Sigma(p)$, the nonlinear structure as shown in Eq. (3.3) must be retained. In this limit of course, even Eq. (3.3) is not completely reliable since the massless approximation for $\Pi(p)$ can no longer be used. To begin, therefore, we make the linearized approximation and examine in turn each of the corrections shown in Fig. 5. The role of the nonlinear structure will be described after considering the linearized problem.

We first recall how the factor of $1/N$ is canceled in the leading-order equation (3.3) corresponding to Fig. 2. For $p \ll \alpha$, it is the near-constant behavior of $\Sigma_1(p)$ that solves the linearized equation (3.7). For $p \gtrsim \alpha$, we have obtained an analytic form ($\sim 1/p^2$) only for the asymptotic regime $p \gg \alpha$. The cancellation of the $1/N$ factor depends on details of the solution $\Sigma(k)$ for k in the neighborhood of α where we have not obtained an analytical solution.

The graphs of Fig. 5 represent higher-order corrections to the gauge-boson propagator. For $k \ll \alpha$, the gauge propagator corrections each give contributions to $\Pi(k)$ of order $(1/N)\ln(\alpha/k)$. In the limit $p \ll \alpha$, the $\ln(\alpha/k)$ factor will play an important role in the dominant integration region from p to α. When convoluted with the leading expression for the insertion $\Sigma_1(k)$ in (3.9), it will in fact give a contribution of order N^2, canceling both the overall factor of $1/N$ and the additional factor of $1/N$ accompanying the $\ln(\alpha/k)$. This is exactly what we did not want to happen. Fortunately, the $\ln(\alpha/k)$ factor can be shown to cancel between the two graphs of Fig. 5(a). The cancellation follows from the Ward identity, and is in fact the same cancellation described at the end of Sec. II. Without it, the effective low-energy theory would not retain its scale invariance in the presence of quantum corrections since α would reenter in the logarithm. Equivalently, the cancellation of the $\ln(\alpha/k)$ terms is essential to maintain the existence of the infrared stable fixed point in the running coupling constant (2.19). After cancellation, the higher-order $1/N$ correction simply shifts the position of the fixed point but does not eliminate it. We conclude that for $p \ll \alpha$, the graphs of Fig. 5(a) combine to give a contribution that is of order $1/N$.

The graphs of Figs. 5(b) and 5(c) must be taken together to produce a correction that is down in the $1/N$ expansion. Figure 5(c) represents the wave-function renormalization correction appearing in Eq. (3.2). For momentum $p \ll \alpha$, it is not difficult to estimate the effect of the correction. The dominant range of integration is then $p \leq k \ll \alpha$, and in this range $A(k)$ takes the form

$$A(k) \propto \frac{1}{N} \ln \frac{k}{\alpha} \ . \tag{5.1}$$

It is not difficult to see that if this correction is inserted into the right-hand side of Eq. (3.3) [more specifically, the upper integral in Eq. (3.4)], it gives a contribution of order unity. The essential integral is proportional to

$$\frac{1}{N} \int_p^\alpha \frac{dk}{k} \frac{1}{N} \ln \frac{k}{\alpha} \frac{1}{k^{8/\pi^2 N}} \propto \frac{1}{p^{8/\pi^2 N}} \ . \tag{5.2}$$

The same result can be seen by making the approximation[9] that $\Sigma(p)$ is constant up to α and cutting the integral off at α. The corrected integral will then be proportional to $(1/N^2)\ln^2(\alpha/m) \simeq 1$.

To eliminate this problem, the Ward identity again comes to the rescue. In the limit $p \ll k \ll \alpha$, the leading k dependence of the vertex correction in Fig. 5(b) is $-A(k)$ [Eq. (5.1)]. Thus the logarithm cancels between these two graphs and there remains nothing to overcome the two factors of $1/N$. The combination of the two graphs will give a contribution to the right-hand side of the Dyson-Schwinger equation that is of order $1/N$.

The remaining correction in the linearized approximation is shown in Fig. 5(d). The $\Sigma(k)$ insertion in this graph sits on the fermion propagator that is common to the two vertex subgraphs. There is therefore no vertex or self-energy subgraph that could lead to a behavior of the form of Eq. (5.1). It is not difficult to see that when $\Sigma(k)$ is convoluted with the two overlapping loop integrals, only one of the two factors of $1/N$ is canceled. One concludes, finally, that for $p \ll \alpha$, each of the contributions of Fig. 5 is down in the $1/N$ expansion relative to the right-hand side of the Dyson-Schwinger equation pictured in Fig. 2. It should not be difficult to extend this argument to higher-order terms in the $1/N$ expansion.

When p becomes as small as $\Sigma(p)$, the nonlinear structure of the Dyson-Schwinger equation must be retained. As already emphasized, Eq. (3.3) is not completely reliable in this regime since the gauge-boson propagator then is sensitive to $\Sigma(p)$ and is not simply given by Eq. (2.18). The expected behavior of $\Sigma(p)$ in this regime, however, is shown in Eq. (3.10). It has a finite value at $p=0$ and then begins to evolve into the slowly falling form $\Sigma_1(p)$ of (3.9) once $p \gg \Sigma(p)$. It is clear from this that the factor of $1/N$ in the Dyson-Schwinger equation (3.3) must be compensated predominantly by the integration range $k \gg \Sigma$, even if $p \leq \Sigma$. The contribution from the nonlinear regime itself will be of order $1/N$ compared to the leading piece. One concludes, therefore, that the contributions of Fig. 5 are indeed higher order in the $1/N$ expansion for all $p \ll \alpha$.

We conclude with a brief discussion of the momentum range $p \geq \alpha$. As described in Sec. III, the $1/p^2$ momentum dependence of $\Sigma(p)$ has nothing to do with the $1/N$ expansion. It emerges from a short-distance analysis of the Dyson-Schwinger equation or, equivalently, from an application of the operator-product expansion. The higher-order contributions of Fig. 5 will not affect this asymptotic behavior. The form of the solution $\Sigma(p)$ for $p \simeq \alpha$ has not yet been determined analytically. The numerical solutions, however, reinforce the qualitative expectation that the solution to Eq. (3.3), $\Sigma(p)$, will evolve smoothly from its slowly falling low-p form into a rapidly damping solution at high p. We expect that, once analytic solutions are found for $p \simeq \alpha$, it should be possible to show that the higher-order corrections of Fig. 5 are suppressed in the transition region, just as they are for both small p/α and large p/α.

VI. CONCLUSION

In this paper we have given a detailed analysis of dynamical chiral-symmetry breaking in QED_3 in the large-N limit. Massless QED_3 has been shown to be infrared finite order by order in the $1/N$ expansion. This is due to an effective scale invariance at low momenta, arising from the existence of an infrared stable fixed point. Therefore chiral-symmetry breaking with ensuing mass generation is not essential for solving the infrared problems.

We have solved the Dyson-Schwinger equations of this theory both analytically and numerically in the large-N limit. To obtain analytical solutions we used the linearized equations. Therefore the overall mass scale of the solution was not fixed. For $p \ll \alpha/8$ there were two solutions, one slowly falling like $p^{-8/(\pi^2 N)}$ and the other rapidly falling like $(1/p)p^{8/(\pi^2 N)}$. We have given arguments that the relevant solution for low p is the slowly falling one. This justifies the constant $\Sigma(p)$ ansatz used in earlier work.[9] For $p \gg \alpha$, the solution to the homogeneous Dyson-Schwinger equation falls like $1/p^2$, corresponding to a dynamically generated mass and the spontaneous breaking of chiral symmetry.

The infinite-α solution $p^{-8(\pi^2 N)}$ can only be relevant to the finite-α theory if there is a large range between $\Sigma(0)$ and α. To establish this hierarchy, the full nonlinear, finite-α equation was solved numerically for values of N from one to three. The numerical evidence, presented in Table I shows that $\Sigma(0)/\alpha$ is small even for $N=1$ and that it falls at least as fast as e^{-N}. The dependence of $\Sigma(0)/\alpha$ on N should be studied further, both numerically and analytically, for larger values of N. In particular, it is important to extend the numerical study to values of N well beyond $32/\pi^2$, where the power-law solution $\Sigma_1(p)$

takes over from the oscillatory solution.

Because $\Sigma(p)$ begins to fall like $1/p^2$ for $p \gg \alpha$, it corresponds to dynamical mass generation and the spontaneous breakdown of chiral symmetry. Even though the actual size of $\Sigma(0)$ is small compared to α, the momentum scale at which the dynamical mass turns on and off is of order α. The slowly falling solution that appears well below α would not correspond to spontaneous chiral-symmetry breaking if it were to persist to infinite p. Since the chiral-symmetry breaking appears at $p \simeq \alpha$, there is no approximate scale invariance to be spontaneously broken along with chiral symmetry.

We have computed the Cornwall-Jackiw-Tomboulis composite operator effective potential for $\langle \bar{\psi}\psi \rangle$, and shown that the symmetry-breaking solution is energetically favored to the symmetric one. Therefore we have established that dynamical chiral-symmetry breaking in fact occurs in QED_3, provided the large-N expansion is justified.

We have also given arguments to show that the $1/N$ expansion is reliable. First we have shown that logarithmic corrections to $\Pi(p)$, of higher order in $1/N$, which could have vitiated the infrared fixed point structure of the theory, do not arise. A careful analysis of the Dyson-Schwinger equation showed that our use of the $1/N$ expansion is consistent. The higher-order vertex corrections, and the wave-function renormalization, do not contribute to $\Sigma(p)$, to leading order in $1/N$. This analysis has so far been carried out only for $p \ll \alpha$. The extension to the region $p \gg \alpha$ was hampered by the fact that we do not yet have analytic solutions for p of the order of α.

It would be nice if one could give a formal proof for the symmetry breaking in QED_3 (Ref. 8). It will also be instructive to investigate the Goldstone sector of QED_3, perhaps deriving the effective nonlinear σ model. Finite-temperature restoration of the broken symmetry could also be studied. Finally, we offer the gratuitous remark that the knowledge gained and the techniques developed in studying this model field theory should be utilized in the investigation of more realistic four-dimensional models.

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