A class of viscous magnetohydrodynamic type-I cosmologies

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Bianchi type-I models with a powers-of-t metric are investigated as solutions of the Einstein field equations for a viscous fluid with or without a magnetic field. Solutions are found which satisfy linear thermodynamic relations as well as all energy conditions. In general, these solutions contain a tilting velocity and depend on two parameters which satisfy certain restrictions. In the magnetohydrodynamic case solutions are shown to exist when the magnetic field and the spatial component of the velocity vector are parallel and also when they are perpendicular.

I. INTRODUCTION

Much of the considerable literature on relativistic cosmological models is confined to investigations of models which satisfy the Einstein field equations for a perfect fluid. In particular, attention is focused on the standard Friedmann-Robertson-Walker (FRW) models, which are the simplest possible perfect-fluid models and the basic models used to relate observations to theory in cosmology, and on the anisotropic spatially homogeneous models; excellent in-depth discussions of the latter models have been given by MacCallum.[1,2]

Misner[3] suggested the existence of strong dissipative mechanisms, such as neutrino-induced viscosity, during the early history of the Universe. Other authors[4–19] have discussed the consequences of neutrino viscosity or other types of viscous dissipative processes; in some of these articles exact solutions have been found of the Einstein field equations with terms representing viscosity. These solutions are, in general, spatially homogeneous models and, in many cases, are of type I (Refs. 1 and 2). The field equations used in the cited articles are those for a fluid with bulk and/or shear viscosity; no attempt is made to include heat-conduction terms or to impose a set of thermodynamic relations to be satisfied by the matter content of the Universe. On the other hand, the various developmental articles in relativistic thermodynamics and kinetic theory,[20–24] some of which refer to cosmological applications, are not concerned with exact solutions of the field equations obtained.

The adiabatic theory of galaxy formation[25] predicts that clusters of galaxies should form a cell type of structure and this prediction has been supported by observations.[26] The adiabatic theory is based on metric perturbations of standard perfect-fluid FRW models[27] whereas it would seem more appropriate to consider perturbations in a model with viscosity, particularly in view of the fact that the observed cell structure requires dissipative forces, such as viscosity, in the early Universe. Furthermore, if dissipative forces are to be considered, it is natural to consider also the thermal behavior of the cosmological fluid by including heat-conduction terms in the stress-energy tensor of the matter and by imposing an appropriate set of thermodynamic conditions to be satisfied by the fluid variables.

Our purpose, then, is to find exact cosmological solutions in which the matter content is an imperfect fluid with viscosity and heat conduction. We shall be particularly interested in models in which the matter content tends to a perfect fluid at the later epochs, but in which the viscosity is significant at earlier epochs. However, before proceedings, we shall make a further physically plausible generalization. It is possible that there may exist at present a very small cosmic magnetic field,[28,29] which may be the remains of a strong magnetic field which existed during the plasma era.[30] Thus, we shall include an electromagnetic component in the stress-energy tensor to be used and, as with viscosity, we shall have particular interest in models with a decaying magnetic field which is of significant magnitude at early epochs.

The field equations to be satisfied are

$$G_{\mu\nu}=E_{\mu\nu}+\rho u_\mu u_\nu+(p-\zeta\theta)h_{\mu\nu}-2\eta\sigma_{\mu\nu}+q_\mu u_\nu+q_\nu u_\mu \ , \tag{1.1}$$

where ρ, p, u^μ, θ, $\sigma_{\mu\nu}$, q_μ, $\eta(\geq 0)$, and $\zeta(\geq 0)$ are, respectively, the density, thermodynamic pressure, fluid velocity vector, expansion, shear tensor, heat-conduction vector, and shear and bulk-viscosity components, and $h_{\mu\nu}$ and $E_{\mu\nu}$ are, respectively, the projection tensor and the electromagnetic stress-energy tensor given by

$$h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu \ , \tag{1.2}$$

$$E_{\mu\nu}=F_{\mu\alpha}F^\alpha{}_\nu-\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \ , \tag{1.3}$$

where $F_{\mu\nu}$ is the Maxwell tensor. Note that the quantities $\sigma_{\mu\nu}$ and q_μ are orthogonal to the four-velocity u^μ, i.e.,

$$\sigma_{\mu\nu}u^\nu=0, \quad q_\mu u^\mu=0 \ , \tag{1.4}$$

and the expression $-2\eta\sigma_{\mu\nu}$ in Eq. (1.1) for the trace-free shear stress tensor is a result of the linearized phenomenological laws based on the assumption of small deviations from equilibrium.[20–24,31]

The thermodynamic conditions that the models will be required to satisfy are those originally proposed by Eckart[32] (see also Refs. 20—24) as follows.

(i) Baryon conservation:

$$(nu^{\mu})_{;\mu}=0\ , \tag{1.5}$$

where n is the particle density.

(ii) Gibb's relation:

$$Td(S/n)=d(\rho/n)+pd(1/n)\ , \tag{1.6}$$

where T is the temperature and S is the entropy density.

(iii) Positive entropy production:

$$S^{\mu}{}_{;\mu}\geq 0\ , \tag{1.7}$$

where $S^{\mu}=Su^{\mu}+T^{-1}q^{\mu}$ is the entropy flux.

(iv) Temperature gradient law:

$$q^{\mu}=-\kappa h^{\mu\nu}(T_{,\nu}+Ta_{\nu})\ , \tag{1.8}$$

where $h^{\mu\nu}=g^{\mu\nu}+u^{\mu}u^{\nu}$ is the projection tensor, a_{ν} is the acceleration vector, and κ is the thermal conductivity. The condition $\kappa\geq 0$ is sufficient to ensure that the condition (1.7) is satisfied.

Solutions of Eqs. (1.1)—(1.8), with or without an electromagnetic field, have been found by Coley and Tupper.[33–38] These solutions have as their spacetime metric that of the FRW models, i.e.,

$$ds^2=-dt^2+R^2(t)[dr^2/(1-kr^2)+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2]\ , \tag{1.9}$$

where $k=0$, ± 1, and the solutions exist because of the fact that a general stress-energy tensor, such as the right-hand side of Eq. (1.1), may be degenerate in the sense that its local tetrad components are of the form

$$T^{ij}=\mathrm{diag}(\bar{\rho},\bar{p},\bar{p},\bar{p})\ , \tag{1.10}$$

where the latin suffixes denote tetrad components. In such a case the stress-energy tensor may be interpreted as that of a perfect fluid. Various reinterpretations of stress-energy tensors have been discussed by Tupper,[39–41] Raychaudhuri and Saha,[42,43] and Carot and Ibañez,[44] and a detailed discussion of the geometric and algebraic conditions for such reinterpretations to be possible has been given by Hall and Negm.[45] In the case of the FRW models, which are usually regarded as solutions of the field equations for a perfect fluid, the reinterpretation as an imperfect fluid requires that the four-velocity u^{μ} must be tilting (i.e., noncomoving) with respect to the hypersurface-orthogonal preferred observer. If the velocity is comoving then the solution becomes the usual perfect-fluid solution with, at most, the addition of a bulk-viscosity term. This reinterpretation of the physical content of the FRW models has been extended to two-fluid models in which one fluid is a comoving radiation fluid, representing the cosmic microwave background, and the other fluid is a tilting viscous fluid representing the cosmic fluid.[46–48]

We would like to extend the investigations of Refs. 31—36 to spatially homogeneous anisotropic models for two reasons. First, it would be interesting to investigate the effect of discarding the assumption of isotropy. Second, the solutions of Refs. 33—38 are the only known solutions of Eqs. (1.1)—(1.8) and all are perfect-fluid spacetimes in which the stress-energy tensor has been reinterpreted as that of an imperfect fluid. In some sense, it would be more satisfactory to have solutions of these equations in which the stress-energy tensor does not have the degenerate form (1.10), so that the spacetime does not admit alternative physical interpretations. For this purpose we shall consider only the simplest of the spatially homogeneous models, namely, the type-I models, the general metric of which can be written in the form

$$ds^2=-dt^2+A^2(t)dx^2+B^2(t)dy^2+C^2(t)dz^2\ . \tag{1.11}$$

However, we shall not use this general metric, partly because the generality of the three unknown functions $A(t)$, $B(t)$, $C(t)$ makes it difficult to obtain specific solutions, but mainly because we want to find solutions which will give us some basis for comparison between the imperfect-fluid FRW models and the type-I models. The investigations of imperfect-fluid FRW models in Refs. 33—38 were largely confined to those zero-curvature FRW models which, in the perfect-fluid form, satisfy an equation of state of the form $p/\rho=\gamma$, constant. The metrics of these models are of the form

$$ds^2=-dt^2+t^{2a}(dx^2+dy^2+dz^2)\ , \tag{1.12}$$

where $\frac{1}{3}\leq a\leq\frac{2}{3}$ for $1\geq\gamma\geq 0$, and so we shall restrict this investigation to the simple type-I generalization of the metric (1.12), viz.,

$$ds^2=-dt^2+t^{2a}dx^2+t^{2b}dy^2+t^{2c}dz^2\ . \tag{1.13}$$

As in the case of the metric (1.12), we shall assume that the fluid has a tilting velocity with spatial component in the direction of the x-coordinate axis. (For a discussion of spatial homogeneous models with tilting velocity see King and Ellis.[49]) The metric (1.13) is well known as that of the Kasner vacuum solution for which the following conditions hold:

$$a+b+c=1\ , \tag{1.14}$$

$$a^2+b^2+c^2=1\ . \tag{1.15}$$

If only the condition (1.14) holds, the metric (1.13) represents a perfect-fluid solution with

$$\rho=p=(1-a^2-b^2-c^2)t^{-2}\ . \tag{1.16}$$

This solution is sometimes known as the Kasner perfect-fluid solution, but is more correctly attributed to Jacobs.[50] From our earlier comments we would expect this perfect-fluid solution to appear in reinterpreted form as one of the imperfect-fluid solutions that we find.

In seeking solutions of Eqs. (1.1)—(1.8) we shall require the models to satisfy the dominant energy condition.[51] For the spacetime with metric (1.13) this condition implies that the following inequalities hold:

$$ab+bc+ca>0\ ,$$
$$(b+c)(a+b+c-1)\geq 0\ ,$$
$$(c+a)(a+b+c-1)\geq 0\ ,$$
$$(a+b)(a+b+c-1)\geq 0\ , \quad (1.17)$$
$$(a+1)(b+c)\geq b^2+c^2\ ,$$
$$(b+1)(c+a)\geq c^2+a^2\ ,$$
$$(c+1)(a+b)\geq a^2+b^2\ ,$$

and these imply

$$a+b+c-1\geq 0\ , \quad (1.18)$$

an inequality which will be useful in our later discussions.

In Sec. II we seek exact solutions, with metric of the form (1.13), of the viscous fluid field equations, i.e., Eq. (1.2) with $E_{\mu\nu}=0$. In Secs. III and IV we consider the cases in which a magnetic field is present and acts parallel to and perpendicular to, respectively, the spatial velocity component. All the solutions that we present here are physically acceptable in the sense that all energy conditions, thermodynamic conditions, and positivity conditions are satisfied. For simplicity, we assume throughout that the bulk-viscosity coefficient ζ is zero.

II. VISCOUS FLUID SOLUTIONS

We shall assume that, in the coordinates used for the metric (1.13), the four-velocity has a spatial component in the x direction: i.e.,

$$u^\mu=(\alpha,\beta t^{-a},0,0)\ , \quad (2.1)$$

where $\alpha(\geq 1)$ and β are functions of t only satisfying

$$\alpha^2-\beta^2=1\ . \quad (2.2)$$

Equations (1.4) and (2.1) suggest the following form for q_μ:

$$q_\mu=Q(\beta,-\alpha t^a,0,0)\ , \quad (2.3)$$

where $Q^2=q_\mu q^\mu$. With these assumptions, and using Eq. (A5), the nontrivial components of the field equations (1.1), with $E_{\mu\nu}=0$, for the metric (1.13) are

$$(ab+bc+ca)t^{-2}=\rho\alpha^2+p\beta^2-\tfrac{2}{3}\eta X\beta^2-2Q\alpha\beta\ , \quad (2.4)$$

$$0=-(\rho+p)\alpha\beta+\tfrac{2}{3}\eta X\alpha\beta+Q(\alpha^2+\beta^2)\ , \quad (2.5)$$

$$(b+c-b^2-bc-c^2)t^{-2}=\rho\beta^2+p\alpha^2-\tfrac{2}{3}\eta X\alpha^2-2Q\alpha\beta\ , \quad (2.6)$$

$$(c+a-c^2-ca-a^2)t^{-2}=p+\tfrac{1}{3}\eta X-\eta(b-c)\alpha t^{-1}\ , \quad (2.7)$$

$$(a+b-a^2-ab-b^2)t^{-2}=p+\tfrac{1}{3}\eta X+\eta(b-c)\alpha t^{-1}\ , \quad (2.8)$$

where

$$X=2\dot\alpha+(2a-b-c)\alpha t^{-1}\ . \quad (2.9)$$

From Eqs. (2.7) and (2.8) we obtain either

$$b=c\ , \quad (2.10)$$

or $2\eta\alpha=(1-a-b-c)t^{-1}$ which, from Eq. (1.18), implies that $\eta\leq 0$, which we reject [note that $\eta=0$ implies that $q_\mu=0$ and leads to the Kasner perfect-fluid solution given by Eqs. (1.14) and (1.16)]. Hence Eq. (2.10) must hold and this together with Eqs. (2.4)–(2.9) yields

$$\rho=[2b(a-b+1)\alpha^2-b(2-3b)]t^{-2}\ , \quad (2.11)$$

$$p=\tfrac{1}{3}[2b(a-b+1)\alpha^2+2a+2b-2a^2-4ab-3b^2]t^{-2}\ , \quad (2.12)$$

$$Q=2b(1+a-b)\alpha\beta t^{-2}\ , \quad (2.13)$$

$$\eta X=[a+b-a^2+ab-2b(a-b+1)\alpha^2]t^{-2}\ , \quad (2.14)$$

$$X=2\dot\alpha+2(a-b)\alpha t^{-1}\ , \quad (2.15)$$

and the dominant energy condition implies that

$$b>0,\ \ a-b+1\geq 0,\ \ a+2b-1\geq 0\ ,$$
$$a+b\geq 0,\ \ a+b-a^2+ab\geq 0\ . \quad (2.16)$$

To complete the solution we need to specify the velocity component α which must be chosen such that η, as given by Eqs. (2.14) and (2.15), is non-negative and Eqs. (1.5)–(1.8) must be satisfied with κ non-negative. There are many possible choices, but we note that one set of inequalities that ensures $\eta\geq 0$ is

$$\dot\alpha<0\ , \quad (2.17)$$

$$b-a\geq 0\ , \quad (2.18)$$

$$\alpha\geq\cosh\phi\ , \quad (2.19)$$

where

$$\cosh^2\phi=\tfrac{1}{2}b^{-1}(a-b+1)^{-1}(a+b-a^2+ab)\ , \quad (2.20)$$

which, from Eqs. (2.16) and (2.18), satisfies $\cosh^2\phi\geq 1$, as required. We shall adopt the inequalities (2.17)–(2.19) together with the conditions[35,36]

$$\alpha\to\infty\ \text{ as } t\to 0\ , \quad (2.21)$$

$$\alpha\to\cosh\phi\ \text{ as } t\to\infty\ . \quad (2.22)$$

From Eqs. (2.12) and (2.13), the condition (2.21) implies that the ratio $p/\rho\to\frac{1}{3}$ as $t\to 0$ while the condition (2.22) implies that $p/\rho\to\gamma$ as $t\to\infty$ where

$$\gamma=(a+b-a^2-ab-b^2)(a-b-a^2+ab+3b^2)^{-1}\ . \quad (2.23)$$

It is reasonable to expect the ratio p/ρ to decrease as $t\to\infty$ but always to remain non-negative, so we must have $0\leq\gamma<\frac{1}{3}$. The conditions $p\geq 0$ and $p<\rho/3\rho$ become, respectively,

$$a+b-a^2-ab-b^2\geq 0\ , \quad (2.24)$$

$$a+2b-a^2-2ab-3b^2<0\ , \quad (2.25)$$

and a linear combination of these two inequalities yields

$$a-b-a^2+ab+3b^2>0\ , \tag{2.26}$$

which, from Eqs. (2.11), (2.20), and (2.22), implies that $\rho>0$ always. Hence, by adopting the conditions (2.17) to (2.16), we have a model which expands out of an initial radiation state toward a final state with $p/\rho=\gamma$ where $0\le\gamma<\frac{1}{3}$.

Equation (1.5) leads to

$$n=n_0\alpha^{-1}t^{-(a+2b)}\ , \tag{2.27}$$

where n_0 is a constant, and Eq. (1.6) implies that $T=T(t)$. Equation (1.8) becomes

$$2b(a-b+1)at^{-2}=\kappa T(\dot{T}T^{-1}+\dot{\beta}\beta^{-1}+at^{-1})\ , \tag{2.28}$$

and the condition $\kappa\ge0$ is satisfied if

$$\dot{T}T^{-1}+\dot{\beta}\beta^{-1}+at^{-1}\ge0\ , \tag{2.29}$$

which, since $\beta\dot{\beta}<0$ and $\dot{T}<0$, shows that we must have

$$a>0\ . \tag{2.30}$$

The values of a and b which give rise to physically acceptable models are those satisfying the inequalities (2.16)–(2.19), (2.24)–(2.26), and (2.30). These values are represented by the points in the region $ABCD$ in Fig. 1. The boundary AB is excluded since this corresponds to $a=0$ and thus, from Eq. (2.29), $\dot{T}=\dot{\beta}=0$ which is contrary to the assumption (2.17). The boundary BC corresponds to $\gamma=\frac{1}{3}$, i.e., radiation, at all times irrespective of the value of α. However, in this case the temperature is given by the Stefan-Boltzmann law $\rho T^{-4}=$ const, so that the condition (2.29) becomes

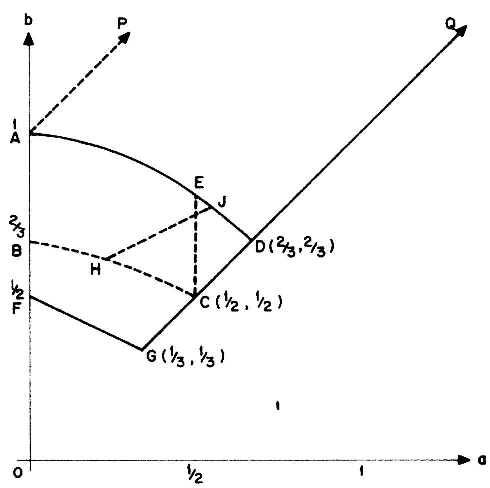

FIG. 1. The interior of the region $ABCD$ together with the open line DA represents the possible values of a and b corresponding to models with viscous fluid only. The interior of the subregion ECD together with ED represents the values of a and b corresponding to the particular choice (2.32) for α and β. The interior of the semi-infinite region $PAFGQ$ together with FG represents models with constant tilt. Models with a magnetic field in the direction of tilt are represented by the horizontal line FC ($b=\frac{1}{2}$) and by the interior of the region $JHCD$ together with the line DJ.

$$(a-\tfrac{1}{2})t^{-1}+[b(a-b+1)\rho^{-1}t^2+\beta^{-2}]\alpha\dot{\alpha}\ge0\ , \tag{2.31}$$

which shows that we must have $a>\frac{1}{2}$. This is beyond the allowable range of values on BC, so the boundary BC must be excluded. The boundary CD corresponds to the $k=0$ FRW models which have been discussed in Refs. 33–38. The boundary DA corresponds to solutions with $p=0$.

As an example of a possible functional form for α satisfying the conditions (2.17), (2.21), and (2.22) we consider

$$\alpha^2=\cosh^2\phi+h^2t^{-2r},\quad \beta^2=\sinh^2\phi+h^2t^{-2r}\ , \tag{2.32}$$

where h and r are positive constant parameters. In order to satisfy condition (2.29) we choose T to be of the form

$$T=T_0t^{-m}\beta^k\ , \tag{2.33}$$

where T_0 and m are positive constants and k is an arbitrary constant. The requirement that T is a decreasing function of t implies that

$$m+kr>0\ . \tag{2.34}$$

Since the matter distribution approaches a radiation state as $t\to0$, we shall assume that the Stefan-Boltzmann law holds as $t\to0$, i.e., $T^4\rho^{-1}\to$ const as $t\to0$. From Eqs. (2.11) and (2.33) this implies that

$$2m+r(2k-1)-1=0\ . \tag{2.35}$$

By picking out powers of h^2t^{-2r} we find that the condition (2.29) is satisfied if

$$a-m\ge0,\quad a-m-r(k+1)\ge0\ . \tag{2.36}$$

Note that the second of these inequalities together with Eq. (2.35) yield

$$r\le\tfrac{1}{3}(2a-1)\ , \tag{2.37}$$

which, since $r>0$, implies that $a>\frac{1}{2}$. Thus models with velocity components given by Eq. (2.32) form a subset of the possible models and are represented in Fig. 1 by the region CDE excluding the line CE.

As an example of a specific model let us choose a set of values for a, b, m, k, and r satisfying all of the above conditions. One such set is

$$a=\tfrac{7}{12},\quad b=\tfrac{2}{3},\quad m=\tfrac{1}{2},\quad k=\tfrac{1}{2},\quad r=\tfrac{1}{18}\ , \tag{2.38}$$

which yields

$$\cosh^2\phi=\tfrac{17}{16},\quad \sinh^2\phi=\tfrac{1}{16}\ , \tag{2.39}$$

and the complete solution is

$$\rho=\tfrac{11}{144}(17+16h^2t^{-1/9})t^{-2}\ , \tag{2.40}$$

$$p=\tfrac{11}{432}(3+16h^2t^{-1/9})t^{-2}\ , \tag{2.41}$$

$$\eta=88h^2t^{-10/9}(17+16h^2t^{-1/9})^{1/2}(51+80h^2t^{-1/9})^{-1}\ , \tag{2.42}$$

$$T=\tfrac{1}{4}T_0t^{-1/2}(1+16h^2t^{-1/9})^{1/2}\ , \tag{2.43}$$

$$\kappa=\tfrac{11}{24}T_0t^{-1/2}(17+16h^2t^{-1/9})^{1/2}(1+16h^2t^{-1/9})^{3/4}\ , \tag{2.44}$$

$$n=4n_0t^{-23/12}(17+16h^2t^{-1/9})^{-1/2}\ , \tag{2.45}$$

and the entropy S can be calculated by integrating Eq. (1.6). Note that all quantities approach infinity as $t\to 0$ and approach zero as $t\to\infty$, and the ratio $p/\rho\to\frac{1}{17}$ as $t\to\infty$.

If we replace the condition (2.17) by $\dot{\alpha}=0$ we can find solutions in which α and β are constants. In this case the condition (2.19) must be replaced by $\alpha>\cosh\phi$ in order to avoid the case $\eta=0$. Thus α and β are of the form

$$\begin{aligned}\alpha^2&=\tfrac{1}{2}b^{-1}(a-b+1)^{-1}(a+b-a^2+ab)+h_0^2\ ,\\ \beta^2&=\tfrac{1}{2}b^{-1}(a-b+1)^{-1}(b-a)(a+2b-1)+h_0^2\ ,\end{aligned} \tag{2.46}$$

where h_0^2 is an arbitrary positive constant. The solution in this case is

$$\rho=[a-b-a^2+ab+3b^2+2b(a-b+1)h_0^2]t^{-2}\ , \tag{2.47}$$

$$p=[a+b-a^2-ab-b^2+\tfrac{2}{3}b(a-b+1)h_0^2]t^{-2}\ , \tag{2.48}$$

$$\eta=2^{1/2}b^{3/2}(a-b+1)^{3/2}(b-a)^{-1}h_0^2t^{-1}\times[a+b-a^2+ab+2b(a-b+1)h_0^2]^{-1/2}\ , \tag{2.49}$$

$$T=T_0t^{-m},\ \ 0\le m\le a\ , \tag{2.50}$$

$$\kappa=T_0^{-1}[2b(a-b+1)]^{1/2}(a-m)^{-1}t^{m-1}\times[a+b-a^2+ab+2b(a-b+1)h_0^2]^{1/2}\ . \tag{2.51}$$

These solutions are valid within the semi-infinite region in Fig. 1 bounded by the straight lines *PA*, *AF*, *GQ* where *P*, $Q\to\infty$; within this region ρ, η, and κ are positive for all values of a, b, and h_0^2. Points within the region bounded by the lines *BF*, *FG*, *GC*, and the arc *CB* correspond to solutions in which $p=\gamma\rho$ where $1>\gamma>\frac{1}{3}$. Points within the region bounded by the lines *PA*, *AB*, *CQ*, and the arc *BC* correspond to solutions for which $\frac{1}{3}>\gamma>0$; those in the region bounded by the lines *PA*, *QD*, and the arc *AD* require h_0^2 to be greater than some nonzero positive number in order that $p\ge 0$. The solutions lying on the arc *BC* represent radiating fluids, i.e., $\rho=3p$, but are excluded because the Stefan-Boltzmann law requires $a\ge\frac{1}{2}$. The solutions represented by *PA* are excluded since they require h_0^2 to be infinite in order that $p\ge 0$. The line *AF* is $a=0$ which requires $m=0$, so points on this line correspond to solutions with constant temperature and infinite conductivity. The solutions corresponding to points on the line *FG* have metrics corresponding to the Kasner perfect-fluid solutions and provide an example of the situation described in Refs. 40 and 41 in which the stress-energy tensor of a perfect fluid may have identical components to the stress-energy tensor of an imperfect fluid. In the viscous fluid interpretation the ratio $\gamma=p/\rho$ varies in the range $1>\gamma>\frac{1}{3}$ as h_0^2 varies in the range $0<h_0^2<\infty$. Note that $h_0^2=0$ yields the perfect-fluid models. The line *GQ* represents the $k=0$ FRW models interpreted as viscous fluids, as discussed in Refs. 33–38. In this case η is infinite. Note that points on the line *DQ* represent models which, in their perfect-fluid form, have negative pressure. However, in their viscous fluid interpretation the pressure is not negative provided that h_0^2 is chosen appropriately. For example, if we take $a=b=1$, Eq. (2.48) leads to $p=(-1+2/3h_0^2)t^{-2}$ so that we must have $h_0^2\ge\frac{3}{2}$ for nonnegative pressure. All other physical quantities are positive.

Finally, we note that points close to, and on the left of, the line $a=b$ represent models which may be regarded as metric perturbations of the FRW models. For example, if $\epsilon>0$ is small, then the point $a=\frac{2}{3}-\epsilon$, $b=\frac{2}{3}$ can be thought of as a small axial perturbation of the Einstein–de Sitter model $a=b=\frac{2}{3}$. Such a model can be considered either with a variable velocity of some form, such as that given by Eq. (2.32), or with a constant velocity. Consider the latter case and suppose that $h_0^2=\epsilon$ then, to the first order in ϵ, we find

$$\begin{aligned}&\rho=(\tfrac{4}{3}+\epsilon)t^{-2},\ \ p=\tfrac{13}{3}\epsilon t^{-2}\ ,\\ &\eta=\tfrac{2}{3}(1-3\epsilon)t^{-1}\ ,\\ &T=T_0t^{-m}(0\le m\le\tfrac{2}{3}-\epsilon)\ ,\\ &\kappa=\tfrac{1}{2}T_0^{-1}(2-3m)^{-2}[8(2-3m)+(22+3m)\epsilon]t^{m-1}\ .\end{aligned} \tag{2.52}$$

Note that as $\epsilon\to 0$, i.e., as the metric approaches that of the Einstein–de Sitter model, both η and κ approach nonzero finite values. This indicates that the Einstein–de Sitter model, and FRW models in general, in its perfect-fluid interpretation possesses a latent shear viscosity coefficient and a latent thermal conductivity which have no effect on the field equations because the models have zero shear, zero heat conductivity, and zero-temperature gradient. None of the solutions described in this section exist if u^μ is comoving, i.e., $\beta=0$.

III. VISCOUS FLUID WITH PARALLEL MAGNETIC FIELD

In this section we generalize the models of the previous section by considering a viscous magnetohydrodynamic fluid, i.e., we assume that the electromagnetic stress-energy tensor on the right-hand side of the field equations (1.1) is nonzero and is due to a magnetic field in the direction of the spatial velocity component, as viewed by a hypersurface orthogonal preferred observer. This assumption implies that the only nonzero component of the Maxwell tensor is F_{23} and, assuming that the magnetic field is a function of t only, the Maxwell equations

$$F_{[\mu\nu;\sigma]}=0,\ \ F^{\mu\nu}{}_{;\nu}=J^\mu\ , \tag{3.1}$$

lead to

$$F_{23}=A_0\ , \tag{3.2}$$

where A_0 is a constant. Using the expression (2.3) for q_μ, the field equations (1.1) again lead to $b=c$ and to the following solution:

$$\rho=[2b(a-b+1)\alpha^2-b(2-3b)]t^{-2}-\tfrac{1}{2}A_0^2t^{-4b}\ , \tag{3.3}$$

$$p=\tfrac{1}{3}[2b(a-b+1)\alpha^2+2a+2b-2a^2-4ab-3b^2]t^{-2} - \tfrac{1}{6}A_0{}^2t^{-4b}, \quad (3.4)$$

$$\eta X=[(b-a)(a+2b-1)-2b(a-b+1)\beta^2]t^{-2} - A_0{}^2t^{-4b}, \quad (3.5)$$

together with Eqs. (2.13) and (2.15).

The dominant energy condition applied to the viscous fluid alone leads to the conditions

$$\begin{aligned} &2b(2a+b)t^{-2}-A_0{}^2t^{-4b}>0 ,\\ &2b(a+2b-a)t^{-2}-A_0{}^2t^{-4b}\geq 0 , \quad (3.6)\\ &(a+b-a^2+ab)t^{-2}-A_0{}^2t^{-4b}\geq 0 . \end{aligned}$$

If $b<\frac{1}{2}$ these conditions, together with the conditions (2.16) which apply to the total stress-energy tensor, imply an upper time limit on the model. On the other hand, if $b=\frac{1}{2}$ the conditions are simply limits on the value of A_0, and if $b>\frac{1}{2}$ they imply a lower time limit on the model. We shall investigate only those models which are valid as $t\to\infty$ and so we discard the case $b<\frac{1}{2}$.

Case 1. $b=\frac{1}{2}$. Equations (2.13), (2.15), and (3.3)–(3.6) become

$$\rho=[(a+\tfrac{1}{2})\alpha^2-\tfrac{1}{4}-\tfrac{1}{2}A_0{}^2]t^{-2} , \quad (3.7)$$

$$p=\tfrac{1}{3}[(a+\tfrac{1}{2})\alpha^2+\tfrac{1}{4}-2a^2-\tfrac{1}{2}A_0{}^2]t^{-2} , \quad (3.8)$$

$$Q=(a+\tfrac{1}{2})\alpha\beta t^{-2} , \quad (3.9)$$

$$\eta X=[a(\tfrac{1}{2}-a)-(a+\tfrac{1}{2})\beta^2-A_0{}^2]t^{-2} , \quad (3.10)$$

$$X=2\dot{\alpha}+(2a-1)\alpha t^{-1} , \quad (3.11)$$

$$(2a+\tfrac{1}{2})>A_0{}^2,\ \ a\geq A_0{}^2,\ \ \tfrac{1}{2}+3a/2-a^2\geq A_0{}^2 . \quad (3.12)$$

From Eqs. (3.10) and (3.11) we see that if $a>\frac{1}{2}$, then ηX is always negative and so $\dot{\alpha}$ must be sufficiently negative to counteract the other (positive) term in the expression for X. This seems to be an unlikely possibility, particularly when $t\to\infty$, so we discard this case in favor of $a<\frac{1}{2}$. Note that the value $a=\frac{1}{2}$ corresponds to the zero-curvature FRW model filled with radiating fluid which has been discussed in Ref. 38.

Since $a<\frac{1}{2}$, the strongest of the conditions (3.12) is $a\geq A_0{}^2$; we shall take A_0 to have its maximum value, i.e.,

$$A_0{}^2=a . \quad (3.13)$$

Equations (3.7), (3.8), and (3.10) now become

$$\rho=\tfrac{1}{4}(2a+1)(2\alpha^2-1)t^{-2} , \quad (3.7a)$$

$$p=\tfrac{1}{12}(2a+1)(2\alpha^2+1-4a)t^{-2} , \quad (3.8a)$$

$$\eta X=-\tfrac{1}{2}(2a+1)(\beta^2+a)t^{-2} . \quad (3.10a)$$

Equations (3.7a) and (3.8a) show that for all α and t

$$1-4a/3\geq p/\rho\geq\tfrac{1}{3} , \quad (3.14)$$

so these solutions lie within the ultrarelativistic range. This is clear from Fig. 1 since the line $b=\frac{1}{2}$ lies below the arc BC which represents solutions satisfying the equation of state $\rho=3p$.

If α and β are constant then the ratio p/ρ is constant and

$$\eta=\tfrac{1}{2}(2a+1)(\beta^2+a)(1-2a)^{-1}t^{-1} , \quad (3.15)$$

which is always positive. If we take T to be of the form $T=T_0t^{-m}$, where m is positive, the condition (2.29) is satisfied provided that $a\geq m$ and the thermal conductivity is given by

$$\kappa=\tfrac{1}{2}T_0{}^{-1}(2a+1)(a-m)^{-1}\alpha t^{m-1} . \quad (3.16)$$

Note that a solution exists if u^μ is comoving, i.e., $\beta=0$. In this case

$$\begin{aligned} &\rho=\tfrac{1}{4}(2a+1)t^{-2} ,\\ &p=\tfrac{1}{12}(2a+1)(3-4a)t^{-2} , \quad (3.17)\\ &\eta=\tfrac{1}{2}(2a+1)a(1-2a)^{-1}t^{-1} , \end{aligned}$$

and the heat conduction and temperature gradient are zero. However, κ is not zero but approaches a finite limit as $\beta\to 0$, namely, the expression (3.16) with $\alpha=1$.

If α is not constant the time derivative of the ratio $\gamma=p/\rho$ is given by

$$\dot{\gamma}=-\tfrac{8}{3}(1-2a)(2\alpha^2-1)^{-2}\alpha\dot{\alpha} , \quad (3.18)$$

so that the desirable requirement $\dot{\gamma}<0$ implies that $\dot{\alpha}>0$. In fact $\alpha\to\infty$ as $\gamma\to\frac{1}{3}$ showing that as the material distribution approaches radiating matter, its four-velocity approaches that of light, i.e., it becomes a null vector. One possible functional form for α satisfying $\dot{\alpha}>0$ and $\alpha\to\infty$ as $t\to\infty$ is

$$\alpha^2=1+h^2t^{2r},\ \ \beta=ht^r , \quad (3.19)$$

where h and r are constants with $r\to 0$. With this choice Eqs. (3.10a) and (3.11) lead to

$$\eta=\tfrac{1}{2}(2a+1)(a+h^2t^{2r})(1+h^2t^{2r})^{1/2} \times[(1-2a)+(1-2a-2r)h^2t^{2r}]^{-1}t^{-1} , \quad (3.20)$$

and η is always non-negative provided that

$$1-2a-2r\geq 0 . \quad (3.21)$$

Equation (2.28) for κ becomes

$$\kappa=\tfrac{1}{2}T_0{}^{-1}(2a+1)(1+h^2t^{2r})^{1/2}h^{-k}t^{m-1-kr} \times(a-m+kr+r)^{-1} , \quad (3.22)$$

and κ will be non-negative if

$$a-m+kr+r\geq 0 . \quad (3.23)$$

Since $\gamma\to\frac{1}{3}$ as $t\to\infty$, we shall assume that the Stefan-Boltzmann law holds as $t\to\infty$, i.e., $T^4\rho^{-1}\to$ const as $t\to\infty$. From Eqs. (3.7) and (3.19) we see that $\rho\sim t^{2r-2}$ as $t\to\infty$ and, assuming the form (2.33) $T\sim t^{-m+kr}$ always. Hence Eq. (2.35) again holds. Since Eq. (3.21) shows that $r<\frac{1}{2}$, Eq. (2.35) implies that T decreases with time. In fact, Eqs. (3.21), (3.23), and (2.35) show that

$$\tfrac{1}{3}(1-2a)\leq r\leq\tfrac{1}{2}(1-2a) . \quad (3.24)$$

As an example, we note a set of values of a, m, k, and r satisfying the required conditions is

$$a=\tfrac{1}{4},\quad m=\tfrac{3}{5},\quad k=1,\quad r=\tfrac{1}{5}\ , \tag{3.25}$$

and the corresponding solution is

$$\begin{aligned} &\rho=\tfrac{3}{8}(1+2h^2t^{2/5})t^{-2},\quad p=\tfrac{1}{4}(1+h^2t^{2/5})t^{-2}\ ,\\ &\eta=\tfrac{15}{8}(1+4h^2t^{2/5})(1+h^2t^{2/5})^{1/2}\\ &\qquad\times(5+h^2t^{2/5})t^{-1}\ ,\\ &T=T_0ht^{2/5},\quad \kappa=15T_0{}^{-1}h^{-1}(1+h^2t^{2/5})t^{-3/5}\ , \end{aligned} \tag{3.26}$$

with metric $ds^2=-dt^2+t^{1/2}dx^2+t(dy^2+dz^2)$. Note that $\gamma\to\frac{2}{3}$ as $t\to 0$ and $\gamma\to\frac{1}{3}$ as $t\to\infty$.

Case 2. $b>\frac{1}{2}$. In this case we shall attempt to find solutions similar to those of Sec. II satisfying the conditions (2.17)—(2.23). For this purpose we again choose α and β to be of the form (2.32). Equations (3.3)—(3.5) become

$$\begin{aligned}\rho=&(a-b-a^2+ab+3b^2)t^{-2}\\ &+2b(a-b+1)h^2t^{-2r-2}-\tfrac{1}{2}A_0{}^2t^{-4b}\ ,\end{aligned} \tag{3.27}$$

$$\begin{aligned}p=&(a+b-a^2-ab-b^2)t^{-2}\\ &+\tfrac{2}{3}b(a-b+1)h^2t^{-2r-2}-\tfrac{1}{6}A_0{}^2t^{-4b}\ ,\end{aligned} \tag{3.28}$$

$$\eta X=-2b(a-b+1)h^2t^{-2r-2}-A_0{}^2t^{-4b}\ . \tag{3.29}$$

The ratio $p/\rho=\gamma$ is given by

$$\begin{aligned}&[a+b-a^2-ab-b^2+\tfrac{1}{3}F(t)]\\ &\qquad\times[a-b-a^2+ab+3b^2+F(t)]^{-1}\ ,\end{aligned} \tag{3.30}$$

where $F(t)=2b(a-b+1)h^2t^{-2r}-\frac{1}{2}A_0{}^2t^{2-4b}$, and $\dot\gamma\le 0$ implies that

$$(a+2b-a^2-2ab-3b^2)\dot F\ge 0\ . \tag{3.31}$$

Now $a+2b-a^2-2ab-3b^2\gtrless 0$ implies that $\gamma\gtrless\frac{1}{3}$ always. We shall confine our attention to models in which $\gamma<\frac{1}{3}$ always so we shall adopt the conditions (2.24) and (2.25) and $\dot F\le 0$, i.e.,

$$-4rb(a-b+1)h^2t^{-2r-1}-(1-2b)A_0{}^2t^{1-4b}\le 0\ . \tag{3.32}$$

This inequality imposes another condition on the initial time, a somewhat different condition to the conditions (3.6). However, this inequality becomes simply a relation between h^2 and $A_0{}^2$ if the powers of t in the two terms are equal, i.e., if

$$r=2b-1\ , \tag{3.33}$$

an identification that we shall use in the remainder of this section. The inequality (3.32) now becomes

$$A_0{}^2\le 4b(a-b+1)h^2\ , \tag{3.34}$$

where equality yields the case $\gamma=$ const.

The conditions (3.6) show that the model starts at some time $t=t_0>0$ given by $t_0{}^{4b-2}\ge C^{-1}A_0{}^2$ where C is the least of the quantities $2b(2a+b)$, $2b(a+2b-1)$, and $a+b-a^2+ab$. The conditions (2.24) and (2.25) confine the solution to the interior of the region $ABCDA$ in Fig. 1, together with the boundary AD. However, the condition (2.29) for positive thermal conductivity, assuming that T is of the form (2.33), leads to the conditions (2.34) and (2.36) again. The latter condition implies that $a-r\ge m+kr>0$ which, from Eq. (3.33) leads to

$$a-2b+1>0\ . \tag{3.35}$$

Since the model is not valid at $t=0$, it never approaches a radiation state with our choice of α and so Eq. (2.35) will not hold in this case and there are no further restrictions.

The line $a-2b+1=0$ cuts the boundary of the region $ABCDA$ at H and J in Fig. 1 so the allowable values of a and b for a valid solution are those at any point in the interior of the region $JHCDJ$ and the arc DJ (we exclude FRW models). In part of this region $2b(a+2b-1)$ is less than $a+b-a^2+ab$ and in the remaining part of the inequality is reversed. The least value taken by either of these expressions within the allowable region and its boundary is $\frac{1}{2}$ [by $2b(a-2b+1)$ at the vertex C]. Hence if we adopt as the initial time the value t_0 given by

$$t_0{}^{4b-2}=t_0{}^{2r}=2A_0{}^2\ , \tag{3.36}$$

we know that the model will be valid at all subsequent times for all values of a and b lying within the allowable region.

In order to provide an example of this type of solution we use the fact that the same spacetime may satisfy the field equations for different matter distributions. Accordingly, we choose the same metric as that used for the example given by Eqs. (2.38)—(2.45), viz,

$$ds^2=-dt^2+t^{7/6}dx^2+t^{4/3}(dy^2+dz^2)\ , \tag{3.37}$$

so that $a=\frac{7}{12}$, $b=\frac{2}{3}$ and, from Eq. (3.33), we have $r=\frac{1}{3}$. A suitable choice of m and k satisfying the inequalities (2.36) is $m=\frac{1}{8}$, $k=\frac{3}{8}$. Equation (3.34) becomes $A_0\le\frac{22}{9}h^2$ so, for the purposes of our example, we shall choose

$$A_0{}^2=2h^2\ . \tag{3.38}$$

Equation (3.36) shows that $A_0{}^2t_0{}^{-2/3}=\frac{1}{2}$, so $h^2t_0{}^{-2/3}=\frac{1}{4}$. The complete solution is then

$$\begin{aligned}&\alpha^2=\tfrac{1}{16}[17+4(t/t_0)^{-23}]\ ,\\ &\beta^2=\tfrac{1}{16}[1+4(t/t_0)^{-2/3}]\ ,\end{aligned} \tag{3.39}$$

$$\rho=\tfrac{1}{144}[187+52(t/t_0)^{-2/3}]t^{-2}\ , \tag{3.40}$$

$$p=\tfrac{1}{432}[33+52(t/t_0)^{-2/3}]t^{-2}\ , \tag{3.41}$$

$$\begin{aligned}\eta=&\tfrac{58}{3}[17+4(t/t_0)^{-2/3}]\\ &\times[17+12(t/t_0)^{-2/3}]^{-1}(t/t_0)^{-2/3}t^{-1}\ ,\end{aligned} \tag{3.42}$$

$$T=T_0 2^{-3/4}[1+4(t/t_0)^{-2/3}]^{3/16}t^{-1/8}\ , \tag{3.43}$$

$$\begin{aligned}\kappa=&\tfrac{1}{3}2^{7/4}[17+4(t/t_0)^{-2/3}]^{1/2}\\ &\times[1+4(t/t_0)^{-2/3}]^{13/16}t^{-7/8}T_0{}^{-1}\ ,\end{aligned} \tag{3.44}$$

$$n=4n_0[17+4(t/t_0)^{-2/3}]^{-1/2}t^{-23/12}\ , \tag{3.45}$$

and the magnitude of the magnetic field as measured by a hypersurface-orthogonal comoving observer is

$$|\mathbf{B}| = A_0 t^{-4/3} = 2^{-1/2}(t/t_0)^{-1/3} t^{-1} . \quad (3.46)$$

Note that $\gamma = 0.118\,55$ when $t = t_0$ and $\gamma \to \frac{1}{17}$ as $t \to \infty$.

Alternatively, using the same values for a, b, r, m, and k, we can obtain a solution in which $\gamma = \frac{1}{17}$ for all t by replacing Eq. (3.38) by

$$A_0{}^2 = \tfrac{22}{9} h^2 \quad (3.47)$$

so that $h^2 t_0{}^{-2/3} = \frac{9}{44}$. In this case the solution is

$$\alpha^2 = \tfrac{1}{176}[187 + 36(t/t_0)^{-2/3}] ,$$
$$\beta^2 = \tfrac{1}{176}[11 + 36(t/t_0)^{-2/3}] , \quad (3.48)$$

$$\eta = 18 \times 11^{1/2}[187 + 36(t/t_0)^{-2/3}]^{1/2} \times [187 + 108(t/t_0)^{-2/3}]^{-1}(t/t_0)^{-2/3} t^{-1} , \quad (3.49)$$

$$T = T_0(176)^{-3/16}[11 + 36(t/t_0)^{-2/3}]^{3/16} t^{-1/8} , \quad (3.50)$$

$$\kappa = \tfrac{128}{3} T_0{}^{-1} \alpha \beta^{-3/16} t^{-7/8} , \quad (3.51)$$

with $|\mathbf{B}|$ again given by Eq. (3.46).

Note that the solution given by Eq. (3.17) is the only solution of this section that exists for a comoving velocity.

IV. VISCOUS FLUID WITH ORTHOGONAL MAGNETIC FIELD

We now consider the case of a viscous magnetohydrodynamic fluid in which the magnetic field is in a direction orthogonal to that of the spatial velocity component. We shall again assume that the four-velocity is of the form (2.1) and we take the magnetic field to act in the direction of the y axis only, as viewed by a hypersurface-orthogonal preferred observer. This implies that the only nonzero component of the Maxwell tensor is F_{31}, and the Maxwell equations (3.1) then lead to

$$F_{31} = A_0 , \quad (4.1)$$

where A_0 is a constant. Assuming that q_μ again has the form (2.3), the field equations (1.1) become

$$(ab + bc + ca)t^{-2} = \tfrac{1}{2} A_0{}^2 t^{-2(a+c)} + \rho\alpha^2 + p\beta^2 - \tfrac{2}{3}\eta X \beta^2 - 2Q\alpha\beta , \quad (4.2)$$

$$0 = -(\rho + p)\alpha\beta + \tfrac{2}{3}\eta X \alpha\beta + Q(\alpha^2 + \beta^2) , \quad (4.3)$$

$$(b + c - b^2 - bc - c^2)t^{-2} = \tfrac{1}{2} A_0{}^2 t^{-2(a+c)} + \rho\beta^2 + p\alpha^2 - \tfrac{2}{3}\eta X \alpha^2 - 2Q\alpha\beta , \quad (4.4)$$

$$(c + a - c^2 - ca - a^2)t^{-2} = -\tfrac{1}{2} A_0{}^2 t^{-2(a+c)} + p + \tfrac{1}{3}\eta X - \eta(b - c)\alpha t^{-1} , \quad (4.5)$$

$$(a + b - a^2 - ab - b^2)t^{-2} = \tfrac{1}{2} A_0{}^2 t^{-2(a+c)} + p + \tfrac{1}{3}\eta X + \eta(b - c)\alpha t^{-1} , \quad (4.6)$$

where X is given by Eq. (2.9).

Applying the dominant energy condition to the viscous fluid alone we obtain

$$2(ab + bc + ca) > A_0{}^2 t^{2(1-a-c)} , \quad (4.7)$$

$$b + c - b^2 - c^2 + ab + ca \geq A_0{}^2 t^{2(1-a-c)} , \quad (4.8)$$

$$(a + c)(a + b + c - 1) \geq A_0{}^2 t^{2(1-a-c)} , \quad (4.9)$$

$$a + b - a^2 - b^2 + bc + ca \geq A_0{}^2 t^{2(1-a-c)} , \quad (4.10)$$

in addition to Eqs. (1.8), which apply to the total stress-energy tensor. We are interested only in models which are valid as $t \to \infty$, so the above equations imply that $a + c - 1 \geq 0$.

From Eqs. (4.5) and (4.6) we obtain

$$A_0{}^2 t^{2(1-a-c)} = (c - b)[(a + b + c - 1) + 2\eta\alpha t] , \quad (4.11)$$

which, from Eq. (1.9), shows that

$$c - b > 0 , \quad (4.12)$$

and it follows that

$$A_0{}^2 t^{2(1-a-c)} > (c - b)(a + b + c - 1) , \quad (4.13)$$

for $\eta \geq 0$ and for this to be valid as $t \to \infty$ we must have $a + c - 1 \leq 0$. Thus all conditions on the time coordinate can be satisfied as $t \to \infty$ if and only if

$$a + c = 1 , \quad (4.14)$$

in which case the inequalities (4.7)—(4.10) and (4.13) serve only to limit the value of A_0 and impose no restrictions on the time coordinate.

Using Eq. (4.14), the field equations (4.2)—(4.6) lead to

$$\rho = [(2a + b - 2a^2 + ab - b^2 - A_0{}^2)\alpha^2 - (a - a^2 + ab - b^2 + \tfrac{1}{2} A_0{}^2)]t^{-2} , \quad (4.15)$$

$$p = \tfrac{1}{3}[(2a + b - 2a^2 + ab - b^2 - A_0{}^2)\alpha^2 - (a^2 + ab + b^2 - a + \tfrac{1}{2} A_0{}^2)]t^{-2} , \quad (4.16)$$

$$Q = (2a + b - 2a^2 + ab - b^2 - A_0{}^2)\alpha\beta t^{-2} , \quad (4.17)$$

$$\eta[2\dot{\alpha} + (3a - b - 1)\alpha t^{-1}] = -(2a + b - 2a^2 + ab - b^2 - A_0{}^2)(\alpha^2 - 1)t^{-2} - \tfrac{1}{2}[b(3a - b - 1)A_0{}^2]t^{-2} . \quad (4.18)$$

Eliminating η between Eqs. (4.11) and (4.18) we obtain the differential equation for α:

$$n^2 t\dot{\alpha}/\alpha = l - h^2\alpha^2 , \quad (4.19)$$

where

$$n^2 = A_0{}^2 - b(1 - a - b) , \quad (4.20)$$

$$h^2 = (1 - a - b)(2a + b - 2a^2 + ab - b^2 - A_0{}^2) , \quad (4.21)$$

$$l = h^2 + (1 - 2a)A_0{}^2 . \quad (4.22)$$

Note that n^2 is positive from Eqs. (4.13) and (4.14) and h^2 is positive from Eqs. (4.8), (4.12), and (4.14).

If we choose $l < 0$, i.e., $l = -m^2$, then the solution to Eq. (4.19) is $\alpha^2 = m^2(C^2 t^{2m^2/n^2} - h^2)^{-1}$, where C^2 is a constant, and the condition $\alpha^2 \geq 1$ implies that $m^2+h^2 \geq C^2 t^{2m^2/n^2} > h^2$, thus imposing both upper and lower limits on the values of t. The choice $l=0$ also imposes an upper limit on t. Since we are interested only in solutions which remain valid as $t \to \infty$, we reject these choices and consider only the case $l=m^2>0$. In this case the solution is

$$\alpha^2=(m^2/h^2)[1-(t/t_0)^{-2m^2/n^2}]^{-1} , \quad (4.23)$$

where t_0 (>0) is a constant of integration that can be identified with the initial time of validity of the model. Note that $\alpha \to \infty$ as $t \to t_0$ and $\alpha^2 \to m^2/h^2$ as $t \to \infty$. This implies that $m^2 \geq h^2$ and hence, from Eq. (4.22),

$$a \leq \tfrac{1}{2} . \quad (4.24)$$

Using Eq. (4.23), we find that $\dot{\beta}/\beta \to -\infty$ as $t \to t_0$, so the condition (2.29) implies that $\dot{T}/T \to \infty$ as $t \to t_0$ and the temperature is initially increasing. Furthermore, a simple analysis using Eqs. (2.29) and (4.23) shows that T is of the form $T=\beta^{-1}F(t)$, where $F(t_0)$ is finite, so that $T=0$ when $t=t_0$. We reject this solution as unsatisfactory since it represents a universe expanding out of an initial radiation state with zero temperature.

The only remaining possible solution of Eq. (4.19) is

$$\alpha=mh^{-1}=\text{const} . \quad (4.25)$$

Since this implies again that $m \geq h$, it follows that Eq. (4.24) holds in this case also.

Using Eq. (4.14), condition (4.12) becomes

$$1-a-b>0 , \quad (4.26)$$

and the inequalities (4.7)—(4.10) and (4.13) satisfied by A_0 are

$$\begin{aligned} &A_0{}^2 < 2(a+b-a^2) , \\ &A_0{}^2 < 2a+b-2a^2+ab-b^2 , \quad A_0{}^2 \leq b , \\ &A_0{}^2 \leq 2a+2b-2a^2-ab-b^2 , \\ &A_0{}^2 \geq b(1-a-b) , \end{aligned} \quad (4.27)$$

and the first four of these are compatible with the fifth provided that $a \geq 0$ and that the dominant energy condition holds, i.e., the conditions (1.8) with Eq. (4.14).

The values of a and b corresponding to physically acceptable models satisfying all necessary energy and positivity conditions are represented by the region enclosed by the quadrilateral $OABC$ in Fig. 2. The boundary OA ($a=0$) is excluded since this leads to α being infinite at all times; the boundary AB ($a+b=1$) is excluded since, from Eqs. (4.21), (4.22), and (4.25), $A_0=0$ along this line. The boundary BC ($a=\frac{1}{2}$) corresponds to $\alpha=1$, $\beta=0$, so a solution with comoving velocity does exist; in this case the heat conduction vector and the projection of the temperature gradient orthogonal to the preferred observer are zero but the thermal conductivity is finite and nonzero, as in the case of the solution given by Eq. (3.17). The boundary CO ($b=0$) is excluded since this corresponds to $A_0=0$. The arc OB is an arc of the circle $a^2+b^2-a=0$; this corresponds to pure radiation models, i.e., models with $\gamma=\frac{1}{3}$. The models are excluded since they require

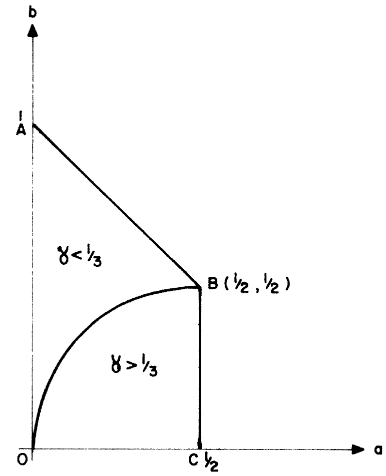

FIG. 2. Models with a magnetic field orthogonal to the direction of tilt are represented by the interior of the quadrilateral $OABC$ together with the open line BC but excluding the circular arc OB.

ρT^{-4} to be constant always, a requirement that does not satisfy the condition (2.29) unless $a \geq \frac{1}{2}$. It follows that the only radiation model in the allowable region is the one represented by the point B, i.e., $a=b=c=\frac{1}{2}$, which is the radiating FRW model and so is excluded. Hence, the allowable models are those represented by the interior of the quadrilateral $OABC$ plus the open line BC but excluding the circular arc OB. The models represented by the regions $OABO$ and $OBCO$ have equations of state $p=\gamma\rho$ with $\gamma<\frac{1}{3}$ and $\gamma>\frac{1}{3}$, respectively, where γ is a constant for all t.

To illustrate these solutions we choose a model represented by a point in the region $OABO$ so that $\gamma<\frac{1}{3}$. Our choice is

$$a=\tfrac{1}{4}, \quad b=\tfrac{1}{2} , \quad (4.28)$$

so that the metric of the model is

$$ds^2=-dt^2+t^{1/2}dx^2+t\,dy^2+t^{3/2}dz^2 . \quad (4.29)$$

The conditions (4.27) lead to $\frac{1}{8}<A_0{}^2<\frac{1}{2}$ and, for the purposes of our example, we choose

$$A_0{}^2=\tfrac{1}{4} . \quad (4.30)$$

With this choice, Eqs. (4.21) and (4.22) give $h^2=\frac{1}{8}$, $m^2=\frac{1}{4}$, so that

$$\alpha=2^{1/2}, \quad \beta=1 , \quad (4.31)$$

and the complete solution is

$$\rho=\tfrac{21}{16}t^{-2}, \quad p=\tfrac{19}{48}t^{-2}, \quad \gamma=\tfrac{19}{63} , \quad (4.32)$$

$$\eta=\tfrac{1}{8}t^{-1}, \quad n=n_0 2^{-1/2}t^{-5/4}, \quad |\mathbf{B}|=2^{-1/2}t^{-3/2} , \quad (4.33)$$

$$Q=t^{-2}, \quad T=T_0 t^{-r}, \quad \kappa=T_0{}^{-1}(\tfrac{1}{4}-r)^{-1}t^{-1+r} , \quad (4.34)$$

where $0<r<\frac{1}{4}$.

Finally, we note that if A_0 takes its minimum possible value which, from Eq. (4.27), is

$$A_0{}^2=b(1-a-b)\ , \tag{4.35}$$

then $\eta=0$ and the model represents a magnetohydrodynamic fluid with heat conduction. As an example of this situation consider the metric used in the previous example, i.e., given by Eqs. (4.28) and (4.29). In this case we find that $A_0{}^2=\frac{1}{8}$ and the complete solution is

$$\alpha^2=\tfrac{7}{5},\ \ \beta^2=\tfrac{2}{5}\ , \tag{4.36}$$

$$\rho=\tfrac{3}{4}t^{-2},\ \ p=\tfrac{1}{8}t^{-2},\ \ \gamma=\tfrac{1}{6}\ , \tag{4.37}$$

$$Q=\tfrac{1}{4}(\sqrt{7/2})t^{-2},\ \ T=T_0t^{-r}\ ,$$
$$\kappa=\tfrac{1}{8}\sqrt{14}(\tfrac{1}{4}-r)^{-1}T_0{}^{-1}t^{-1+r}\ , \tag{4.38}$$

$$n=n_0(\tfrac{5}{7})^{1/2}t^{-5/4},\ \ |\mathbf{B}|=0.418t^{-3/2}\ . \tag{4.39}$$

V. CONCLUSION

We have shown that the simple class of Bianchi type-I universes with metric (1.13) contains many possible models which represent viscous fluids with or without an axial magnetic field which may be either parallel to or orthogonal to the spacelike direction of the tilting velocity. All of these models satisfy appropriate thermodynamic conditions and satisfy the dominant energy condition at all times. Some of these models expand out of an initial radiation state into a matter-dominated state, others have a constant value for the ratio p/ρ, while still others describe an ultrarelativistic state; i.e., the models cover virtually the entire gamut of possibilities of the equation of state between p and ρ.

Contrary to the findings of Belinskii and Khalatnikov,[15] whose investigations of type-I models led to models which, near the initial state, possess a material energy density which vanishes and later increases, all of our models have a material energy density which is always decreasing. In most cases this density is infinite at the initial time $t=0$, but when a magnetic field is present some models are valid only for $t\geq t_0>0$, in which case the material energy density is finite at the initial time $t=t_0$.

The most notable difference between the investigation of zero-curvature FRW models satisfying the viscous magnetohydrodynamic field equations[36] and the present investigation is the existence of type-I models in which a magnetic field is orthogonal to the spatial component of the velocity vector. For such models the (2,2) and (3,3) components of the field equations are not identical so we have an additional independent equation which enables us to find the velocity components, rather than by simply choosing a suitable function. Unlike the FRW models, these type-I models are not, in general, reinterpretations of perfect-fluid models, the sole exception being the Kasner model which was found in Sec. II. They are thus the first known solutions of Eqs. (1.1)–(1.8) which are uniquely viscous magnetohydrodynamic models.

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APPENDIX

The volume expansion Θ, shear tensor $\sigma_\mu{}^\nu$, and acceleration four-vector a^μ, corresponding to a unit timelike vector u^μ are defined as

$$\Theta=u^\mu{}_{;\mu}\ , \tag{A1}$$

$$\sigma_\mu{}^\nu=\tfrac{1}{2}h_\mu{}^\alpha h^{\nu\beta}(u_{\alpha;\beta}+u_{\beta;\alpha})-\tfrac{1}{3}h_\mu{}^\nu\Theta\ , \tag{A2}$$

$$a^\mu=u^\mu{}_{;\nu}u^\nu\ . \tag{A3}$$

For the metric (1.13) and the four-velocity given by Eq. (2.2), i.e.,

$$u^\mu=(\alpha,\beta t^{-a},0,0)\ ,$$

Θ and the nonzero components of $\sigma_\mu{}^\nu$ and a^μ are given by

$$\Theta=\dot\alpha+(a+b+c)\alpha t^{-1}\ , \tag{A4}$$

$$\sigma_0^0=-\tfrac{1}{3}\beta^2X,\ \ \sigma_0{}^1=-\tfrac{1}{3}\alpha\beta t^{-a}X\ ,$$
$$\sigma_1{}^1=\tfrac{1}{3}\alpha^2X,\ \ \sigma_2{}^2=\sigma_3{}^3=-\tfrac{1}{6}X\ , \tag{A5}$$

where X is given by Eq. (2.9), and

$$a^0=\beta(\dot\beta+a\beta t^{-1}),\ \ a^1=\alpha t^{-a}(\dot\beta+a\beta t^{-1})\ . \tag{A6}$$

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