Cosmic strings with cosmological constant

Qingjun Tian*
The University of Chicago, Enrico Fermi Institute, 5640 Ellis Avenue, Chicago, Illinois 60637
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All static and cylindrically symmetric vacuum solutions of Einstein's field equation with cosmological constant Λ are found. We use these solutions to represent the exterior metric of a cosmic string. If Λ is negative, the exterior approaches the anti–de Sitter metric away from the string. If Λ is positive, the exterior metric is either an $\mathbf{R}^3\times S^1$ universe with a curvature singularity, an $\mathbf{R}^2\times S^2$ universe with a conical singularity, or an $\mathbf{R}^2\times S^2$ universe with no singularity.

I. INTRODUCTION

In recent years, there has been much interest in cosmic strings. According to gauge theories with spontaneous symmetry breaking, the Universe may have undergone a number of phase transitions since the big bang. Cosmic strings, one of the topological structures produced in these phase transitions, possibly survive to the present day.[1] Strings could provide the fluctuations of the density of the Universe necessary to form galaxies,[2] and also could act as gravitational lenses.[3,4] The gravitational effects have been investigated by a number of authors,[5] and they have shown that the geometry outside a string with uniform energy density is conical with a deficit angle (compared with the flat space) of $8\pi\mu$, where μ is the linear mass density of the string. Garfinkle[6] has treated the string more generally as a self-interacting scalar field minimally coupled to a U(1) gauge field, and has shown that there exists a class of static, cylindrically symmetric solutions to the equations of those fields asymptotically approaching Minkowski space with the same deficit angle as above. In light of recent developments regarding the properties of the very early Universe, it is now believed that the cosmological constant Λ may have been in the past as large as 10^{120} times that of the present epoch.[7] Therefore, it is natural to ask whether or not the above "deficit angle" feature of the space-time metric of a cosmic string still exists. The aim of this work is to answer this question. We shall show that, with nonzero Λ, the stress-energy components $T_{\phi\phi}$ and T_{rr} cannot both vanish, as they can in the case $\Lambda=0$. Consequently, the exterior metric for a string may or may not, in a certain sense, be conical. If Λ is negative, the exterior, having the topology $\mathbf{R}^3\times S^1$, will approach the anti–de Sitter metric as one goes away from the string. If Λ is positive, the exterior must be either with the topology $\mathbf{R}^3\times S^1$ and a curvature singularity, or with the topology $\mathbf{R}^2\times S^2$ and either with or without a conical singularity.

In Sec. II we obtain a one-parameter family of solutions to the vacuum Einstein's field equation with cosmological constant. We shall, in Sec. III, show that, under some reasonable assumptions about the cosmic strings, the geometry of the string has a boost symmetry along its axis and that, in the presence of a cosmological constant, the components $T_{\phi\phi}$ and T_{rr} of the stress-energy tensor cannot both vanish. In the same section we discuss, by means of the O'Brien-Synge-Lichnerowicz jump condition, the geometrical properties of the spacetime of the string. In Sec. IV we illustrate these remarks with an example.

II. STATIC, CYLINDRICALLY SYMMETRIC SPACETIMES WITH Λ

A spacetime is said[8] to be static and cylindrically symmetric if it possesses three commuting Killing vector fields of which one, $(\partial/\partial t)^a$, is timelike, while the other two, $(\partial/\partial z)^a$ and $(\partial/\partial\phi)^a$, are spacelike such that any two are orthogonal to each other and each is hypersurface orthogonal. We can write such a spacetime metric, in terms of the t, z, and ϕ above, as

$$ds^2=-e^{A}dt^2+e^{B}dz^2+e^{C}d\phi^2+dr^2\ , \tag{1}$$

where r is the spatial distance in the direction orthogonal to the three Killing fields, and A, B, and C are arbitrary functions of r.

Since the Einstein tensor G_{ab} is diagonal in these coordinates, the stress-energy tensor T_{ab} must, from Einstein's equation, also be diagonal. That is, we have

$$T_{ab}=\epsilon_1 t_a t_b-\epsilon_2 z_a z_b-\sigma\phi_a\phi_b-\delta r_a r_b\ , \tag{2}$$

where ϵ_1, ϵ_2, σ, and δ are functions of r only, t^a, z^a, and ϕ^a are unit vector fields lying in the respective directions of the Killing fields $(\partial/\partial t)^a$, $(\partial/\partial z)^a$, and $(\partial/\partial\phi)^a$, and r^a is a unit vector field orthogonal to all three of these Killing fields.

Einstein's field equation (with $G=c=1$) with cosmological constant Λ now reduces to the following system of ordinary differential equations:

$$\tfrac{1}{2}(B''+C'')+\tfrac{1}{4}(B'^2+C'^2+B'C')=-(8\pi\epsilon_1+4\beta^2/3)\ , \tag{3a}$$

$$\tfrac{1}{2}(A''+C'')+\tfrac{1}{4}(A'^2+C'^2+A'C')=-(8\pi\epsilon_2+4\beta^2/3)\ , \tag{3b}$$

$$\tfrac{1}{2}(A''+B'')+\tfrac{1}{4}(A'^2+B'^2+A'B')=-(8\pi\sigma+4\beta^2/3) \tag{3c}$$

and

$$\tfrac{1}{4}(A'B'+B'C'+C'A')=-(8\pi\delta+4\beta^2/3)\;, \tag{3d}$$

where a prime means d/dr, and we have set $\beta=\sqrt{3\Lambda}/2$.

We first claim that a one-parameter family of static and cylindrically symmetric spacetime metrics in a vacuum with cosmological constant is given by[9]

$$ds^2=-[\tan\beta(r+\hat{r})]^{\gamma_1}[\sin 2\beta(r+\hat{r})]^{2/3}dt^2+[\tan\beta(r+\hat{r})]^{\gamma_2}[\sin 2\beta(r+\hat{r})]^{2/3}dz^2+[\tan\beta(r+\hat{r})]^{\gamma_3}[\sin 2\beta(r+\hat{r})]^{2/3}d\phi^2 +dr^2\;, \tag{4}$$

where $\hat{r}$ is an arbitrary real number, and γ_1, γ_2, and γ_3 are real numbers in the interval $[-\frac{4}{3},\frac{4}{3}]$, satisfying the algebraic equations

$$\gamma_1+\gamma_2+\gamma_3=0,\quad \gamma_1\gamma_2+\gamma_2\gamma_3+\gamma_3\gamma_1=-\tfrac{4}{3}\;. \tag{5}$$

To show this, first set the stress-energy components to zero in (3a)–(3d). Next, add (3a), (3b), and (3c), use (3d), and set $\Sigma=A+B+C$ to obtain $\Sigma''+\frac{1}{2}\Sigma'^2=-8\beta^2$. A two-parameter family of general solutions of this equation is $\Sigma=\Sigma_0+2\ln\sin 2\beta(r+\hat{r})$, where Σ_0 and $\hat{r}$ are arbitrary constants. Next, subtract (3a) from the sum of (3b) and (3c) and use (3d) to get $A''+\frac{1}{2}A'\Sigma'=-\frac{8}{3}\beta^2$. Solve this equation for A. Do the same for B and for C. These solutions result, on rescaling the coordinates, in the metric (4). The algebraic equations (5) are the result of substituting the expressions for A, B, and C into (3d). In Sec. III we shall use the metric (4) to find the exterior metric for a cosmic string.

We further claim that, for Λ negative, there is one additional solution—not within the general class (4)—with the property $A''=B''=C''=0$. This solution yields the anti–de Sitter metric[10] (i.e., a spacetime with constant curvature and negative scalar curvature):

$$ds^2=e^{\pm 2\kappa r}(-dt^2+dz^2+d\phi^2)+dr^2\;, \tag{6}$$

where we have set $\kappa=(|\Lambda|/3)^{1/2}$. We observe that the metric (4) with negative Λ—and so with β purely imaginary and the functions in (4) hyperbolic—approaches the metric (6) as $r\to\infty$ or $\hat{r}\to\infty$. We note in passing that, if Λ is positive, the metric (4) never approaches a spacetime time with constant curvature.[11]

Finally, we remark that the metric (4) is a generalization of the Levi-Civita vacuum spacetime. One sees this as follows. First replace dt^2, dz^2, and $d\phi^2$ by $(4\beta^2)^{-(\gamma_1+2/3)/2}dt^2$, $(4\beta^2)^{-(\gamma_2+2/3)/2}dz^2$, and $(4\beta^2)^{-(\gamma_3+2/3)/2}d\phi^2$, respectively. Then replace r in favor of $\rho=r^{-Q}$ where $Q=m^2-m+1$, $m\in[0,2]$. Finally, set constants

$$\gamma_1=\frac{2m}{Q}-\frac{2}{3},\quad \gamma_2=\frac{2m^2-2m}{Q}-\frac{2}{3}\;,$$

and

$$\gamma_3=\frac{2-2m}{Q}-\frac{2}{3}$$

and let $\beta\to 0$. The result of these steps is the Levi-Civita metric in the form given in Ref. 12.

III. THE GEOMETRY OF COSMIC STRINGS

In this section we shall explore the geometrical properties of the solution for a cosmic string. Let us first describe our coordinates for a string. The cosmic string is to be a static, cylindrically symmetric configuration with stress-energy tensor independent of time, positions on its axis, and orientations around its axis. It follows from these features that the spacetime for a string also has the symmetries above and can be described by our coordinates—the string's symmetry axis lying in the direction of the Killing field $(\partial/\partial z)^a$, the Killing field $(\partial/\partial\phi)^a$ having closed orbits around the axis, and the parameter r starting from the axis. We also identify $\phi=0$ and $\phi=2\pi$, so the range of ϕ is from 0 through 2π. The stress-energy tensor is still in the form of (2). We further assume $\epsilon_1=\epsilon_2$ in (2) and denote the common value by ϵ. (Indeed, this assumption can be derived from a new Lagrangian by adding a term of Λ into the Lagrangian in Ref. 6.)

We now show, using the same argument as that[6] for $\Lambda=0$, that the assumptions above still permit one, even with $\Lambda\neq 0$, to choose $A=B$ in the metric (1). This choice reflects that the geometry of the string must possess a further symmetry—a boost symmetry—along the axis. To show that this can be done, first subtract (3a) from (3b), and integrate the difference over $r\in[0,\infty)$ once to obtain $B'-A'=k\,e^{-(A+B+C)/2}$, where k is a constant. Next note that $e^{-C/2}\to\infty$ as $r\to 0$, since the orbits of $(\partial/\partial\phi)^a$ are closed. This implies that the norm of the Killing field $(\partial/\partial\phi)^a$ vanishes on the axis. But now, by the smoothness of the metric, it follows that the constant k must be zero and, therefore, that $B-A$ must be constant. Since the two Killing fields $(\partial/\partial t)^a$ and $(\partial/\partial g)^a$ can still be rescaled, one is permitted to choose $A=B$. We will use this choice afterward.

Set $U=e^{3A/4}$ and $V=e^{C/2+A/4}$, so U is strictly positive and V non-negative. The metric (1) then becomes

$$ds^2=U^{4/3}(-dt^2+dz^2)+V^2U^{-2/3}d\phi^2+dr^2\;, \tag{7}$$

where U and V satisfy the new version of the Eqs. (3a)–(3d):

$$V''+\Theta V=0\;, \tag{8a}$$

$$U''+\Gamma U=0\;, \tag{8b}$$

and

$$U'V'+\Delta UV=0\;, \tag{8c}$$

where we have set $\Theta=8\pi\epsilon-2\pi\sigma+\beta^2$, $\Gamma=6\pi\sigma+\beta^2$, and

$\Delta=6\pi\delta+\beta^2$. Since these are three equations on two metric functions V and U, there must exist a relation between the stress-energy components ϵ, σ, and δ. Unfortunately, this relation is apparently too complicated to be useful. A more useful relation is the conservation of stress-energy, which here takes the form

$$\frac{4}{3}(\epsilon-\delta)\frac{U'}{U}+\left[\frac{V'}{V}-\frac{1}{3}\frac{U'}{U}\right](\sigma-\delta)=\delta' . \quad (9)$$

A cosmic string is frequently described by a stress-energy tensor in the form (2) under the assumption that components $T_{\phi\phi}$ and T_{rr} vanish. But now, in the presence of the cosmological constant, this assumption is no longer tenable. Indeed, were σ and δ both to vanish, then Eq. (9) would require that $\epsilon U'=0$, i.e., that $\epsilon=0$ (implying no string at all) or $U'=0$ [implying, from applying (8b) in a vacuum, that $\Lambda=0$].

We then make some further assumptions to rule out unphysical stress energies. We first demand that the weak energy condition[13] hold, i.e., that, in the case at hand, ϵ be positive and greater than both σ and δ. We next demand that there be no conical singularity on the axis. We shall regard the string as a concentrated configuration of matter, i.e., as having the stress-energy components take some appropriate value within the string and vanish outside. Therefore, another condition is needed to join these two parts properly. We assume that the O'Brien-Synge-Lichnerowicz jump condition holds to guarantee that there be no surface layer of stress-energy on the boundary.

Regard the metric (7) as the interior metric for a string. We first write out the boundary condition on the axis. Since the range of ϕ is 0 to 2π, our boundary condition of no conical singularity on the axis can be written as $\lim_{r\to 0}(VU^{-1/3}/r)=1$. But this condition is equivalent to the following two conditions:

$$V_0=0 \text{ and } V_0'=U_0^{1/3} , \quad (10)$$

where the subscript 0 denotes evaluation on the axis. Joining Eqs. (10) and (7b) with (7c) and its derivative, we obtain the two requirements

$$U_0'=0 \text{ and } \sigma_0=\delta_0 . \quad (11)$$

Consider next the exterior metric for a string. One class of exteriors can be obtained by either solving (8a)–(8c) in a vacuum or by imposing the boost symmetry, i.e., $\gamma_1=\gamma_2=\pm\frac{2}{3}$, and $\gamma_3=\mp\frac{4}{3}$, on the metric (4). The result is

$$ds^2=[a\cos\beta(r+\hat{r})]^{4/3}(-dt^2+dz^2)+\frac{d^2a^{-2/3}}{\beta^2}[\sin\beta(r+\hat{r})]^2\times[\cos\beta(r+\hat{r})]^{-2/3}d\phi^2+dr^2 , \quad (12)$$

where a, d, and $\hat{r}$ are constants to be determined later. This metric is applicable to the exterior of a string.

There is still one other possible exterior, provided Λ is negative. This exterior is the limit of the metric (4) as $|\beta\hat{r}|\gg 1$. We shall discuss this case at the end of this section.

Now we consider the jump condition on the surface, $r=r_s$. The O'Brien-Synge-Lichnerowicz jump condition—the application of Einstein's field equation to the surface—requires that both the metric and the extrinsic curvature of the surface be continuous across the surface. This condition, in the case at hand, means that both the metric and the derivatives of the metric functions are continuous. On using the metric (12) for the exterior, this condition turns out to be

$$a^2=U_s{}^2+(U_s'/\beta)^2 , \quad (13)$$

$$U_s'V_s'+\beta^2U_sV_s=0 , \quad (14)$$

$$\beta\tan\beta(r_s+\hat{r})=-U_s'/U_s , \quad (15)$$

and

$$d^2=V_s'^2+(\beta V_s)^2 , \quad (16)$$

where the subscript s denotes evaluation on the surface bounding the string. These jump conditions indicate how those constants in the metric (12) are related to the interior metric (7). Additionally, the interior metric is dominated by a given stress-energy tensor through Einstein's field equations (8a)–(8c). Therefore the conditions (13)–(16) are essentially the relations between the constants in the exterior metric (12) and the stress-energy components. So far, we have obtained all the equations (8a)–(8c) and boundary conditions (10), (11), and (13)–(16) needed to determine the spacetime metric for a given string. In the next few paragraphs we shall discuss the meaning of the jump conditions in some detail. But before we do that, first let us get some idea about the interior.

Recall that there is a relation between the three components of a given stress-energy tensor ϵ, σ, and δ. Therefore, given only two of them, say ϵ and σ, one can solve (8a) and (8b) for V and U. These two solutions are involved with four integral constants: two of them are multiplicative, say, V_1 and U_1; the other two may be called V_2 and U_2 for V and U, respectively. Substitute the solutions into Eq. (8c) to obtain the component δ. Using the boundary condition (10) and (11) on the axis, one can determine the integral constants V_2, U_2 and the ratio of U_1 to V_1. Then, only one arbitrary constant, say V_1, still remains. However, it can be shown that this arbitrariness will have no effect on the spacetime solution—or more specifically, not on the norm of the Killing field $(\partial/\partial\phi)^a$ (since only the range of ϕ has been restricted). Indeed, let V be replaced by V_1V in the norm of $(\partial/\partial\phi)^a$, i.e., in $g_{\phi\phi}{}^{1/2}=VU^{-1/3}$. The nonconical singularity condition (10) on the axis requires that U in $g_{\phi\phi}{}^{1/2}$ be replaced by $V_1{}^{-3}U$. The norm of $(\partial/\partial\phi)^a$, therefore, must be independent of V_1. There is, therefore, a unique spacetime solution for a given stress-energy tensor. This argument, of course, works for both interior and exterior metrics. For convenience, however, we shall deal with the quantity $\Delta\phi$ in the exterior, where $\Delta\phi=2\pi(1-da^{-1/3})$, and call it the deficit angle, rather than with the norm of the Killing fields $(\partial/\partial\phi)^a$.

Let us next investigate the jump conditions in detail. The simplest one is Eq. (13), which specifies how the scale of the coordinates t and z of the exterior is related to that of the interior. One can choose any convenient value of, say, V_1, to fix the coordinate's scale of t and z of the exterior.

The condition (14) is just an application of Eq. (8c) on the surface. This condition implies that the component δ of the stress-energy tensor must be continuous on the surface, i.e., $\delta \to 0$ as $\gamma \to r_s$. For given values of stress-energy components ϵ and σ, the condition (14) can be used for finding the diameter value r_s of the string.

We next consider the most crucial condition, (15). This condition suggests that the parameter $\hat{r}$ in the metric (12) depends only on the stress-energy tensor, but not [like the other two constants a and d in (12)] on the free choice of multiplicative constant V_1, namely, that $\hat{r}$ be fully determined by a given stress-energy tensor. The importance of this condition involves the singular behavior of the spacetime metric. The vacuum metric (12) with positive cosmological constant has a real singularity; i.e., there exists an incomplete timelike geodesic along which the curvature scale $R_{abcd}R^{abcd}$ will become infinite, unless $\hat{r}$ takes the value of r_c, where $r_c = \pi/2\beta$. We note in passing that there is no such geodesic for $\Lambda < 0$. If the parameter $\hat{r}$ does take the value of r_c, the singularity will disappear from the exterior and the metric (12) will describe a part of a "closed" spacetime with the range of variable r from r_s to r_c. Let us name this spacetime $M_{\rm cl}$. For the interior part of the whole Universe, the presence of the string along the z axis will "smear out" what would otherwise have been a singularity on the axis. The topology for this sort of closed spacetime is $\mathbb{R}^2 \times \mathbb{S}^2$.

Finally, consider the condition (16). This condition, along with the condition (13), fixes the quantity $da^{-1/3}$ relating to the norm of $(\partial/\partial\phi)^a$, or equivalently, the deficit angle $\Delta\phi$ defined above in the exterior. We call $\Delta\phi$ a deficit angle just because it can be thought of as a generalization of the case $\Lambda = 0$. In general, however, the word "deficit" has no meaning unless there is a natural definite vacuum space to compare with, like the Minkowski space in the case $\Lambda = 0$. Here we say the metric (12) with a deficit angle $\Delta\phi$ only in the sense of comparing this metric with the vacuum metric (12) with $\Delta\phi = 0$ and in the absence of string in the spacetime. It is easy to see that, in the absence of string along the z axis, the vacuum metric (12) has no conical singularity anywhere, provided $\Delta\phi = 0$ and $\hat{r} = 0$. Let us name this vacuum spacetime M_0. The presence of string may have both $\hat{r}$ and $\Delta\phi$ nonvanishing, as we have argued that both $\hat{r}$ and $\Delta\phi$ are fully determined by a given stress-energy tensor. Let us first consider the case $\hat{r} = 0$. $\Delta\phi \neq 0$ means that, on the appearance of a string along the z axis, there is an angle $\Delta\phi$ which has been "cut out from" (if $\Delta\phi > 0$) or "plugged into" (if $\Delta\phi < 0$) the spacetime M_0. Otherwise, $\Delta\phi = 0$ means that the appearance of string along the z axis does not make this kind of angle change in the exterior. Calling $\Delta\phi$ a "deficit" angle makes sense only in this case. The more interesting case, however, is that $\hat{r} = r_c$, i.e., the closed spacetime defined above. Here, the deficit angle $\Delta\phi \neq 0$ means that there is a conical singularity at $r = r_c$, and $\Delta\phi = 0$ simply means without a conically singular point in the exterior of $M_{\rm cl}$. We claim that it is possible to have $\Delta\phi$ either positive, negative, or zero by adjusting the value of the stress-energy components.

To understand our claim, one needs the useful relation

$$\frac{\Delta\phi}{2\pi} = 1 - \frac{\left| 1 - 4\mu + \int_0^{r_s} [2\pi(\sigma+\delta) - 2\beta^2/3] V U^{-1/3} dr \right|}{[\cos\beta(r_s + \hat{r})]^{2/3}} , \tag{17}$$

where $\mu = 2\pi \int_0^{r_s} \epsilon V U^{-1/3} dr$ represents the mass per unit length, or linear mass density, of the string. To prove this relation, first multiply Eq. (8a) by $U^{-1/3}$, then (8c) by $U^{-4/3}$, and, taking the difference and integrating it from $r = 0$ to $r = r_s$, we obtain

$$(V' U^{-1/3}) \big|_0^{r_s} = -4\mu + \int_0^{r_s} [2\pi(\sigma+\delta) - 2\beta^2/3] V U^{-1/3} dr . \tag{18}$$

Evaluate the left-hand side of (18) by using the boundary conditions (10) on the axis as well as (13)–(16) on the surface to obtain (17) through the definition of $\Delta\phi$.

By virtue of the relation (17), one can see that, in general, $\Delta\phi$ has a wide range of values. The value and sign of $\Delta\phi$ are mainly determined by the choice of both parameter $\hat{r}$ and the difference $(2\epsilon - \sigma - \delta)$, for β (compared to ϵ, σ, and δ) is usually quite small. (Note in passing that by means of the weak energy condition, this difference should always be positive.) We shall give an example in Sec. IV to illustrate that by adjusting the parameter in the components of a stress-energy tensor we can obtain a closed ($\mathbb{R}^2 \times \mathbb{S}^2$) universe $M_{\rm cl}$, defined above, without any singularity.

We next discuss two limiting cases of the relation (17). Set the cosmological constant to be zero and let the components σ and δ vanish. Then (17) reduces to $\Delta\phi = 8\pi\mu$. This limit is the result proven by Vilenkin,[3] Gott,[4] and Hiscock:[5] the exterior metric is the Minkowski space with a deficit angle $8\pi\mu$. The second limit is what has been called the classical limit, i.e., the diameter $r_s \to 0$. The relation (17) in this limit reduces to

$$\Delta\phi/2\pi = 1 - |1 - 4\mu| (\cos\beta\hat{r})^{-2/3} .$$

In the case of $\hat{r} = 0$, Λ being either positive or negative, the exterior (12) is always conical in the sense that the metric is approaching a conical singularity as $r_s \to 0$. In the case of $\hat{r} = r_c$ and Λ being positive (i.e., the closed Universe $M_{\rm cl}$), the exterior is the limit of (12) as $\mu \to \frac{1}{4}$, and the norm of $(\partial/\partial\phi)^a$ has no definite value. Hence there is no way to say whether the exterior is conical or not in this limit.

So far, the discussions about the properties of the exterior of the string have been based on considering only the metric (12) as the exterior metric. However, there is one other possible exterior metric, provided $\Lambda < 0$, namely, the anti–de Sitter metric (6)

$$ds^2 = e^{\pm 2\kappa r} [a_1^{4/3}(-dt^2 + dz^2) + d_1^{2/3} d\phi^2] + dr^2 ,$$

which is the limit of the metric (4) as $|\beta\hat{r}|$ gets larger and larger. Match the interior (7) with the anti–de Sitter metric (6) on the surface by means of the O'Brien-Synge-

Lichnerowicz jump condition to obtain the following boundary conditions:

$$U_s = a_1 e^{\pm 3\kappa r_s/2}, \tag{19}$$

$$U_s' = \pm \tfrac{3}{2}\kappa U_s, \tag{20}$$

$$V_s = (a_1 d_1)^{1/3} e^{\pm \kappa r_s/2}, \tag{21}$$

$$V_s' = \pm \tfrac{3}{2}\kappa V_s, \tag{22}$$

where a_1 and d_1 are constants in the anti–de Sitter metric written as above. These conditions (19)—(22) can also be obtained from the boundary condition (13)—(16) by setting $|\beta\hat{r}|$ to be much larger than 1.

There is, in this case, a formula for the deficit angle analogous to (17). It can be obtained by substituting the boundary condition (19)—(22) into (18). The result is

$$\frac{\Delta\phi}{2\pi} \equiv (1 - d_1{}^{1/3}) = 1 - e^{\mp\kappa r_s}\left[1 - 4\mu + \int_0^{r_s} [2\pi(\sigma+\delta) + 3\kappa^2/2] V U^{-1/3} dr\right], \tag{23}$$

where $\kappa = (|\Lambda|/3)^{1/2}$. According to this relation (and using the same argument as before), one can see that $\Delta\phi$ is still fully determined by a given stress-energy tensor, and that it could take a value either positive, negative, or zero.

IV. EXAMPLE

In this section we shall give an example of a solution for a cosmic string to illustrate the geometrical properties described in Sec. III.

Let Λ be positive. Choose the components, ϵ and σ, of stress-energy tensor T_{ab} in the form of (2) to be

$$\epsilon = \epsilon_1 = \epsilon_2 = [(\alpha^2 - 1)\beta^2 + 2\pi\sigma]/8\pi \tag{24}$$

and

$$\sigma = -\beta^2 \left[1 + \frac{n^2(1-p)\cos 2\beta\hat{r}\cos n\beta r}{2p + (p-1)\cos 2\beta\hat{r}\cos n\beta r}\right] \Big/ 6\pi, \tag{25}$$

where $\alpha = (1 - \hat{r}/r_c + \hat{r}/\lambda)$ with $r_c = \pi/2\beta$; $\hat{r}$, λ, and n are free parameters, and p is some constant ultimately to be expressed in terms of $\hat{r}$, λ, and n.

One can now proceed in the usual way. Solve (8a) and (8b) with (24) and (25) for V and U. Substitute these solutions into Eq. (8c) to obtain the component δ of the stress-energy tensor. Then impose the boundary conditions (10) and (11) to fix the integral constants in V, U, and δ. Only one constant, say V_1, is left and we set it to be 1. The resulting interior metric is

$$ds^2 = [2p + (p-1)\cos 2\beta\hat{r}\cos n\beta r]^{4/3}(-dt^2 + dz^2) + U_0{}^{2/3}\left[\frac{\sin\alpha\beta r}{\alpha\beta}\right]^2 [2p + (p-1)\cos 2\beta\hat{r}\cos n\beta r]^{-2/3} d\phi^2 + dr^2, \tag{26}$$

where $U_0 = 2p + (p-1)\cos 2\beta\hat{r}$. For example, $U_0 = p + 1$ if $\hat{r} = r_c$, or $U_0 = 3p - 1$ if $\hat{r} = 0$. The component δ is

$$\delta = -\frac{\beta^2}{6\pi}\left[1 + \frac{\alpha n(1-p)\cos 2\beta\hat{r}\sin n\beta r\cot\alpha\beta r}{2p + (p-1)\cos 2\beta\hat{r}\cos n\beta r}\right]. \tag{27}$$

Here, the parameter $\hat{r}$ in (25)—(27) has been treated as the same parameter $\hat{r}$ in the exterior metric (12). Of course, the way in which $\hat{r}$ appears in the stress-energy components is not unique. The only criterion for setting the parameter $\hat{r}$ is to keep the jump condition (13)—(16) satisfied. Given a value of $\hat{r}$, the stress-energy tensor (and therefore, the interior metric) has two free parameters: λ and n. And furthermore, for the given values of both λ and n, one can obtain a unique interior metric through (26). Match this interior with the metric (12) to obtain a unique exterior metric.

Let us now consider the most interesting case, i.e., the closed spacetime $M_{\rm cl}$ where $\hat{r}$ takes the value r_c. The interior metric, with $\alpha = \pi/2\beta\lambda$, now becomes

$$ds^2 = [2p - (p-1)\cos n\beta r]^{4/3}(-dt^2 + dz^2) + (1+p)^{2/3}\left[\frac{\sin\alpha\beta r}{\alpha\beta}\right]^2 [2p - (p-1)\cos n\beta r]^{-2/3} d\phi^2 + dr^2. \tag{28}$$

Now the exterior (12), describing a part of a closed universe with topology $\mathbf{R}^2 \times \mathbf{S}^2$, becomes

$$ds^2 = (a\sin\beta r)^{4/3}(-dt^2 + dz^2) + \frac{d^2 a^{-2/3}}{\beta^2}(\cos\beta r)^2 (\sin\beta r)^{-2/3} d\phi^2 + dr^2, \tag{29}$$

where the constants a and d in (29) and p in (28) are given, through the jump condition (13)—(16), by the expressions

$$a^2 = [2p + (p-1)\cos n\beta r_s]^2 + n^2(1-p)^2(\sin n\beta r_s)^2, \tag{30}$$

$$\tan\beta r_s \tan\frac{\pi r_s}{2\lambda} = -\frac{\pi}{2\beta\lambda}, \tag{31}$$

$$p = \frac{\cos n\beta r_s + n\sin n\beta r_s \tan\beta r_s}{\cos n\beta r_s + n\sin n\beta r_s \tan\beta r_s - 2}, \tag{32}$$

and

$$da^{-1/3}=\frac{2\beta\lambda\sin(\pi r_s/2\lambda)}{\pi\left[\left[\frac{\pi}{2\beta\lambda}-1\right]n\sin n\beta r_s\right]^{1/3}(\cos\beta r_s)^{2/3}}\,. \tag{33}$$

Letting $\hat{r}=r_c$ and p be given by (32), one can examine (24), (25), and (27) to see that the weak energy condition is satisfied, i.e., $\epsilon>0$ and $(2\epsilon-\delta-\sigma)>0$. For simplicity, one may assume $n\beta r_s \ll 1$ to approximate the right-hand side of the relation (33). The deficit angle $\Delta\phi$ is given in the expression

$$\frac{\Delta\phi}{2\pi}=1-|da^{-1/3}|\doteq 1-\left[\frac{2}{\pi}\right]^{4/3}\frac{\beta\lambda^{4/3}}{n^{2/3}r_s^{1/3}}\,. \tag{34}$$

Note that by adjusting the parameter n one can make $\Delta\phi$ either positive, negative, or zero. We therefore have illustrated that, for a cosmic string with a particular set of the stress-energy components, its spacetime geometry, if Λ is positive, could be a closed universe ($\mathbf{R}^2\times\mathbf{S}^2$) without any singularity.

Finally, following the argument of Vilenkin,[14] let us indicate the numerical values of the above parameters. The diameter of the string is of the order m^{-1} where m is the characteristic mass of the field. For the grand-unification string, say, $m\sim 10^{15}$ GeV $\sim 10^{-4}$ in Planck units. Thus r_s is of the order 10^4 in Planck units. For these values of m and r_s of the string—actually for any kind satisfying $\beta r_s \ll 1$—the parameter λ, according to (31), is very slightly smaller than r_s. On the other hand, Guth[15] has pointed out that during the phase transitions the cosmological constant assumed is of the order of constant energy density of the Universe, i.e., $\beta=\sqrt{3\Lambda}/2\sim 10^{-10}$ at that time. Substitute the values of r_s, β, and λ into (34) and let $\Delta\phi=0$. The required value of n is about 4×10^{-10}. The corresponding value of ϵ_0 is of the order 2.3×10^{-8}, of $\sigma_0=\delta_0$ the order -5.3×10^{-10}, and of μ the order 5.4, all in Planck units. The component δ will go to zero as $r\rightarrow r_s$, as required in general, while σ remains negative. The situation seems to be the following. The pressures in both the ϕ and r directions are positive in order to balance the negative pressure caused by the positive cosmological constant background.

In summary, the components $T_{\phi\phi}$ and T_{rr} of the stress-energy tensor for a cosmic string, in the presence of the cosmological constant, cannot both be assumed to vanish. In light of the O'Brien-Synge-Lichnerowicz jump condition, the norm of the Killing field $(\partial/\partial\phi)^a$ or, equivalently, the deficit angle $\Delta\phi$, is fixed by a given stress-energy tensor. If Λ is negative, the exterior is either the metric (12) or the anti—de Sitter metric (6) depending on the value of a certain parameter involved in the stress-energy tensor. If Λ is positive, the situation for the exterior depends also on a certain parameter's value related to the stress-energy tensor. The exterior, therefore, could be a part of either a universe ($\mathbf{R}^3\times\mathbf{S}^1$) approaching a curvature singularity, or a universe ($\mathbf{R}^2\times\mathbf{S}^2$) with or without a conical singularity.

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*Permanent address: Department of Physics, Tianjin Normal University, Tianjin, China.

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[8] The spacetime defined at the beginning of Sec. II may be called a static plane symmetric, since it is the universal covering of static and cylindrically symmetric spacetime which requires the additional property that one spacelike Killing field must have closed orbits. However, we do not need the property until Sec. III.

[9] See, also, B. Linet, report, 1985 (unpublished).

[10] To verify that the metric (6) is the anti—de Sitter space metric, one can directly compute its constant curvature and negative scalar curvature. D. Garfinkle has shown (private communication) the coordinate transformation from (6) to the usual form given in Ref. 13, p. 131.

[11] That the solution (4) for positive Λ never approaches de Sitter metric does not mean that it is impossible for any string to exist in de Sitter space. Indeed, consider the metric

$$ds^2=-dt^2+\exp[2(\Lambda/3)^{1/2}t](dr^2+dz^2+b_{(r)}{}^2d\phi^2)\,,$$

where $b(r)$ is any positive smooth function of r satisfying (1) $db/dr\,|_{r=0}=1$, (2) $db/dr=\lambda$ for $r\geq r_s$ (λ is a constant less than 1), and (3) $b^{-1}d^2b/dr^2<0$ for all r. Such functions clearly exist. The stress-energy for this spacetime is that of a string sitting along the z axis. The only nonzero components of the stress-energy tensor are the positive energy density T_{00} and the equal negative pressure in the direction of the axis. Define $\mu=\int\int T_{00}\sqrt{g_{rr}}dr\sqrt{g_{\phi\phi}}d\phi$, the mass per unit length of the string. Computing out this integral, we obtain $\mu=(1-\lambda)/4$, independent of the t. The spacetime, nonsingular everywhere, is de Sitter for $r>r_s$, with a deficit angle $\Delta\phi=8\pi\mu$. Alternatively, consider the metric

$$ds^2=-dt^2+\frac{3}{\Lambda}\cosh^2(\Lambda/3)^{1/2}t\,[d\chi^2+\sin^2\chi(d\theta^2+b_{(\theta)}{}^2d\phi^2)]$$

with $b(\theta)$ satisfying conditions similar to those above. The stress-energy for this metric will still be that of a string, now sitting along the χ axis. The properties of the stress-energy are exactly as above. The spacetime, again nowhere singular, is also de Sitter ($\kappa=1$) outside the string, with a deficit angle $\Delta\phi$ expressed similar to the above in terms of the mass per unit length of the string.

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