Does statistical mechanics equal one-loop quantum field theory?

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The "thermodynamic" partition function $Z^T(\beta)=\sum_n \exp(-\beta E_n)$ is compared to the Euclidean "quantum" path integral $Z^Q(\beta)=\int d[\phi]\exp(-S)$ over (anti)periodic fields $\phi(\tau+\beta)=\pm\phi(\tau)$. We assume (1) free spin-0 or spin-$\frac{1}{2}$ fields and (2) an ultrastatic spacetime. Our main result is that $Z^T(\beta)$ does not equal $Z^Q(\beta)$. Nevertheless, they are simply related: we prove that $\ln Z^Q(\beta)=\ln Z^T(\beta)+(A+B\ln\mu^2)\beta$. Thus, the logarithms of the two partition functions differ only by a term proportional to β. The constant A arises from vacuum energy and the constant B from the renormalization-scale (μ) dependence of Z^Q. We derive a simple formula for A and B in terms of the "energy" ζ function $\zeta^E(z)=\sum_k E_k{}^{-z}$. In particular we show that A and B are determined by the behavior of the energy ζ function near $z=-1$: for small ϵ, $\pm\zeta^E(-1+\epsilon)=\frac{1}{4}B\epsilon^{-1}+\frac{1}{2}A+O(\epsilon)$ (where the upper sign applies to bosons and the lower sign applies to fermions). We also give a high-temperature expansion of $Z(\beta)$ in terms of $\zeta^E(z)$. Finally we argue that Z^T and Z^Q are interchangeable in any situation where gravitational effects are unimportant. This is because adding a term linear in β to $\ln Z$ is equivalent to shifting all energies by a constant; but if gravity is neglected, then the physics only depends upon the difference between energies, which is unchanged.

I. INTRODUCTION

The most important quantity in statistical mechanics is the thermodynamic partition function of the canonical ensemble:[1]

$$Z^T(\beta)=\sum_n e^{-\beta E_n} . \tag{1.1}$$

The index n denotes all distinct (many-particle) states of the system and E_n their energy. The temperature is $1/k\beta$ and the superscript T means "thermodynamic."

The most important quantity in quantum field theory is the amplitude, which may be expressed as a path integral:[2]

$$Z=\int d[\phi]e^{-iS[\phi]} . \tag{1.2}$$

Here $d[\phi]$ is a measure of the space of fields ϕ, and $S[\phi]$ is the action of the field configuration $\phi(x)$. The amplitude (1.2) depends upon the boundary conditions. For example, in a scattering problem one sums (1.2) over all field configurations consistent with a fixed initial state ϕ_i at time t_i, and a fixed final state ϕ_f at time t_f.

Because (1.1) and (1.2) appear similar, they are often held to be "the same thing." This suspicion is borne out by the fact that many of the formal manipulations carried out with (1.1) and (1.2) are the same. For example, the "free energy" $F=-\beta^{-1}\ln Z^T$ is analogous to the "effective potential density" $V=-(\mathrm{vol})^{-1}\ln Z^Q$. More precisely, one defines the path integral (1.2) by its Euclidean continuation, and integrates over all Euclidean fields (anti)periodic in Euclidean time τ with period β (Ref. 3). Thus the quantum analog of (1.1) is

$$\begin{aligned} &Z^Q(\beta)=\int d[\phi]\exp[-S(\phi)] ,\\ &\phi(\tau)\pm\phi(\tau+\beta) . \end{aligned} \tag{1.3}$$

Here the superscript Q means "quantum," the upper sign is bosons, and the lower sign is fermions. We will see that although $Z^T(\beta)$ does not equal $Z^Q(\beta)$, they are closely related. This paper extends the results of Gibbons,[4] who used a similar formalism, but only established the connection between Z^Q and Z^T in the $\beta\to\infty$ limit. Related results have been obtained by Dowker,[5,6] and others.

One reason that Z^T and Z^Q cannot be the same is that (in general) Z^Q depends upon the choice of some regularization mass parameter μ. Another reason is that, as we will see, the quantum partition function includes a vacuum-energy contribution which is not present in the thermodynamic partition function.

The paper is in five sections. In Sec. II we introduce the energy ζ-function and show how it is related to Z^T by a Mellin transform. In Sec. III we define Z^Q by using ζ-function regularization, and obtain an expression for Z^Q in terms of the four-dimensional ζ function. In Sec. IV we establish the exact connection between Z^Q and Z^T. In Sec. V we invert the Mellin transform representation of Z^T and obtain a simple high-temperature approximation for it. This is followed by a short conclusion.

Throughout this paper, we treat the bosonic and fermionic cases simultaneously, and often label corresponding quantities by B and F, respectively, and the corresponding equations by a and b.

II. THE ENERGY ζ FUNCTION AND THE THERMODYNAMIC PARTITION FUNCTION

Consider the following two infinite products, for bosons and fermions, respectively:

$$Z_B^T(\beta)=\prod_k \left[\sum_{p=0}^{\infty} \exp(-\beta p E_k)\right]$$
$$=(1+e^{-\beta E_1}+e^{-2\beta E_1}+\cdots)$$
$$\times(1+e^{-\beta E_2}+e^{-2\beta E_2}+\cdots)\cdots , \quad (2.1a)$$

$$Z_F^T(\beta)=\prod_k \left[\sum_{p=0}^{1} \exp(-\beta p E_k)\right]$$
$$=(1+e^{-\beta E_1})(1+e^{-\beta E_2})\cdots . \quad (2.1b)$$

Here E_k are the energies of the *single*-particle states (or modes). If we assume that the fields are free (or noninteracting) then it is easy to see that these infinite products (2.1) are exactly the partition functions (1.1). For example, if one takes the first term from each expression in parentheses, then their product is $e^{-\beta E_g}=1$, where $E_g=0$ is the ground-state energy. This corresponds to the many-particle state with no quanta in any mode. Similarly, the product of the second term from each expression in parentheses corresponds to a many-particle state containing one quanta in every mode.

It is important to note that the thermodynamic partition functions (2.1) correspond to the trace of $\exp(-\beta{:}H{:})$ where $:H:$ is the *normal-ordered* Hamiltonian. If one did not normal order H then $Z_B^T(\beta)$ would be

$$\prod_k \left[\sum_{p=0}^{\infty} \exp[-\beta(p+\tfrac{1}{2})E_k]\right] \quad (2.2)$$

which is identically *zero* for all $\beta>0$. Similarly, if H was not normal ordered, the fermionic partition function $Z_F^T(\beta)$ would be *infinite* for all $\beta>0$. Thus we see that in the very definition of the thermodynamic partition function, it is necessary to "throw away" the vacuum energy "by hand." The consequences of this will become clear later.

If we sum the geometric series in (2.1a) we then obtain extremely simple and well-known expressions for Z^T:

$$Z_B^T(\beta)=\prod_k [1-\exp(-\beta E_k)]^{-1} , \quad (2.3a)$$

$$Z_F^T(\beta)=\prod_k [1+\exp(-\beta E_k)] . \quad (2.3b)$$

Another very useful expression for Z^T can be obtained in terms of the energy (or three-dimensional) ζ function. This is defined by

$$\zeta^E(z)=\sum_k {E_k}^{-z} \quad (2.4)$$

for $\mathrm{Re}(z)>3$, and by analytic continuation elsewhere. It is also convenient to define the energy kernel $Y(t)$:

$$Y(t)=\sum_k \exp(-tE_k) . \quad (2.5)$$

Note that although (2.5) appears similar to (1.1) they are *not* the same. Here E_k are the energies of the single-particle states and not the energies of the distinct multiparticle states.

One may easily prove that the energy ζ function and kernel are related by the Mellin transform

$$\Gamma(z)\zeta^E(z)=\int_0^{\infty} t^{z-1}Y(t)dt . \quad (2.6)$$

One can also see that, by expanding $\ln(1-x)=-x-\frac{1}{2}x^2-\cdots$,

$$\ln Z_B^T(\beta)=-\sum_k \ln(1-e^{-\beta E_k})=\sum_{n=1}^{\infty}\frac{1}{n}Y(\beta n) , \quad (2.7a)$$

$$\ln Z_F^T(\beta)=\sum_k \ln(1+e^{-\beta E_k})=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}Y(\beta n) . \quad (2.7b)$$

From the Mellin transform (2.6) one can see that

$$n^{-z}\Gamma(z)\zeta^E(z)=\int_0^{\infty}\beta^{z-1}Y(n\beta)d\beta . \quad (2.8)$$

Combining (2.7) and (2.8) and summing over n one finally obtains the Mellin transform of Z^T as

$$\zeta^E(z)\zeta(z+1)\Gamma(z)=\int_0^{\infty}\beta^{z-1}\ln Z_B^T(\beta)d\beta , \quad (2.9a)$$

$$(1-2^{-z})\zeta^E(z)\zeta(z+1)\Gamma(z)=\int_0^{\infty}\beta^{z-1}\ln Z_F^T(\beta)d\beta , \quad (2.9b)$$

where $\zeta(z)=\sum_{n=1}^{\infty}n^{-z}$ is the Riemann ζ function. These formulas (2.9) will become very useful later.

III. THE QUANTUM PARTITION FUNCTION

We assume that the spacetime has an "ultrastatic" metric of the form

$$ds^2=-dt^2+{}^3g_{ab}dx^a dx^b , \quad (3.1)$$

with a time-independent three-metric ${}^3g_{ab}$. With this metric we can obtain a real Euclidean section by the Wick rotation $t\to i\tau$. The scalar wave equation and the (squared) Dirac equation take the form[4]

$$S\phi(t,x)=\left[-\frac{\partial^2}{\partial t^2}+H^2\right]\phi(t,x)=0 , \quad (3.2)$$

where the Hamiltonian $H^2=-{}^3\Box+M^2+\xi R$ does not depend on time. From the three-dimensional energy eigenfunctions and eigenvalues,

$$H^2\psi_k(x)={E_k}^2\psi_k(x) , \quad (3.3)$$

it is easy to construct the four-dimensional eigenfunctions $\psi_k(x)\exp(imt)$ of S.

The simplest way to determine the quantum partition function (1.3) is with ζ-function regularization.[7] (This is equivalent to other well-known methods, for example, dimensional regularization.) Because we are considering free fields, the one-loop approximation is exact, and Z can be expressed as a functional determinant. This functional determinant, which is formally infinite, can in turn be defined by a ζ function, yielding[7,8]

$$\ln Z_B^Q(\beta)=\tfrac{1}{2}\zeta_B'(0,\beta)+\tfrac{1}{2}\zeta_B(0,\beta)\ln(\mu^2) , \quad (3.4a)$$

$$\ln Z_F^Q(\beta)=-\tfrac{1}{2}\zeta_F'(0,\beta)-\tfrac{1}{2}\zeta_F(0,\beta)\ln(\mu^2) . \quad (3.4b)$$

Here a prime denotes d/dz. The four-dimensional ζ functions $\zeta_B(z,\beta)$ and $\zeta_F(z,\beta)$ are treated in some detail in Refs. 5–9. They are the sum of eigenvalues $\sum_n \lambda_n{}^{-z}$ corresponding to eigenfunctions of S which are, respectively, periodic or antiperiodic with Euclidean period β (Ref. 4):

$$\zeta_B(z,\beta)=\sum_k \sum_{m=-\infty}^{\infty} \left[E_k{}^2+\left[\frac{2\pi m}{\beta}\right]^2\right]^{-z}, \tag{3.5a}$$

$$\zeta_F(z,\beta)=\sum_k \sum_{m=-\infty}^{\infty} \left[E_k{}^2+\left[\frac{2\pi m+\pi}{\beta}\right]^2\right]^{-z}. \tag{3.5b}$$

These sums converge for $\mathrm{Re}(z)>3$, and the functions are defined on the rest of the complex-z plane by analytic continuation. The mass μ is a regularization mass which must be introduced to define the functional measure in the path integral (1.3) (Ref. 7).

If we introduce the kernel $Q(t)$ associated with the squared energies [compare to (2.5)]

$$Q(t)=\sum_k \exp(-tE_k{}^2) \tag{3.6}$$

and the θ functions[10]

$$\theta_B(t)=\sum_{m=-\infty}^{\infty} \exp[-(2\pi)^2 m^2 t]\ , \tag{3.7a}$$

$$\theta_F(t)=\sum_{m=-\infty}^{\infty} \exp[-(2\pi)^2 (m+\tfrac{1}{2})^2 t]$$

$$=\theta_B(t/4)-\theta_B(t)\ . \tag{3.7b}$$

Then, from (3.5),

$$\zeta_{B(F)}(z,\beta)\Gamma(z)=\int_0^\infty t^{z-1}Q(t)\theta_{B(F)}(\beta^{-2}t)dt\ . \tag{3.8}$$

The Mellin transform of the kernel $Q(t)$ is the energy ζ-function $\zeta_E(2z)$:

$$\zeta^E(2z)\Gamma(z)=\int_0^\infty t^{z-1}Q(t)dt\ . \tag{3.9}$$

In order to establish the relationship between Z^T and Z^Q it will be necessary to exchange the order of two integrals. To do this, we need to first define a "modified" four-dimensional ζ function.

The modified ζ functions are defined by

$$\bar\zeta_{B(F)}(z,\beta)\Gamma(z)=\int_0^\infty t^{z-1}Q(t)[\theta_{B(F)}(\beta^{-2}t) - \beta(4\pi t)^{-1/2}]dt \tag{3.10}$$

and from (3.8) and (3.9) they are more simply expressed in terms of the original four-dimensional ζ functions as

$$\bar\zeta_{B(F)}(z,\beta)\Gamma(z)=\zeta_{B(F)}(z,\beta)\Gamma(z) -(4\pi)^{-1/2}\beta\zeta^E(2z-1)\Gamma(z-\tfrac{1}{2})\ . \tag{3.11}$$

The second term in (3.11) will eventually turn out to be the vacuum-energy contribution to Z^Q.

Consider the Mellin transforms of $\bar\zeta$ with respect to β:

$$I_{B(F)}(z,s)=\int_0^\infty \beta^{s-1}\bar\zeta_{B(F)}(z,\beta)d\beta\ . \tag{3.12}$$

Because of the remarkable unimodular transformation property of the θ function[10,11]

$$\theta_B(t)=(4\pi t)^{-1/2}\theta_B((16\pi^2 t)^{-1}) \tag{3.13}$$

we can transform the integrand of (3.10) and substituting it into (3.12) obtain

$$I_B(z,s)\Gamma(z)=\int_0^\infty \beta^{s-1}\left\{\int_0^\infty t^{z-1}Q(t)\beta(4\pi t)^{-1/2}\left[\theta_B\left[\frac{\beta^2}{16\pi^2 t}\right]-1\right]dt\right\}d\beta\ , \tag{3.14a}$$

$$I_F(z,s)\Gamma(z)=\int_0^\infty \beta^{s-1}\left\{\int_0^\infty t^{z-1}Q(t)\beta(4\pi t)^{-1/2}\left[2\theta_B\left[\frac{\beta^2}{4\pi^2 t}\right]-\theta_B\left[\frac{\beta^2}{16\pi^2 t}\right]-1\right]dt\right\}d\beta\ . \tag{3.14b}$$

The integrands in (3.14) fall off as $\sim e^{-\beta^2/\pi t}$ as $\beta\to\infty$, so one can exchange the order of integration in (3.14) and obtain

$$I_B(z,s)\Gamma(z)=2^s\pi^{-1/2}\zeta(s+1)\Gamma\left[\frac{s+1}{2}\right] \times\int_0^\infty t^{z+s/2-1}Q(t)dt\ , \tag{3.15a}$$

$$I_F(z,s)\Gamma(z)=(1-2^s)\pi^{-1/2}\zeta(s+1)\Gamma\left[\frac{s+1}{2}\right] \times\int_0^\infty t^{z+s/2-1}Q(t)dt\ . \tag{3.15b}$$

We have also used

$$\int_0^\infty t^{z-1}\left[\theta_B\left[\frac{t}{4\pi^2}\right]-1\right]dt=2\zeta(2z)\Gamma(z) \tag{3.16}$$

in obtaining (3.15). Now using (3.9) we obtain the following useful closed forms from (3.15):

$$I_B(z,s)\Gamma(z)=\pi^{-1/2}2^s\Gamma\left[\frac{s+1}{2}\right]\Gamma\left[z+\frac{s}{2}\right] \times\zeta(s+1)\zeta^E(s+2z)\ , \tag{3.17a}$$

$$I_F(z,s)\Gamma(z)=\pi^{-1/2}(1-2^s)\Gamma\left[\frac{s+1}{2}\right]\Gamma\left[z+\frac{s}{2}\right] \times\zeta(s+1)\zeta^E(s+2z)\ . \tag{3.17b}$$

We can use these expressions to relate Z^Q and Z^T.

IV. COMPARING THE THERMODYNAMIC AND QUANTUM PARTITION FUNCTIONS

The quantum partition function Z^Q has been defined (3.4) in terms of the four-dimensional ζ function. We thus require $\zeta(0,\beta)$ and $\zeta'(0,\beta)$. We can obtain them from (3.11) in terms of $\bar{\zeta}(z,\beta)$ and $\zeta^E(z)$.

The integral that appears in (3.10) is finite for all values of z. This is because the integrand vanishes sufficiently fast as $t\to 0^+$ and as $t\to+\infty$ [as $t^{z-1}\exp(-\beta^2/t)$ and $t^{z-1}\exp(-E_0{}^2t)$, respectively]. Since $\Gamma(z)\sim 1/z$ as $z\to 0$ this implies that

$$\bar{\zeta}(0,\beta)=0\ . \tag{4.1}$$

We also need to know something about the behavior of the energy ζ function near $z=-1$. If the spatial sections Σ are compact then $\zeta^E(z)$ has only simple poles as its possible singularities.[9] Therefore near $z=-1$ it is of the form

$$\zeta^E(z)=\frac{R_{-1}}{z+1}+C+O(z+1)\ . \tag{4.2}$$

If $\zeta^E(z)$ is regular at $z=-1$ then the residue R_{-1} vanishes, and $C=\zeta^E(-1)$.

We can now obtain the desired expressions for $\zeta(0,\beta)$ and $\zeta'(0,\beta)$. From (3.11), (4.1), and (4.2) we see that

$$\zeta_{B(F)}(0,\beta)=-\tfrac{1}{2}\beta R_{-1}\ , \tag{4.3}$$

$$\zeta'_{B(F)}(0,\beta)=\bar{\zeta}'_{B(F)}(0,\beta)-\beta C\ . \tag{4.4}$$

Now, from (3.12) and (3.17),

$$\int_0^\infty \beta^{s-1}\bar{\zeta}'_B(0,\beta)d\beta=2\zeta(s+1)\Gamma(s)\zeta^E(s)\ , \tag{4.5a}$$

$$\int_0^\infty \beta^{s-1}\bar{\zeta}'_F(0,\beta)d\beta=(2^{1-s}-2)\zeta(s+1)\Gamma(s)\zeta^E(s)\ , \tag{4.5b}$$

where we have used $\Gamma(s/2)\Gamma(s/2+\frac{1}{2})=\pi^{1/2}2^{1-s}\Gamma(s)$. The right-hand side of (4.5) is identical to the Mellin transform of $\ln Z^T(\beta)$ in (2.9). If the Mellin transforms of two functions are equal, then those functions are equal (except possibly on a set of measure zero).[12] Consequently from (4.3) and (4.4) and the definition of Z^Q in (3.1) we obtain

$$\ln Z_B^Q(\beta)=\ln Z_B^T(\beta)-\tfrac{1}{2}\beta(C+\tfrac{1}{2}R_{-1}\ln\mu^2)\ , \tag{4.6a}$$

$$\ln Z_F^Q(\beta)=\ln Z_F^T(\beta)+\tfrac{1}{2}\beta(C+\tfrac{1}{2}R_{-1}\ln\mu^2)\ . \tag{4.6b}$$

This completes the proof.

We can now distinguish two possible cases.

Case 1. $\zeta^E(z)$ is regular at $z=-1$. In this case $R_{-1}=0$ and $C=\zeta^E(-1)$. The quantum partition function is therefore independent of the renormalization scale μ. The only remaining difference between Z^Q and Z^T is the vacuum-energy term which, as pointed out in Sec. II, was "dropped" in defining Z^T:

$$\ln Z^Q_{B(F)}=\ln Z^T_{B(F)}\mp\tfrac{1}{2}\beta\zeta^E(-1)\ . \tag{4.7}$$

Here and elsewhere the upper sign refers to bosons and the lower sign to fermions.

Case 2. $\zeta^E(z)$ has a pole at $z=-1$. In this case the quantum partition function depends upon the renormalization scale μ, and the "residual vacuum energy" is scale dependent. It is

$$\pm\tfrac{1}{2}(C+\tfrac{1}{2}R_{-1}\ln\mu^2)\ . \tag{4.8}$$

This vacuum energy can also be written from (4.2) as

$$\pm\frac{1}{2}\left[\frac{d}{dz}+\frac{1}{2}\ln\mu^2\right]z\zeta^E(z-1)\Bigg|_{z=0}\ . \tag{4.9}$$

This formula applies in both case 1 and case 2.

V. HIGH-TEMPERATURE EXPANSION OF THE THERMODYNAMIC PARTITION FUNCTION

We can obtain a useful high-temperature approximation to $\ln Z^T(\beta)$ by inverting its Mellin transform. This yields an expression in terms of the energy ζ function. To approximate the inverse Mellin transform (a contour integral) one shifts its contour of integration to the left past several poles of the integrand and uses Cauchy's theorem. At high temperatures the dominant contributions come from the residues of the poles, and the remaining part of the integral can be neglected.

To invert the Mellin transform of $\ln Z^T(\beta)$, choose d a real constant $d>3$. Then inverting (2.9) one obtains

$$\ln Z_B^T(\beta)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\beta^{-s}\zeta(s+1)\Gamma(s)\zeta^E(s)ds\ , \tag{5.1a}$$

$$\ln Z_F^T(\beta)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\beta^{-s}(1-2^{-s})\zeta(s+1)\Gamma(s)\zeta^E(s)ds\ . \tag{5.1b}$$

In general the integrand of (5.1) has poles at $s=3,2,1,\ldots$. If we deform the contour of integration past these poles and apply Cauchy's theorem, one obtains the following high-temperature $T\to\infty$ or $\beta\to 0^+$ expansions:

$$\ln Z_B^T(\beta)=\frac{\pi^4}{45}R_3\beta^{-3}+\zeta(3)R_2\beta^{-2}+\frac{\pi^2}{6}R_1\beta^{-1}-\zeta^E(0)\ln\beta+\zeta^{E'}(0)+O(\beta\ln\beta)\ , \tag{5.2a}$$

$$\ln Z_F^T(\beta)=\left[\frac{7}{8}\right]\frac{\pi^4}{45}R_3\beta^{-3}+(\tfrac{3}{4})\zeta(3)R_2\beta^{-2}+\left[\frac{1}{2}\right]\frac{\pi^2}{6}R_1\beta^{-1}+\zeta^E(0)\ln 2+O(\beta\ln\beta)\ . \tag{5.2b}$$

Here R_k refers to the residues of $\zeta^E(z)$ at $z=k$, i.e.,

$$\zeta^E(z)=\frac{R_3}{z-3}+\frac{R_2}{z-2}+\frac{R_1}{z-1}+f(z)\ , \tag{5.3}$$

where $f(z)$ is regular at $z=1$, 2, and 3. We have also assumed that $\zeta^E(z)$ is regular at $z=0$ (Ref. 9). It is possible to express the residues as spacetime integrals of local quantities involving the curvature and mass. For example, R_3 is proportional to the volume of the spatial section Σ (Ref. 9). Thus the leading β^{-3} term has the appearance (vol) T^3 associated with a gas of massless radiation.

VI. SUMMARY AND DISCUSSION

We have shown that

$$\ln Z^{Q}(\beta)-\ln Z^{T}(\beta)=\mp\frac{1}{2}\beta\left[\frac{d}{dz}+\tfrac{1}{2}\ln\mu^{2}\right]z\zeta^{E}(z-1)\Big|_{z=0},$$

where Z^Q and Z^T are the quantum and thermodynamic partition functions and $\zeta^E(z)$ is the energy ζ function. This means that if $\zeta^E(z)$ is regular at $z=-1$ then Z^Q is independent of the renormalization mass μ^2 and is dimensionless. In that case, $\pm\frac{1}{2}\zeta^E(-1)$ can be interpreted as a vacuum-energy contribution. If $\zeta^E(z)$ has a pole at $z=-1$, then Z^Q depends upon the renormalization scale μ and has an "anomalous dimension."

One can ask if the difference between Z^T and Z^Q means that one would obtain different physical predictions from them. This is not the case. The reason is that the addition to $\ln Z(\beta)$ of a term proportional to β simply shifts the definition of energy $E=-\partial\ln Z/\partial\beta$ and free energy $F=-\beta^{-1}\ln Z$ by a constant, and leaves the entropy unchanged. This does not affect thermodynamic processes, because they only involve a *change* in the energy, and not the overall value of the energy. However, it does affect the value of the renormalized cosmological constant in gravity. In that instance it seems more correct to use Z^Q.

Finally, we have obtained a high-temperature expansion of $\ln Z^T(\beta)$. That expansion shows that for $n=1,2,3$ the term in $\ln Z^T(\beta)$ proportional to β^{-n} is also proportional to the residue of $\zeta^E(z)$ at $z=n$.

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