Nonequilibrium truncation scheme for the statistical mechanics of relativistic matter

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In this paper we apply a new self-consistent truncation scheme to the infinite coupled hierarchy of transport equations that describe a relativistic plasma. In particular we describe in detail the truncated transport equations for a charged scalar plasma and a neutral scalar plasma in the pairing-approximation limit. We include a brief discussion of particle production within our language. The quantum-mechanical dynamical equations for a positive-energy electron plasma are also cast in a form suitable for approximation within our truncation scheme.

I. INTRODUCTION

The study of relativistic kinetic theories[1] and generalizations that include the mechanism of particle production continue to be topics of interest. In addition to the general theoretical motivation in developing a self-consistent truncation scheme for the relativistic kinetic theories there is considerable experimental effort in studying particle production phenomena from both two- and many-body collisions. Of particular interest to us are heavy-ion collisions in the energy range 1—200 GeV/A. These are currently used or proposed as tools with which to study both transport properties and particle production. At intermediate energies, i.e., 1 GeV/A, pion production from heavy-ion collisions is usually calculated using thermal equilibrium models[2,3] applied at the final stage of the nuclear expansion phase. At fully relativistic energies, i.e., heavy-ion collisions at 10 GeV/A and beyond, there is considerable speculation on the formation of a quark-gluon plasma.[4] Once again theoretical studies of the evolution of this plasma are typically based on classical hydrodynamics together with thermal equilibrium assumptions.[5] In addition to these assumptions the particle production mechanisms that would signal the transition from a quark-gluon plasma to the hadron phase remain unknown. It does seem clear that any measured signature would reflect collective transport of color degrees of freedom thus making transport theories based on QCD relevant. To date, classical relativistic theories of the quark-gluon plasma have been formulated[6] but the absence of a consistent truncation scheme has meant it is difficult to gauge the level of validity of earlier classical or equilibrium models.

In this paper we develop an approximation scheme for the quantum statistical mechanics of relativistic matter that does not assume the concept of local thermodynamic equilibrium or assume the existence of a convergent iterative series via Green's functions for the scattering amplitude.[7] Our approximation scheme exploits the language associated with the covariant-transport-equation approach to multiparticle production. Within this language the quantum-field equations for the amplitudes are replaced by so-called transport equations for the covariant phase-space distribution functions $f(R,P)$ where for our fully relativistic problem R and P are the four-vectors for position and momenta, respectively.

By rewriting the dynamical field equations as transport equations we have developed a self-consistent approximation scheme which describes the evolution of relativistic matter including particle production and which makes full use of the classical insight provided by the phase-space representation. That is, we have developed an approximation scheme that allows the introduction of quantum corrections to any order of truncation and does not require the concept of thermal equilibrium to achieve closure. The approximation scheme discussed in this paper should enable theoretical questions regarding the validity of earlier models to be addressed. We expect our approach to be useful in describing transport problems involving large numbers of degrees of freedom, that is, when classical limits associated with coherentlike states[8] are expected to be quite accurate. Although it is well known that reactions between composite fermion systems with many degrees of freedom exhibit a large degree of classical behavior many quantum features will not be negligible.

In a series of papers Carruthers and Zachariasen[9] have shown how the field-theoretic description of multiparticle production processes can be recast in the form of relativistic transport theory. In particular they show how the phase-space distribution functions are directly related to observable single, double, etc., inclusive differential cross sections. Most of the discussion in these papers has centered on the Klein-Gordon field equation for the neutral pion with a generalized source function $j(x)$. Although these authors note that a phase-space representation should allow a full use of classical intuition they prefer not to exploit this aspect at all and develop an approximation scheme for the transport equations that goes beyond an iterative or Born-series expansion for the scattering amplitude. We do not want to utilize the Born-series approach to simplify the transport equations.

The work presented here developed from an earlier study[10] of the nonrelativistic N-body problem. By rewriting the Schrödinger equation for the N-body amplitude as a quantum Liouville equation for the density matrix the usual transport equation for the Wigner function[11] can be defined. Using the phase-space representation an identity was introduced[10] in which the off-diagonal elements of the density matrix were expanded in deviations from the mean momentum value. On truncating this expansion it was possible to derive a hierarchy of closed self-consistent dynamical equations for the deviations which is exact to the order of truncation. Hence using the phase-space language directly enabled the quantum corrections for the time evolution of the classical density and momenta to be described. By working with the transport equation for the Wigner function we found a closed approximation hierarchy which does not assume thermal equilibrium but rather makes full use of the classical analogy provided by the phase-space variables. The classical solution was restored by looking at the lowest order in the expansion of the quantum density matrix.

In this paper we extend these ideas to relativistic matter considering both scalar and spinor fields. For the scalar field we extend the problem studied by Carruthers and Zachariasen;[9] that is, a transport equation for both neutral and charged plasma based on the Klein-Gordon equation with an arbitrary source function $j(x)$. In this way it is possible to qualitatively compare our truncation scheme with that suggested earlier by these authors. For the spinor field we study the problem of a relativistic quantum electron gas embedded in a strong magnetic field.[12] This problem has many similarities to the transport problem of QCD which will be investigated in a later publication. Hakim and Sivak have studied the electron gas problem by expanding around the thermal equilibrium[13] limit. As these authors point out, relativistic magnetized matter occurs in astrophysical situations with white dwarfs, neutron stars, and possibly the earlier Universe. The development of a transport-equation truncation scheme for this system which does not assume thermal equilibrium and would further enable the equilibrium assumption to be studied is of current interest.

This paper is organized as follows. In Sec. II the general considerations of our truncation scheme are developed and the common notations and definitions are introduced. In Sec. III the scalar and spinor field amplitude equations are written as transport equations in the phase-space language. In Sec. IV the closed set of self-consistent equations are derived in lowest order of quantum correction for the neutral- and charged-scalar-particle transport problem. In addition we briefly describe the role of our truncation scheme directly on particle production. In Sec. V a discussion of our results is given including reference to future applications of our ideas to the very complicated non-Abelian QCD plasma transport problem.

II. GENERAL CONSIDERATIONS

For reasons of clarity and notation we introduce our truncation scheme by briefly reviewing[10] the N-body one-dimensional problem in the nonrelativistic limit. We will also restrict ourselves to the Hartree approximation and assume our N particles to be distinguishable. In addition to providing insight into our truncation scheme for the fully relativistic problem, reviewing the nonrelativistic Hartree approximation provides a useful analogy to the form we will finally assume for the source functions in the relativistic problem. Although in this paper we only consider transport problems where phase-space distribution functions are defined in terms of a single set of phase-space variables, extensions of our truncation scheme to include distribution functions of the form $f(R_1P_1;R_2P_2)$, etc., have been derived and will be published shortly. We define the n-particle Wigner function as

$$f_N^{(n)}(x_1p_2,x_2p_2,\dots,x_np_n;t)=\int\prod_{j=n+1}^{N}(d^3p_jd^3x_j)f_N(x_1p_1,x_2p_2,\dots,x_Np_N;t)\ , \tag{2.1}$$

where in the Hartree approximation

$$f_N^{(n)}(x_1p_1,x_2p_2,\dots,x_np_n;t)=\prod_{i=1}^{n}f_N^{(1)}(x_ip_i;t)\ . \tag{2.2}$$

In the nonrelativistic domain the transport equation in the Hartree approximation for the one-body reduced Wigner function $f_N^{(1)}(x_jp_jt)$ is given by[10]

$$\frac{\partial f_N^{(1)}}{\partial t}+\frac{p_j}{m}\frac{\partial f_N^{(1)}}{\partial x_j}+\frac{2}{\hbar}\sin\left[\frac{\hbar}{2}\frac{\partial^{(v)}}{\partial x_j}\frac{\partial^{(f)}}{\partial p_j}\right]v_{\text{eff}}(x_j)f_N^{(1)}=0\ , \tag{2.3}$$

where $f_N^{(1)}$ is related to the one-body density matrix ρ by[10]

$$f_N^{(1)}(x_jp_jt)=\frac{1}{(2\pi\hbar)^3}\int d^3y_je^{p_jy_j/\hbar}\times\rho(x_j-\tfrac{1}{2}y_j;x_j+\tfrac{1}{2}y_j;t)\ . \tag{2.4}$$

Equation (2.3) can be derived by inverting Eq. (2.4) and substituting in the quantum Liouville equation for the density matrix in the usual way. In Eq. (2.3) x_j and p_j are the position and momentum coordinates, respectively, for particle j, $\partial^{(v)}/\partial x_j$ means this operator acts on function v_{eff} only, and $v_{\text{eff}}(x_j)$ is the effective one-body mean field in the Hartree approximation for particle j. At this point we drop label j because each one-particle density matrix may be treated as an independent function in the Hartree approximation.

We develop our approximation scheme for the transport equation (2.3) by defining the moment function $\langle p^n(x,t)\rangle$ in one dimension as[14]

$$\rho_0(x,t)\langle p^n(x,t)\rangle = \left[\frac{\hbar}{2i}\right]^n \left[\left[\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_k}\right]^n \times\rho(x_k,x_i,t)\right]_{x_i=x_k=x} \tag{2.5}$$

$$= \int f(x,p)p^n dp \ , \tag{2.6}$$

where $\rho_0(x,t)$ is the diagonal density and where for instance $n=1$ corresponds to the time-dependent momentum distribution function $\langle p(x,t)\rangle$. Using definition (2.6) the transport equation (2.3) may be recast[10] exactly as an infinite set of coupled equations for the moments $\langle p(x,t)\rangle$, i.e.,

$$\frac{\partial}{\partial t}(\rho_0\langle p^n\rangle) = -\frac{1}{m}\frac{\partial}{\partial x}(\rho_0\langle p^{n+1}\rangle) + \frac{4\rho_0}{\hbar^2}\sum_{\text{odd } k=1}^{n}\left[\frac{-\hbar}{2i}\right]^{k+1}\binom{n}{k}\langle p^{n-k}\rangle \times\frac{\partial^k v_{\text{eff}}}{\partial x^k} \ , \tag{2.7}$$

where $n=0,1,2\ldots$.

Equation (2.7) is an alternative representation of Eq. (2.3). It is this set of infinite coupled equations that we close using our truncation scheme. We suggest that the most physical truncation scheme is not to simply say $\langle p^n(x,t)\rangle=0$ for all $n\geq m$, where m is the order of truncation and n the running index of (2.7), but rather to use $\langle (p-\langle p\rangle)^n\rangle=0$ for all $n\geq m$. The motivation for this arose partly from the following identity we have introduced for the density matrix:[10]

$$\rho(x,x',t)=\rho_0\left[\frac{x+x'}{2},t\right]\exp\left[\frac{-i}{\hbar}(x'-x)\langle p\rangle\right] \times\left\langle\exp\left[\frac{-i}{\hbar}(x'-x)(p-\langle p\rangle)\right]\right\rangle \ , \tag{2.8}$$

where our notation means

$$\left\langle\exp\left[\frac{-i}{\hbar}(x'-x)(p-\langle p\rangle)\right]\right\rangle = 1-\frac{i}{\hbar}(x'-x)(\langle p\rangle-\langle p\rangle) - \frac{1}{\hbar^2}(x'-x)^2\langle (p-\langle p\rangle)^2\rangle+\cdots \tag{2.9}$$

and

$$\langle (p-\langle p\rangle)^n\rangle=\rho_0{}^{-1}\int f(x,p)(p-\langle p\rangle)^n dp \ ,$$

Equation (2.8) represents an expansion of the density matrix in powers of $(x'-x)^n\langle (p-\langle p\rangle)^n\rangle/\hbar^n$ for $n=0,1,2,3\ldots$, and where p is the momentum coordinate. Using this expression we have introduced a power-series expansion for the off-diagonal matrix elements of the density matrix which is directly related to the deviations from the mean momentum value. By recasting the transport equation in the form (2.7) and truncating the density matrix at some order in $(x'-x)^n\langle (p-\langle p\rangle)^n\rangle/\hbar^n$ it is possible[10] to derive a closed self-consistent set of dynamical equations for the diagonal density and the deviations $\langle (p-\langle p\rangle)^n\rangle$ from the mean momentum value. That is by putting $\langle (p-\langle p\rangle)^n\rangle=0$ for all $n\geq m$ where m is the order of truncation in (2.8) and n is the running index in (2.7) we have shown it is possible to close the set of infinite coupled equations (2.7) without assuming the existence of local thermal equilibrium. This is not possible for a simple moment expansion in terms of $\langle p^n(x,t)\rangle$. On writing the density matrix in the form (2.8) the dynamical evolution equations describing the deviations may be realized by recasting the Liouville equation in the forms (2.3) and (2.7). In addition the coupled equations may be closed self-consistently using the identity (2.8).

In the classical limit we require $\langle (p-\langle p\rangle)^n\rangle=0$ for all n. In this limit the diagonal density in Eq. (2.8) may be called the classical density. In this limit we also obtain the result $\langle p^n\rangle=\langle p\rangle^n$; a result often associated with the classical limit of coherent states. This result is also not realized if the simple moment expansion is truncated for $n\geq m$. When our condition for a classical limit is applied to the set (2.7) the closed classical equations of motion may be realized.

Further insight into the nature of the classical limit may be gained by substituting Eq. (2.8) into the one-particle Wigner-function definition (2.4). In this way the function $f_N^{(1)}(x_jp_jt)$ is given by the series

$$f_N^{(1)}(x_jp_jt)=\rho(x_j)\left[\left\langle\exp\left[-(p_j-\langle p_j\rangle)\frac{\partial}{\partial p_j}\right]\right\rangle \times\delta(p_j-\langle p_j\rangle)\right] . \tag{2.10}$$

Hence in the classical limit writing the diagonal density matrix for a point particle as $\delta(x_j-x_j')$ and setting $\langle (p_j-\langle p_j\rangle)^n\rangle=0$ for all n in the phase-space function becomes

$$f_N^{(1)}(x_jp_jt)=\delta(x_j-x_j')\delta(p_j-p_j') \tag{2.11}$$

as required. The symmetry of the phase-space variables in the definition of $f_N^{(1)}$ also suggests the existence of the series

$$f_N^{(1)}(x_jp_jt)=\rho(p_j)\left[\left\langle\exp\left[-(x_j-\langle x_j\rangle)\frac{\partial}{\partial x_j}\right]\right\rangle \times\delta(x_j-\langle x_j\rangle)\right] . \tag{2.12}$$

The lowest quantum correction in (2.8) has terms up to $\langle (p-\langle p\rangle)^2\rangle(x'-x)^2/\hbar^2$. The closed self-consistent dynamical equations for ρ_0, $\langle p\rangle$, and $\langle (p-\langle p\rangle)^2\rangle$ together with a constraint equation are derived in Ref. 10 explicitly for the nonrelativistic N-body problem in the Hartree approximation. For this order of truncation the $n=0,1$ equations from the set (2.7) remain unchanged. The $n=2$ equation describes the evolution of the deviation $\langle (p-\langle p\rangle)^2\rangle$ and the $n=3$ equation reduces to a constraint equation relating this deviation to the effective potential and diagonal density.[10] The $n=4$ equation thus closing the set of coupled equations. For higher-order truncation of (2.8) the pattern described here is repeated. The mechanics of this is shown in some detail in Sec. IV.

At this point we mention that the dynamical equations associated with expansion (2.8) always respect the conservation of current conditions for the problem of interest. For instance the self-consistent equations describing the evolution of a charged-scalar plasma conserve charge through the well-known continuity equation. The properties of the unusual strictly neutral plasma are also naturally reflected in a modified form of Eq. (2.8). For this plasma Eq. (2.5) shows all moments corresponding to odd n are identically zero. This is correctly reflected by the complete absence of a dynamical equation for conservation of either charge or mass. These points are also shown in more detail in Sec. IV.

For relativistic matter the truncation scheme is analogous to that discussed above for the nonrelativistic problem. The field equations are replaced by a relativistic Liouville equation which is recast as a transport equation in the phase-space functions. A relativistic generalization of Eqs. (2.5)—(2.9) leads to a closed form for the infinite coupled set of relativistic moment equations. The four-vector nature of the density matrix expansion considerably complicates the truncation procedure. For the positive-energy electron plasma it is not desirable to work with a single Liouville equation for the density matrix because of the negative-energy component properties of the Dirac spinors. For this problem the single Liouville equation is represented by conjugate transport equations. For the spinor problem it will also prove useful to expand the Wigner function on a basis of the independent representations of the Dirac γ matrices.

In the same spirit as the Hartree approximation we will restrict ourselves in this paper to the collisionless plasma. Extending this approximation is fairly straightforward and would permit a systematic study of quantum effects on the collision integral.

From a general point of view we do not propose explicitly calculating the phase-space function for any order of approximation as in the conventional moment method.[1] Instead we have utilized the classical intuition provided by the phase-space representation to approximate the time evolution of the density matrix.

III. TRANSPORT EQUATION FOR SCALAR AND SPINOR PARTICLES

A. Scalar particles

In the same spirit as Carruthers and Zachariasen[9] we consider the evolution of spinless particles during a two-body collision. We consider separately the charged and neutral spinless particles. Extending the definition of these authors of a covariant one-particle distribution function a relativistic generalization of (2.4) for charged spinless particles is given by

$$f(R,p)=\frac{1}{(2\pi)^4}\int d^4r\, e^{ip\cdot r} \times\langle\psi|\phi^\dagger(R-\tfrac{1}{2}r)\phi(R+\tfrac{1}{2}r)|\psi\rangle\,, \tag{3.1}$$

where $|\psi\rangle$ is the normalized incoming state, ϕ is the field operator for the produced charged particle and antiparticle, p is the relative momentum four-vector, R,r are the center-of-mass and relative position four-vectors, respectively, and $p\cdot r$ is equivalent to $p^\mu r_\mu$. For the relativistic problem we assume $\hbar=c=1$. For the neutral spinless plasma, Eq. (3.1) is simply modified to read $\phi(R-\frac{1}{2}r)\phi(R+\frac{1}{2}r)$. The choice of $|\psi\rangle$ depends on the problem of interest. For particle production $|\psi\rangle$ would be the two incident particles that collide, i.e., two nucleons or two heavy ions. The field equation is taken to be a solution of

$$(\Box+\mu^2)\phi(x)=j(x)\,, \tag{3.2}$$

where μ^2 is a mass term, $j(x)$ is a source function to be specified later, and $\Box$ is taken for the D'Alembertian. The form of $j(x)$ is related to the interaction Lagrangian. For low-energy heavy-ion collisions and so-called subthreshold pion production the single source of pions has been associated with bremsstrahlung.[15] For this and more general problems it would not be possible to express the source $j(x)$ in terms of a single-particle distribution function. However, in order to draw a close analogy to earlier work on the nonrelativistic N-body Hartree problem and to set a basis for future developments including correlation functions we choose our interaction Lagrangian as one in which the Hartree limit may be realized. More sophisticated interaction Lagrangians including the coupling of baryons and mesons could be accommodated but would complicate the overall development introduced in this paper. Constructing a Liouville-type equation from (3.2) we find

$$(\Box_2-\Box_1)\langle\phi^\dagger(x_1)\phi(x_2)\rangle=\langle\phi^\dagger(x_1)j(x_2)\rangle-\langle j^\dagger(x_1)\phi(x_2)\rangle \tag{3.3}$$

or

$$2\frac{\partial}{\partial R}\cdot\frac{\partial}{\partial r}\left\langle\phi^\dagger\left[R-\frac{r}{2}\right]\phi\left[R+\frac{r}{2}\right]\right\rangle =\left\langle\phi^\dagger\left[R-\frac{r}{2}\right]j\left[R+\frac{r}{2}\right]\right\rangle -\left\langle j^\dagger\left[R-\frac{r}{2}\right]\phi\left[R+\frac{r}{2}\right]\right\rangle\,, \tag{3.4}$$

where $R=(x_1+x_2)/2$, $r/2=(x_2-x_1)/2$.

Hence using (3.1), Eq. (3.4) can be converted into a transport equation for the phase-space distribution function $f(R,p)$ giving

$$2ip\cdot\frac{\partial}{\partial R}f(R,p)=\int d^4r\, e^{ip\cdot r}\left[\left\langle j^\dagger\left[R-\frac{r}{2}\right]\phi\left[R+\frac{r}{2}\right]\right\rangle-\left\langle\phi^\dagger\left[R-\frac{r}{2}\right]j\left[R+\frac{r}{2}\right]\right\rangle\right]\,. \tag{3.5}$$

In order to study our truncation scheme for a simple but nontrivial example we choose the interaction Lagrangian to be of the form

$$L_I = \frac{1}{n}\lambda \phi^{\dagger n}\phi^n \,, \tag{3.6}$$

where $n=2$ but we retain generality for insight into the pairing approximation.[9] By assuming this approximation may be applied, a close analogy can be drawn between our final relativistic transport equation and the nonrelativistic N-body problem in the Hartree limit. Like the Hartree limit the pairing approximation is based on the idea that correlations are weak. Using Eq. (3.6) the transport equation becomes

$$\begin{aligned} 2ip\frac{\partial}{\partial R}f(R,p) = \lambda \int d^4r\, e^{ip\cdot r}\Bigg[&\left\langle \phi^{\dagger n}\left[R-\frac{r}{2}\right]\phi^{n-1}\left[R-\frac{r}{2}\right]\phi\left[R+\frac{r}{2}\right]\right\rangle \\ &- \left\langle \phi^{\dagger}\left[R-\frac{r}{2}\right]\phi^{\dagger(n-1)}\left[R+\frac{r}{2}\right]\phi^{n}\left[R+\frac{r}{2}\right]\right\rangle \Bigg] \,. \end{aligned} \tag{3.7}$$

The pairing approximation assumes the expectation value of a product of fields can be expressed as a sum of the expectation values of all possible pairings, i.e.,

$$\begin{aligned} &\left\langle \phi^{\dagger n}\left[R-\frac{r}{2}\right]\phi^{n-1}\left[R-\frac{r}{2}\right]\phi\left[R+\frac{r}{2}\right]\right\rangle - \left\langle \phi^{\dagger}\left[R-\frac{r}{2}\right]\phi^{\dagger(n-1)}\left[R+\frac{r}{2}\right]\phi^{n}\left[R+\frac{r}{2}\right]\right\rangle \\ &= (n-1)\left[\left\langle \phi^{\dagger(n-1)}\left[R-\frac{r}{2}\right]\phi^{n-1}\left[R-\frac{r}{2}\right]\right\rangle - \left\langle \phi^{\dagger(n-1)}\left[R+\frac{r}{2}\right]\phi^{n-1}\left[R+\frac{r}{2}\right]\right\rangle\right]\left\langle \phi^{\dagger}\left[R-\frac{r}{2}\right]\phi\left[R+\frac{r}{2}\right]\right\rangle \,. \end{aligned} \tag{3.8}$$

If we define

$$v\left[R-\frac{r}{2}\right] = \lambda(n-1)\left\langle \phi^{\dagger(n-1)}\left[R-\frac{r}{2}\right]\phi^{n-1}\left[R-\frac{r}{2}\right]\right\rangle \,, \tag{3.9}$$

$$v\left[R+\frac{r}{2}\right] = \lambda(n-1)\left\langle \phi^{\dagger(n-1)}\left[R+\frac{r}{2}\right]\phi^{n-1}\left[R+\frac{r}{2}\right]\right\rangle \,, \tag{3.10}$$

then the transport equation (3.7) for the charged spinless plasma can be reduced to final form

$$2ip\cdot\frac{\partial}{\partial R}f(R,p) = \int d^4r\, e^{ip\cdot r}\left[v\left[R-\frac{r}{2}\right] - v\left[R+\frac{r}{2}\right]\right]\phi^{\dagger}\left[R-\frac{r}{2}\right]\phi\left[R+\frac{r}{2}\right] \tag{3.11}$$

$$= -\frac{2}{\hbar}\sin\left[\frac{\hbar}{2}\frac{\partial^{(v)}}{\partial R}\frac{\partial^{(f)}}{\partial p}\right]v(R)f(R,p) \,, \tag{3.12}$$

where Eq. (3.1) has been used in going from (3.11) to (3.12).

The transport equation for the neutral-scalar plasma is of the same form as (3.12) but the function v is defined by

$$v\left[R-\frac{r}{2}\right] = \lambda(n-1)\left\langle \phi^{n-2}\left[R-\frac{r}{2}\right]\right\rangle \,, \tag{3.13}$$

$$v\left[R+\frac{r}{2}\right] = \lambda(n-1)\left\langle \phi^{n-2}\left[R+\frac{r}{2}\right]\right\rangle \,, \tag{3.14}$$

for an interaction Lagrangian ($n=4$)

$$L_I = \frac{1}{n}\lambda\phi^n \,. \tag{3.15}$$

The analogy between Eqs. (3.12) and (2.3) is now clear. If the pairing approximation had not been used then the transport equation would be defined in terms of the many-particle distribution function $f(R_1p_1;R_2p_2)$, etc. We shall not try to justify this well-known technique[9] but assume correlations are weak enough for the Hartree limit to be realized. Within this context we have extended the truncation scheme described in this paper to reduce the more general N-body problem to one involving correlations up to and including N-particle-reduced Wigner function.[16] The scheme discussed here has been applied to more general problems which include correlation functions.

An earlier suggestion[9] for calculating (3.12) considered v to be slowly varying and hence only the first term of the operator expansion was kept. This simplified equation was then recast as an integral equation which was expressed as a Born or iterative series in v. We suggest however that a more physical truncation scheme may be developed by first rewriting (3.12) as an infinite set of coupled equations for the relativistic moments and then developing the closed set of equations for the density and deviations as for the nonrelativistic problem.

B. Dirac particles

We now consider the transport equations for the problem of a relativistic quantum electron plasma embedded in a strong external magnetic field. In analogy to the nonrelativistic Hartree problem and the pairing approximation for scalar particles we replace the mutual interactions between the electrons by a single field $A(x)$. Because of the spin degrees of freedom it will prove more convenient to expand the 16-component Wigner function in the usual basis of 16 independent combinations of γ matrices. In addition the constant external magnetic field allows the introduction of a simplifying gauge condition. All of these features will play a central role in future discussions of relativistic QCD plasma dynamics.

In this section we utilize the derivation of Hakim and Sivak for the multicomponent Wigner function. These authors were interested in developing a Chapman-Enskog–type series by expanding around a thermal equilibrium form. We do not want to assume this condition as part of our approximation scheme but develop a formalism in the same spirit as earlier sections of this paper.

The 16-component covariant Wigner function is defined as the normal-ordered product:

$$F_{\alpha,\beta}(R,p)=\frac{1}{(2\pi)^4}\int d^4r\, e^{-i\pi\cdot r}\times\left\langle :\bar{\psi}_\beta\left[R+\frac{r}{2}\right]\otimes\psi_\alpha\left[R-\frac{r}{2}\right]:\right\rangle , \tag{3.16}$$

where $\pi\cdot r=\pi_\mu r^\mu$, p and π are the momentum four-vectors, r and R are the position four-vectors, (α,β) are spinor indices, and the bra-ket stands for statistical averaging. The Dirac fields $\psi\bar{\psi}$ satisfy the equations

$$(i\gamma\cdot\vec{\partial}-m)\psi=0 , \tag{3.17}$$

$$\bar{\psi}(i\gamma\cdot\overleftarrow{\partial}+m)=0 , \tag{3.18}$$

where $\pi^\mu=p^\mu+eA^\mu_{\rm ext}$ and $A^\mu_{\rm ext}$ is the electromagnetic four-potential corresponding to an external magnetic field. Using (4.1) the four-current is defined as

$$J^\mu(R)=e\,{\rm Tr}\int d^4p\, \gamma^\mu F(R,p) \tag{3.19}$$

and the canonical momentum-energy tensor is given by

$$T^{\mu\nu}={\rm Tr}\int d^4p\, \pi^\mu\gamma^\nu F(R,p) , \tag{3.20}$$

where Tr means trace over spinor indices. By defining our Wigner function in terms of the canonical momenta π rather than the kinetic momentum p has the advantage that Eqs. (3.19) and (3.20) are formally identical with the nonquantum expressions.[12] It is much more convenient to expand the matrix $F(R,p)$ on the basis of the 16 γ^A matrices written as

$$\gamma^A=\left[I,\gamma^\mu,\sigma^{\mu\nu}=\tfrac{1}{2}[\gamma^\mu,\gamma^\nu],\gamma^5=\frac{i}{4!}e_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta,\gamma_5\gamma^\mu\right] \quad (A=1,2,\ldots,16) , \tag{3.21}$$

i.e.,

$$F(R,p)=\tfrac{1}{4}[f(R,p)I+f_\mu(R,p)\gamma^\mu+\tfrac{1}{2}f_{\mu\nu}(R,p)\sigma^{\mu\nu}+f_5(R,p)\gamma^5+f_{5\mu}(R,p)\gamma^\mu\gamma_5] , \tag{3.22}$$

where

$$\gamma^0=\begin{bmatrix} I & 0\\ 0 & -I\end{bmatrix},\quad \gamma^i=\begin{bmatrix} 0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix},\quad \gamma^5=\begin{bmatrix} 0 & -I\\ -I & 0\end{bmatrix} . \tag{3.23}$$

In this way the transport equations for each of the components $f_A(R,p)$ can be formulated without explicit reference to the γ matrices. Using the definition (3.16) the inhomogeneous field equations of interest here can be recast as transport equations in the form

$$(\vec{K}\cdot\gamma+2im)F=\gamma\cdot MF , \tag{3.24}$$

$$F(\gamma\cdot\overleftarrow{K}^*-2im)=M^*\cdot\gamma F , \tag{3.25}$$

where for arbitrary $\chi(R,p)$

$$K_\mu\chi(R,p)=\left[\partial_\mu-2ip_\mu-e\left[\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(\chi)}}{\partial p}\right]A^{\rm ext}_\mu\right]\chi(R,p) \tag{3.26}$$

and the source term of interest here is given as

$$M_\mu\chi(R,p)=\frac{-2ie}{(2\pi)^4}\int d^4R'd^4p'e^{-i(p-p')\cdot R'}\times A(R-R')\chi(R,p') . \tag{3.27}$$

Equations (3.24) and (3.25) are equivalent to the single Liouville equation discussed earlier for both the relativistic scalar field and the nonrelativistic Hartree problem. The spinor nature of the field equations (3.17) and (3.18) dictates the use of two transport equations for $F_{\alpha\beta}(R,p)$. Hakim and Sivak choose the Lorentz gauge for the external magnetic field, i.e.,

$$A^\mu_{\rm ext}=-\tfrac{1}{2}F^\mu_{{\rm ext},\beta}R^\beta , \tag{3.28}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. This linear gauge which is valid for a constant external field simplifies the full operator expansions in the homogeneous part of (3.24) or (3.25), as can be seen in Eq. (3.26).

Expanding $F(R,p)$ on the basis of 16 γ^A matrices and substituting in Eqs. (3.24) and (3.25) yields a set of dynamical equations for the 16 components f_A defined by

$$f_A={\rm Tr}(\gamma_A F) . \tag{3.29}$$

These dynamical equation have been tabulated elsewhere.[13] At this point we depart from Hakim and Sivak and rewrite these dynamical equations for the components of F in the following form:

$$D_i=\left[\partial_i+eF^\alpha_{{\rm ext},i}\frac{\partial}{\partial p^\alpha}\right] , \tag{3.30}$$

$$D_\mu f^\mu = -2e \sin\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A_\mu(R) f^\mu \,, \tag{3.31}$$

$$D_\mu f = -2e \sin\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A_\mu(R) f \,, \tag{3.32}$$

$$D^\tau f_{\mu\tau} = -2e \sin\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A^\tau(R) f_{\mu\tau} + 2e \sin\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A_\mu(R) f \,, \tag{3.33}$$

$$D^\mu f_{5\mu} = -2e \sin\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A^\mu(R) f_{5\mu} \,, \tag{3.34}$$

$$D_\mu f^5 = 0 \,, \tag{3.35}$$

$$p_\mu f^\mu = mf + e \cos\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A_\mu(R) f^\mu \,, \tag{3.36}$$

$$p_\mu f = mf_\mu + e \cos\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A_\mu(R) f \,, \tag{3.37}$$

$$p_\nu f_\mu + \epsilon_{\tau\lambda\mu\nu} p^\tau f_5^\lambda + m f_{\mu\nu} = e \cos\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A_\nu(R) f_\mu + \epsilon_{\tau\lambda\mu\nu} e \cos\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A^\tau(R) f_5^\lambda \,, \tag{3.38}$$

$$p^\mu f_{5\mu} = e \cos\left[\frac{1}{2}\frac{\partial^{(A)}}{\partial R}\cdot\frac{\partial^{(f)}}{\partial p}\right] A^\mu(R) f_{5\mu} + m f^5 \,, \tag{3.39}$$

$$p_\mu f^5 = 0 \,. \tag{3.40}$$

Equations (3.30)—(3.35) were introduced from the sum and difference of the tabulated dynamical equations for the components[13] f_A. In this form several redundant equations may be realized and have been omitted from this list. The format of these equations is now close to that studied extensively in previous sections and represents the most useful form of the transport equations for our truncation scheme. Equations (3.36)—(3.40) can be approximated using the truncation scheme introduced in Sec. II. Note that for many of the components the transport equations are not coupled suggesting that each component of F may be studied independently.

Equations (3.33)—(3.37) were also derived from the sum and difference of the tabulated equations and can be thought of as constraint conditions for the electron plasma that relate the various spinor components of the expansion (3.22). In addition, if the field $A_\mu(R)$ can be associated with the Lorentz gauge (3.28) for the external magnetic field the first two constraint equations can be written as

$$(p_\mu - A_\mu^{\text{ext}}) f^\mu = mf \,, \tag{3.41}$$

$$(p_\mu - A_\mu^{\text{ext}}) f = mf_\mu \,. \tag{3.42}$$

Using this approximation the momentum-energy tensor (3.20) becomes

$$T^{\mu\lambda}(R) = \frac{1}{m}\int d^4p (p^\mu - eA^\mu)(p^\lambda - eA^\lambda) f(R,p) \tag{3.43}$$

which is similar to the classical relativistic expression. In practice if $A_\mu(R)$ can be approximated as a Taylor-series expansion in R^μ then these constraint equations may save much computational labor. This could be important for the QCD plasma description when many more components for F would be introduced.

IV. TRUNCATION SCHEME FOR SCALAR TRANSPORT EQUATIONS

A. Charged scalar particles

In this section we extend the ideas introduced for the nonrelativistic N-body problem to approximate the scalar transport equation (3.12). Scalar particle production will be addressed at the end of this section. The spinor transport equations (3.30)—(3.40) are now written in the same form as (3.12) and can be approximated along the same lines as the techniques introduced here for the scalar problem. The main difference arises from the requirement that each dynamical equation for the component of $F_{\alpha\beta}(R,p)$ be treated separately.

By analogy with Eq. (2.6) we write the relativistic moments as

$$\rho_0 \langle p^n \rangle = \int d^4p\, p^n f(R,p) \tag{4.1}$$

and in addition write the relativistic density matrix as

$$\rho(x,x') = \rho_0\left[\frac{x+x'}{2}\right] \exp[-i(x'-x)\langle p\rangle] \times \langle \exp[-i(x'-x)(p-\langle p\rangle)]\rangle \,, \tag{4.2}$$

where $x' = R + r/2$ and $x = R - r/2$ are four-vectors and ρ_0 now represents the charge-density distribution.

Equation (4.2) is valid for the charged spinless plasma when all moments $\langle p^n(x)\rangle$ should contribute in principle. However for the neutral plasma the field ϕ is real and hence Eq. (2.5) shows $\langle p^n(x)\rangle = 0$ for all n odd. For the neutral plasma only Eq. (4.2) reduces to

$$\rho(x,x') = \rho_0\left[\frac{x+x'}{2}\right] \langle \cos(x-x')p\rangle \,. \tag{4.3}$$

Unlike the density matrix expansion which is valid in the nonrelativistic domain, truncating the relativistic expression (4.2) or (4.3) at some order n in $(x'-x)^n \langle (p-\langle p\rangle)^n\rangle / \hbar^n$ suggests the deviations from the mean four-momenta must now satisfy the identity

$$\langle (p_0 - \langle p_0\rangle)^{k_0} (p_1 - \langle p_1\rangle)^{k_1} (p_2 - \langle p_2\rangle)^{k_2} (p_3 - \langle p_3\rangle)^{k_3}\rangle = 0 \,, \tag{4.4}$$

where $k_0 + k_1 + k_2 + k_3 \geq n$. In other words for some order n in the truncation scheme all combinations of k_0, k_1, k_2, k_3 satisfying this inequality need to be taken into account. In Eqs. (4.2) and (4.4) we have introduced the notation $(p_0, \mathbf{p})$ for the momentum four-vector.

We now want to develop our truncation hierarchy for the transport equation (3.12) using the relations (4.1) and (4.4). In this paper we consider only expanding (4.2) to lowest order in the quantum corrections. That is we truncate Eq. (4.2) at order $n=2$ in $(x'-x)^n\langle(p-\langle p\rangle)^n\rangle/\hbar^n$. This order of truncation will enable us to show in some detail the mechanics of our approximation scheme. Generalization to higher n is straightforward and the dynamical equations corresponding to truncation up to and including $N=7$ have been worked out and will be published elsewhere.[16]

Using Eq. (3.14) for the definition of ρ_0, $\langle p\rangle,\langle p^2\rangle,\langle p^3\rangle$ the four relevant coupled moment equations for the $n=2$ truncation are easily derived. For ease of writing we work in 1 + 1 dimensions and remove the subscript from the diagonal density matrix. Choosing p_3 as the relevant spatial dimension the equations we wish to close to order $n=2$ are given explicitly as [compare with (2.7)]

$$\frac{\partial}{\partial x_0}(\rho\langle p_0\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_3\rangle)=0\,, \tag{4.5}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0{}^2\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_0p_3\rangle)=-\frac{1}{2}\rho\frac{\partial v}{\partial x_0}\,, \tag{4.6}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0p_3\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_3{}^2\rangle)=-\frac{1}{2}\rho\frac{\partial v}{\partial x_3}\,, \tag{4.6a}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0{}^3\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_0{}^2p_3\rangle)=-\frac{1}{2}\rho\left[2\frac{\partial v}{\partial x_0}\langle p_0\rangle\right], \tag{4.7}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0{}^2p_3\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_0p_3{}^2\rangle)=-\frac{1}{2}\rho\left[\frac{\partial v}{\partial x_0}\langle p_3\rangle+\frac{\partial v}{\partial x_3}\langle p_0\rangle\right], \tag{4.7a}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0p_3{}^2\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_3{}^3\rangle)=-\frac{1}{2}\rho\left[2\frac{\partial v}{\partial x_3}\langle p_3\rangle\right], \tag{4.7b}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0{}^4\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_0{}^3p_3\rangle)=-\frac{1}{2}\rho\left[3\frac{\partial v}{\partial x_0}\langle p_0{}^2\rangle-3\frac{1}{12}\frac{\partial^3 v}{\partial x_0{}^3}\right], \tag{4.8}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0{}^3p_3\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_0{}^2p_3{}^2\rangle)=-\frac{1}{2}\rho\left[2\frac{\partial v}{\partial x_0}\langle p_0p_3\rangle+\frac{\partial v}{\partial x_3}\langle p_0{}^2\rangle-\frac{3}{12}\frac{\partial^3 v}{\partial x_0{}^2\partial x_3}\right], \tag{4.8a}$$

$$\frac{\partial}{\partial x_0}(\rho\langle p_0{}^2p_3{}^2\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_0p_3{}^3\rangle)=-\frac{1}{2}\rho\left[\frac{\partial v}{\partial x_0}\langle p_3{}^2\rangle+2\frac{\partial v}{\partial x_3}\langle p_0p_3\rangle-\frac{3}{12}\frac{\partial^3 v}{\partial x_0\partial x_3{}^2}\right], \tag{4.8b}$$

$$\frac{\partial}{\partial x_3}(\rho\langle p_0p_3{}^3\rangle)+\frac{\partial}{\partial x_3}(\rho\langle p_3{}^4\rangle)=-\frac{1}{2}\rho\left[3\frac{\partial v}{\partial x_3}\langle p_3{}^2\rangle-\frac{3}{12}\frac{\partial^3 v}{\partial x_3{}^3}\right]. \tag{4.8c}$$

Equation (4.5) is the continuity equation showing explicitly charge conservation. Now let us introduce the constraints $\langle(p-\langle p\rangle)^3\rangle=\langle(p-\langle p\rangle)^4\rangle=0$ which are consistent with truncating (4.2) at $n=2$. These constraints together with Eq. (4.4) lead to a set of relations between the moments which are tabulated in Appendix A. For the order of truncation considered here Eqs. (4.5) for the charge conservation and (4.6) and (4.6a) remain unchanged. The results from Appendix A are used to eliminate extraneous variables from Eqs. (4.7)–(4.8c). For instance substituting (A1) and (A2) into Eq. (4.6) leads to the equation

$$\frac{\partial}{\partial x_0}[\rho\langle p_0\rangle\langle(p_0-\langle p_0\rangle)^2\rangle]+\frac{1}{2}\frac{\partial}{\partial x_0}(\rho\langle p_0\rangle\langle p_0{}^2\rangle)+(\rho\langle p_0\rangle)\frac{\partial v}{\partial x_0}=-\frac{\partial}{\partial x_3}[\rho\langle p_0\rangle\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]-\frac{1}{2}\frac{\partial}{\partial x_3}(\langle p_3\rangle\langle p_0{}^2\rangle)\,. \tag{4.9}$$

In a likewise manner substituting the results of Appendix A into Eqs. (4.7)–(4.8c) leads to the following set of reduced equations tabulated in the same order as the set above:

$$\frac{\partial}{\partial x_0}[\rho\langle p_0\rangle\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]+\frac{1}{2}\frac{\partial}{\partial x_0}(\rho\langle p_3\rangle\langle p_0{}^2\rangle)+\tfrac{1}{4}(\rho\langle p_3\rangle)\frac{\partial v}{\partial x_0}=-\frac{\partial}{\partial x_3}[\rho\langle p_3\rangle\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]-\frac{1}{2}\frac{\partial}{\partial x_3}(\rho\langle p_0\rangle\langle p_3{}^2\rangle)-\tfrac{1}{4}(\rho\langle p_0\rangle)\frac{\partial v}{\partial x_3}\,, \tag{4.9a}$$

$$\frac{\partial}{\partial x_0}[\rho\langle p_3\rangle\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]+\frac{1}{2}\frac{\partial}{\partial x_3}(\rho\langle p_0\rangle\langle p_3{}^2\rangle)=-\frac{\partial}{\partial x_3}[\rho\langle p_3\rangle\langle(p_3-\langle p_3\rangle)^2\rangle]-\frac{1}{2}\frac{\partial}{\partial x_3}(\rho\langle p_3\rangle\langle p_3{}^2\rangle)-(\rho\langle p_3\rangle)\frac{\partial v}{\partial x_3}\,, \tag{4.9b}$$

$$6\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle)^2\langle(p_0-\langle p_0\rangle)^2\rangle]+\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle)^4]+\tfrac{3}{2}\rho\langle p_0{}^2\rangle\frac{\partial v}{\partial x_0}-\frac{1}{8}\rho\frac{\partial^3 v}{\partial x_0{}^3}+3\frac{\partial}{\partial x_3}[\rho(\langle p_0\rangle)^2\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]+3\frac{\partial}{\partial x_3}[\rho\langle p_0p_3\rangle\langle(p_0-\langle p_0\rangle)^2\rangle]+\frac{\partial}{\partial x_3}[\rho\langle p_3\rangle(\langle p_0\rangle)^3]=0\,, \tag{4.10}$$

$$3\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle)^2\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]+3\frac{\partial}{\partial x_0}[\rho\langle p_0p_3\rangle\langle(p_0-\langle p_0\rangle)^2\rangle]+\frac{\partial}{\partial x_0}[\rho\langle p_3\rangle(\langle p_0\rangle)^3]$$
$$+\rho\langle p_0p_3\rangle\frac{\partial v}{\partial x_3}+4\frac{\partial}{\partial x_3}[\rho\langle p_0p_3\rangle\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]+\frac{\partial}{\partial x_3}[\rho(\langle p_0\rangle)^2\langle(p_3-\langle p_3\rangle)^2\rangle]$$
$$+\frac{\partial}{\partial x_3}[\rho(\langle p_3\rangle)^2\langle(p_0-\langle p_0\rangle)^2\rangle]+\frac{\partial}{\partial x_3}[\rho(\langle p_0\rangle)^2(\langle p_3\rangle)^2]+\tfrac{1}{2}\rho\langle p_0{}^2\rangle\frac{\partial v}{\partial x_3}-\frac{1}{8}\rho\frac{\partial^3 v}{\partial x_0{}^2\partial x_3}=0\ , \quad (4.10a)$$

$$4\frac{\partial}{\partial x_0}[\rho\langle p_0p_3\rangle\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]+\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle)^2\langle(p_3-\langle p_3\rangle)^2\rangle]+\frac{\partial}{\partial x_0}[\rho(\langle p_3\rangle)^2\langle(p_0-\langle p_0\rangle)^2\rangle]$$
$$+\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle)^2(\langle p_3\rangle)^2]+\tfrac{1}{2}\rho\langle p_3{}^2\rangle\frac{\partial^3 v}{\partial x_0{}^3}+3\frac{\partial}{\partial x_3}[\rho(\langle p_3\rangle)^2\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]$$
$$+3\frac{\partial}{\partial x_3}[\rho\langle p_0p_3\rangle\langle(p_3-\langle p_3\rangle)^2\rangle]+\frac{\partial}{\partial x_3}[\rho\langle p_0\rangle(\langle p_3\rangle)^3]+\rho\langle p_0p_3\rangle\frac{\partial v}{\partial x_3}-\frac{1}{8}\rho\frac{\partial^3 v}{\partial x_0\partial x_3{}^2}=0\ , \quad (4.10b)$$

$$3\frac{\partial}{\partial x_0}[\rho(\langle p_3\rangle)^2\langle(p_0-\langle p_0\rangle)(p_3-\langle p_3\rangle)\rangle]+3\frac{\partial}{\partial x_0}[\rho\langle p_0p_3\rangle\langle(p_3-\langle p_3\rangle)^2\rangle]+\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle\langle p_3\rangle)^3]$$
$$+6\frac{\partial}{\partial x_3}[\rho(\langle p_3\rangle)^2\langle(p_3-\langle p_3\rangle)^2\rangle]+\frac{\partial}{\partial x_3}(\rho\langle p_3\rangle)^4+\tfrac{3}{2}\rho\langle p_3{}^2\rangle\frac{\partial v}{\partial x_3}-\frac{1}{8}\rho\frac{\partial^3 v}{\partial x_3{}^3}=0\ . \quad (4.10c)$$

Equations (4.5)—(4.6a) and (4.9)—(4.10c) represent the closed set of transport equations for the charged-scalar plasma up to the lowest order of quantum correction. The next set of equations analogous to $n=4$ in Eq. (2.7) gives identically $0=0$. The closed set of equations given here are complicated but suggest the important result that expansion (4.2) and the corresponding cutting condition (4.4) are the correct relativistic generalization of the technique introduced for the nonrelativistic N-body problem. The truncation scheme developed here allows for a nonequilibrium approximation for the statistical mechanics of relativistic matter. Comparing the set of equations above with the nonrelativistic equations of Ref. 10 reveals the role played by the time variables x_0. Although this variable results in more terms in the dynamical equations there is still a similarity in form with the nonrelativistic dynamical equations for the Hartree problem. Equations (4.5)—(4.6a) and (4.9)—(4.9b) represent dynamical equations for the variables $\rho,\langle p_0\rangle,\langle p_3\rangle$, $\langle p_0{}^2\rangle,\langle p_0p_3\rangle$, $\langle p_3{}^2\rangle$. The function v defined by Eq. (3.9) for the charged scalar plasma is in general related to ρ. In analogy to the nonrelativistic problem Eqs. (4.10)—(4.10c) are constraint equations that relate the function v with the moments for this order of truncation. These equations are not to be utilized directly in solving Eqs. (4.5)—(4.9b), but rather represent relations which can be used to test the numerical accuracy of various calculated moments at this order of truncation. If these relations are badly violated it may be necessary to go to higher order in the truncation scheme and use the constraint equations valid at this higher order to test overall accuracy.

In Appendix B the transport equations for this plasma in the classical limit are given where this limit corresponds to $\langle(p-\langle p\rangle)^n\rangle=0$ for all n. In addition the cutting and isolation of the moment equations is shown explicitly for this limit.

B. Neutral-scalar particles

For the strictly neutral plasma all odd moments are zero and Eq. (4.3) defines the density-matrix expansion. On putting all odd moments equal to zero in Eqs. (4.5)—(4.8c) we find the transport equations valid to the same order as the charged plasma to be given by

$$\langle p_3{}^2\rangle\frac{\partial}{\partial x_0}(\rho\langle p_0{}^2\rangle)=-\frac{\rho}{8}\frac{\partial^3 v}{\partial x_0\partial x_3{}^2}\ , \quad (4.11)$$

$$\langle p_0{}^2\rangle\frac{\partial}{\partial x_3}(\rho\langle p_3{}^2\rangle)=-\frac{\rho}{8}\frac{\partial^3 v}{\partial x_0{}^2\partial x_3}\ , \quad (4.11a)$$

$$3\langle p_0{}^2\rangle\frac{\partial^3 v}{\partial x_0\partial x_3{}^2}=\langle p_3{}^2\rangle\frac{\partial^3 v}{\partial x_0{}^3}\ , \quad (4.12)$$

$$3\langle p_3{}^2\rangle\frac{\partial^3 v}{\partial x_0{}^2\partial x_3}=\langle p_0{}^2\rangle\frac{\partial^3 v}{\partial x_3{}^3}\ . \quad (4.12a)$$

Because $\langle p_0\rangle=\langle p_3\rangle=0$ there is no continuity equation for this plasma. Once again Eqs. (4.12) and (4.12a) represent constraint conditions between v and the moments of interest. The simplifying moment condition suggests the resulting transport equations can be solved numerically.

The formula for particle production in a neutral-scalar plasma has been worked out by Carruthers and Zachariasen. The inclusive differential cross section is generally given by

$$2\omega\frac{dN}{d^3p}=\frac{1}{(2\pi)^3}\left[(p^2-\mu^2)\int d^4R\,f(R,p)\right]_{p^2=m^2}\ , \quad (4.13)$$

where ω is a normalization factor. The relativistic generalization of Eq. (2.10) or (2.12) shows the particle production mechanism is directly dependent on the zeroth moment only, i.e., the diagonal density. Of course this moment is coupled to higher moments through the closed set of transport equations.

V. DISCUSSION

In this paper we have introduced a nonequilibrium truncation scheme for relativistic matter and applied our truncation scheme to both scalar and spinor plasmas. For the charged-scalar and neutral-plasma problem the closed set of dynamical equations valid to lowest quantum order were explicitly derived. For the spinor problem it was shown that decomposing the Wigner function into 16 components led to a set of dynamical equations for these components similar in form to those encountered in the scalar or nonrelativistic formalism. In principle these equations could be cut using an equation like (4.2); however, the definition of the current (3.14) together with the close connection between the first moment $\langle p(x)\rangle$ and the current [i.e., Eq. (2.5)] suggests a slight reinterpretation of expansion (4.2) will be necessary for spinor particles, i.e., each spinor component has its own density-matrix expansion.

The ability to cut the infinite set of moment equations using (2.8) and form a closed self-consistent set of dynamical relations for the diagonal density and momentum fluctuations strongly suggests that Eq. (2.8) reflects the underlying physical content of the infinite set of coupled moment equations. In particular it is possible to gain further understanding of Eq. (2.8) by noting from Eq. (2.5) that $\rho(x)\langle p(x)\rangle = j(x)$ where $j(x)$ is the current. Thus Eq. (2.5) can be thought of as an expansion around the current density that respects the overall current continuity equation for the problem of interest. This was shown in detail for the neutral-scalar plasma where no continuity equation should exist and this was found to be the case for our truncation scheme. The problems tackled here centered on the Hartree limit or collisionless plasma and thus were limited to the one-body Wigner function. We are currently investigating the neutral-scalar plasma including many-body dynamics as explicit two-point Wigner functions.[16,17]

Other applications of our truncation scheme under investigation include the transport equations for quarks with color degrees of freedom. For this problem the nature of the gauge condition leads to an additional equation for the generators of the color field that is coupled to the quark transport equations. Thus for positive-energy quarks there will be three dynamical equations to cut. In addition there is strong evidence[18] that the plasma is unstable with respect to small fluctuations of the glue field. This suggests that the classical or quantum mean-field limit may not be a good approximation. Hence inclusion of higher-order terms in the so-called Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy may also be necessary in describing the evolution of the QCD plasma.

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APPENDIX A

In this appendix we list the required relations between moments for the truncation order of interest in Sec. III. The subscripts i or j stand for either timelike subscript 0 or spacelike subscript 3:

$$\langle (p_i - \langle p_i\rangle)^3\rangle = 0\ ,\qquad \langle p_i^3\rangle = 3\langle p_i\rangle\langle p_i^2\rangle - 2(\langle p_i\rangle)^3\ , \tag{A1}$$

$$\langle (p_i-\langle p_i\rangle)^2(p_j-\langle p_j\rangle)\rangle = 0\ ,\qquad \langle p_i^2 p_j\rangle = 2\langle p_i p_i p_j\rangle + \langle p_j\rangle\langle p_i^2\rangle - 2\langle p_j\rangle(\langle p_i\rangle)^2\ , \tag{A2}$$

$$\langle (p_i-\langle p_i\rangle)^4\rangle = 0\ ,\qquad \langle p_i^4\rangle = 6(\langle p_i\rangle)^2\langle p_i^2\rangle - 5(\langle p_i\rangle)^4\ , \tag{A3}$$

$$\langle (p_i-\langle p_i\rangle)^3(p_j-\langle p_j\rangle)\rangle = 0\ ,\qquad \langle p_i^3 p_j\rangle = 3(\langle p_i\rangle)^2\langle p_i p_j\rangle + 3\langle p_i^2\rangle\langle p_i p_j\rangle - 5(\langle p_i\rangle)^3\langle p_j\rangle\ , \tag{A4}$$

$$\langle (p_i-\langle p_i\rangle)^2(p_j-\langle p_j\rangle)^2\rangle = 0\ ,\qquad \langle p_i^2 p_j^2\rangle = 4\langle p_i p_j p_i p_j\rangle + (\langle p_i\rangle)^2\langle p_j^2\rangle - 5(\langle p_i\rangle)^2(\langle p_j\rangle)^2 + \langle p_i^2\rangle(\langle p_j\rangle)^2\ . \tag{A5}$$

APPENDIX B

The classical limits for the transport equations may be realized by substituting the identities between the moments that result from the relations $\langle (p_i-\langle p_i\rangle)^2\rangle = 0$ and $\langle (p_i-\langle p_j\rangle)(p_j-\langle p_j\rangle)\rangle = 0$. For this limit Eq. (4.5) remains unchanged but Eq. (4.6) becomes

$$\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle)^2] + \frac{1}{2}\rho\frac{\partial v}{\partial x_0} + \frac{\partial}{\partial x_0}(\rho\langle p_0 p_3\rangle) = 0$$

or

$$\frac{\partial}{\partial x_0}[(\langle p_0\rangle)^2 + v] = -2\langle p_3\rangle\frac{\partial\langle p_0\rangle}{\partial x_3}\ , \tag{B1}$$

where Eq. (4.5) has been used. Using symmetry arguments Eq. (4.6a) reduces to a similar form, i.e., subscript 0 and 3 are interchanged.

In this classical limit Eqs. (4.7), (4.7a), and (4.7b) yield $0=0$ identically and thus cut the infinite coupled set. This is realized by the additional conditions $\langle (p_i-\langle p_i\rangle)^n\rangle = 0$, where the consequence of this is given in Appendix A. Substituting the relations between moments in Eq. (4.7) gives

$$\frac{\partial}{\partial x_0}[\rho(\langle p_0\rangle)^3] + (\rho\langle p_0\rangle)\frac{\partial v}{\partial x_0} + \frac{\partial}{\partial x_3}[\rho\langle p_3\rangle(\langle p_0\rangle)^2] = 0\ . \tag{B2}$$

This may be easily reduced to $0=0$ using Eq. (B1). The symmetry between Eq. (4.7) and (4.7a)/(4.7b) obviously leads to the same result for these equations.

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