Cosmic strings as random walks

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We present a random-walk model for the formation of cosmic strings in the early Universe. Analytic results are given for the scaling of string length with straight-line end-to-end separation, the length distribution of closed loops, and the fraction of total string length in infinite strings. We explain why the string network has the statistical properties of a set of Brownian, rather than self-avoiding, random walks, even in models in which string intersections in the initial configuration are impossible. It is found that the number of closed loops per unit volume with length between l and $l+dl$ is $dn = C\xi^{-3/2}l^{-5/2}dl$, while the scaling of the straight-line distance R between two points on a string separated by a length l is $R \sim (l\xi)^{1/2}$. The fraction of string length in infinite strings is $\gtrsim \frac{2}{3}-\frac{3}{4}$. These results are in reasonable agreement with previous Monte Carlo simulations and are confirmed by our own computer simulations in which string intersections in the initial configuration do not occur. The prevalence of infinite strings is a natural feature in the random-walk model.

I. INTRODUCTION

Strings are topological defects which can form when a local or global symmetry is spontaneously broken in a phase transition.[1] When a symmetry group G is broken to a subgroup H, strings form if the fundamental group $\pi_1(G/H)$ (loosely, the group of mappings of the manifold of degenerate vacua onto the circle S^1) is nontrivial, i.e., if the coset space G/H is multiply connected. They are of cosmological interest because the strings formed in the breakdown of a class of grand unified theories may act as seeds for galaxy formation.[2] In the simplest toy gauge theory with strings, the Abelian Higgs model, a string is present if $\theta(x)$, the phase of the Higgs-field expectation value, changes by 2π as one traces out a closed contour in space. This can happen because, when the phase transition occurs in the early Universe, θ is uncorrelated over scales larger than some characteristic length ξ, where ξ is less than or on the order of the horizon size ct at the time of the phase transition.

Vachaspati and Vilenkin and Albrecht and Turok[3] have developed computer simulations of the formation of cosmic strings. They divide the Universe into cubic cells of size ξ and assign a random value to θ from the set $0,2\pi/3,4\pi/3$ at each vertex. A string is assumed to pass through the face of a cell when the values of θ on the vertices surrounding that face run sequentially around the face. Using a Monte Carlo simulation with this model, the length distribution of closed loops is found to be[3]

$$dn = Cl^{-2.6\pm0.1}dl \ , \tag{1}$$

where dn is the number of closed loops per unit volume with length between l and $l+dl$, and C is a constant. (This length distribution can be derived by assuming that the initial string configuration is scale invariant.[3,4]) For any two points along a string, the distance between the points measured along the string, l, scales with the straight-line separation between the points, R, as

$$l = (0.97\pm0.03)R^{2.00\pm0.07}/\xi \ . \tag{2}$$

Vachaspati and Vilenkin also find that only $\sim 20\%$ of the total length of the strings is in closed loops, and they suggest that this result is unexpected for scale-invariant strings.

We show that these results can be derived in a straightforward way using a random-walk model for string formation. This model is particularly fruitful because random walks have been investigated in detail.[5-7] In Sec. II we discuss the statistical properties of the initial string configuration produced by the phase transition, including the scaling of string length with straight-line end-to-end distance and the length distribution of closed loops. We show that, in a physically reasonable model, the strings form a set of self-avoiding random walks, but the string network has the statistical properties of a set of Brownian walks. In Sec. III we estimate the fraction of string length in infinite strings, and our conclusions are summarized briefly in Sec. IV.

II. STATISTICAL PROPERTIES OF THE INITIAL STRING CONFIGURATION

Assume that an initial string configuration has been set down on an arbitrary lattice as described in Ref. 3, and consider an observer located at an arbitrary point on the string, which we will define as the origin of the lattice. We assume that the Higgs field is assigned to the cells of the lattice, and strings form along the lattice edges; this is opposite to the convention of Ref. 3. As the observer traces the string from the origin along the lattice, the string will perform a random walk along the edges of the cells, with the restriction that the string is not allowed to recross a previously traced edge of the lattice. (Although the simulations of Ref. 3 allow for self-intersections in the initial string configuration, this problem does not occur on a more physically reasonable lattice; see the next paragraph.) Placing the Higgs field at random on the cells of a lattice will lead to correlations in the direction of the string over scales on the order of the cell size. Over large scales, however, these short-range correlations are unimportant, and we expect the string to perform a random walk in space. For random-walk lengths much greater than the step size, the statistical properties of the random walk are independent of the shape of the lattice on which the random walk is conducted. (In this model, the step size corresponds to the Higgs-field correlation length ξ.) The string we are tracing can rejoin its other end only at the origin, so any strings which do not return to the origin are infinite; if a string does return to the origin, then we assume that it forms a closed loop. It is important to note that this is a purely heuristic argument; the underlying physical reality is that strings form at boundaries of the Higgs-field domains. However, by recognizing that each string traces a random walk in space, one can derive analytically various properties of the initial string configuration.

In a realistic simulation for the formation of cosmic strings, one expects the strings to correspond to a network of self-avoiding random walks. (A self-avoiding random walk is a random walk which is never allowed to intersect itself except to close at the origin.) A "random" division of space into cells yields a lattice in which every edge is bounded by three cells, every vertex is bounded by four cells, and four edges meet at each vertex (see, for example, Ref. 8). Suppose that we divide space up into cells satisfying the above conditions, and assign to the Higgs field in each cell a random value θ between 0 and 2π. Then an edge contains a string segment if the value of the Higgs field changes by 2π as we pass sequentially through the three cells bounding the edge. In passing from one cell to the next, the Higgs field can rotate in two different directions in the field space; we assume that it changes in the direction which minimizes $|\Delta\theta|$. The direction of a string segment along an edge is determined by the direction in which the Higgs field rotates in field space. Then it is easy to show that at any vertex, the four edges meeting at the vertex must either contain exactly one ingoing and one outgoing string, or else be completely unoccupied. Thus, in the initial configuration produced by this model, the strings do not intersect themselves or each other, so each string corresponds to a self-avoiding random walk.

Self-avoiding random walks have been extensively studied in connection with polymer physics.[6] It is thought that the probability u_j for such a random walk to return to the origin (and therefore terminate) on the jth step scales as[6]

$$u_j \sim j^{-23/12} \tag{3}$$

for large j in three dimensions. To derive a value for dn, note that the probability of choosing a particular segment on a particular string is proportional to the length of the string, and the length of the string l will be proportional to the number of steps taken ($l = j\xi$). Thus, $dn \sim [(u_j/l)/\xi^3]dl$, or

$$dn = C\xi^{-13/12} l^{-35/12} dl \ , \tag{4}$$

where C is a dimensionless constant. The number density dn must also include a factor giving the probability that a given lattice edge will contain a string; this factor has been absorbed into the constant in Eq. (4), since it does not affect the scaling of dn with string length. A self-avoiding random walk also has a characteristic scaling of straight-line end-to-end distance with length. If R is the root-mean-square end-to-end distance of a self-avoiding random walk, and l is the length measured along the walk, then in three dimensions[6]

$$R \sim l^{3/5}\xi^{2/5} \ . \tag{5}$$

Although the computer simulations of Ref. 3 give results in disagreement with the self-avoiding random-walk model [Eqs. (4) and (5)], these simulations allow string intersections in the initial configuration, and it is possible that the Brownian spectrum obtained in these simulations is a result of the fact that such intersections are allowed. In order to test whether the absence of such intersections would lead to a configuration with the statistical properties of a set of self-avoiding random walks, we have performed a computer simulation in which such string intersections do not occur. We use a tetrakaidekahedral lattice,[9] in which every edge is bounded by three 14-sided cells, every vertex is bounded by four cells, and four edges meet at each vertex; our lattice size is $84\times84\times84$, in units in which the length of a lattice edge is $2^{1/2}$. The Higgs field is assigned a random value between 0 and 2π; this is taken to be a "continuous" random variable rather than a discrete variable as in Ref. 3. The determination of whether a string passes through a given edge is as described above, and there are no string intersections in the initial configuration.

Our numerical results, however, indicate that the network of strings in this case has the statistical properties of a set of Brownian random walks rather than self-avoiding walks. (A Brownian random walk is an "ordinary" random walk which is allowed to intersect itself.) In our simulation, the rms distance R for two points separated by a length l on an infinite string scales as

$$R^2 = A\xi^2(l/\xi)^n \ , \tag{6}$$

with $A = 2.10\pm0.05$, $n = 0.992\pm0.006$, over lengths from 10 to 200 steps, while the root-mean-square end-to-end distance for a Brownian random walk in three dimensions scales as

$$R \sim (l\xi)^{1/2} \ . \tag{7}$$

Thus, we have an apparent paradox: although the string configuration is not self-intersecting, it has the statistical properties of a set of Brownian, rather than self-avoiding, random walks. To see why this is so, consider a single self-avoiding random walk. Since the walk is not allowed to intersect itself, the walker experiences a repulsion away from the volume of space occupied by the walk. This gives a larger value of R for a given l ($R \sim l^{3/5}$) than is the case for an ordinary random walk ($R \sim l^{1/2}$). However, in the case of the initial string configuration, a single string, as we trace along its path, experiences a repulsion not only from the rest of the string, but from all of the segments belonging to other strings. By construction, the set of all string segments is distributed uniformly in space, so a given step of the random walk a sufficiently large distance from the origin will have no net bias in the step direction relative to the origin, as is the case for a Brownian walk. This effect has been known in polymer physics for some time.[10] A polymer in a dilute solution (i.e., one which interacts only with itself) will adopt a configuration corresponding to a self-avoiding random walk, while in a dense system of polymers, each one has the structure of a Brownian random walk. Thus, one would expect the scaling of the string size R with length l in the initial configuration of cosmic strings to correspond to a self-avoiding random walk on scales smaller than the mean separation between different strings, and to a Brownian walk on scales larger than this; in the cosmic string models discussed here, the mean separation between different strings is of the order of the step size ξ, so the statistical properties of the strings are Brownian on all scales.

The scaling given by Eq. (6) does not correspond exactly to a Brownian walk, because we would expect $A=1$ instead of $A=2.1$. This discrepancy is due to the local interaction of the random walk with itself: immediate reversals of the random walk are prohibited. If ϕ is the angle between successive steps in a random walk, and we allow an arbitrary distribution for ϕ, subject only to the constraint that the distribution for each step be azimuthally symmetric with respect to the previous step, then it is easy to show that A is a function only of $\overline{\cos\phi}$, the value of $\cos\phi$ averaged over the distribution for ϕ:

$$A=(1+\overline{\cos\phi})/(1-\overline{\cos\phi})\ . \tag{8}$$

For our lattice, $\overline{\cos\phi}=\frac{1}{3}$, so $A=2$, while a completely Brownian walk has $\overline{\cos\phi}=0$, $A=1$. In general, A is a lattice-dependent parameter, but we do not expect it to have any physical significance, as the value of A can always be absorbed into the definition of ξ.

Using a set of 20 runs of our numerical simulation, we find that the number density of closed loops of a given length l scales as

$$dn\sim l^{-2.44\pm0.04}dl\ , \tag{9}$$

over lengths from 10 to 50 steps. For a Brownian random walk, the probability u_j that the string returns to the origin on the jth step is, for large j (Ref. 11),

$$u_j\sim j^{-3/2} \tag{10}$$

for a lattice in three dimensions. Then our previous argument gives

$$dn=C\xi^{-3/2}l^{-5/2}dl\ , \tag{11}$$

in agreement with the computer simulations. [Equation (11) can also be derived by assuming that the initial string configuration is scale invariant.[3,4]] However, Eq. (10) applies to the set of all random walks of length j, including those which intersect themselves and have multiple returns to the origin. We assume that Eq. (10) can also be used for our set of nonintersecting random walks which terminate the first time they return to the origin; the basis for this assumption is the argument given above that the self-avoiding strings should display the statistical behavior of Brownian walks.

III. INFINITE STRINGS

The question of the fraction of string length contained in infinite strings arises naturally in random-walk models. It was first noted by Pólya[12] that a random walk in one or two dimensions will always return to the origin after an arbitrarily large number of steps, while a three-dimensional random walk has a nonzero probability of never returning to the origin. Let Π be the probability that a random walk on a given three-dimensional lattice ever returns to the origin. A method for calculating Π for ordinary random walks (i.e., random walks which are allowed to intersect themselves) is given in detail in Ref. 5; we will simply quote the results. For simple cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices, we have

$$\Pi=0.34\ \ (\text{sc})\ , \tag{12a}$$

$$\Pi=0.28\ \ (\text{bcc})\ , \tag{12b}$$

$$\Pi=0.26\ \ (\text{fcc})\ . \tag{12c}$$

Forbidding string intersections, particularly immediate reversals of the string, will reduce the return probability, so in our model in which string intersections do not occur in the initial configuration, Π will be smaller than the values quoted in Eq. (12). The probability Q that a given string segment will belong to a closed loop is

$$Q=\Pi\ , \tag{13}$$

and it is clear that Q gives the total length of string contained in closed loops. The physical significance of Eq. (13) is unclear, as Π is a function of the particular lattice under consideration. However, it does indicate that a large fraction ($\gtrsim\frac{2}{3}-\frac{3}{4}$) of the total length of strings should be in infinite strings rather than closed loops, which explains the results of Ref. 3; our own simulations on the tetrakaidekahedral lattice yield $\sim26\%$ of string length in closed loops. It is clear from Eq. (11) that the probability for a random walk to return to the origin is dominated by the large number of returns after a fairly small number of steps. Almost all of the string length in closed loops is in very small loops of length $l\sim\xi$, so the fraction of the total length of all strings in closed loops of length $l\gg\xi$ is effectively zero. This result is independent of both the type of random walk and the particular lattice under consideration. From this, one might naively expect the fraction of length in infinite strings to be negligible instead of $\sim\frac{2}{3}-\frac{3}{4}$. This paradox is resolved by realizing that infinite strings are not obtained as the limit of very large loops; on the contrary, as one goes to longer and longer walks, one is much more likely to be on an infinite walk than on a large loop.

IV. CONCLUDING REMARKS

Our heuristic treatment of the formation of cosmic strings as a random-walk problem gives results in good agreement with previous computer simulations and provides a natural explanation for the prevalence of infinite strings noted in such simulations. The initial string configuration has the statistical properties of a set of Brownian random walks even in initial configurations in which the strings are not allowed to intersect themselves or each other. This is due to the fact that in a uniformly distributed network of self-avoiding random walks, the repulsive "force" which a self-avoiding random walk exerts on itself is exactly canceled by the repulsive "forces" due to all of the other random walks. A similar effect arises in a dense solution of polymers, in which each polymer has the structure of a Brownian random walk.

John Preskill[13] has independently arrived at some of the results presented here.

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