Quantum stress-energy tensors and the weak energy condition

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Certain processes predicted by quantum field theory, such as the Hawking black-hole evaporation process and radiation by moving mirrors, involve stress-energy tensors which exhibit peculiar properties from the classical point of view. More specifically, these stress-energy tensors do not obey the weak energy condition because they involve negative-energy densities and we show that, as a result, they are nondiagonalizable by a local Lorentz transformation under certain circumstances. In addition, we show that $T_{ab}U^aU^b$ is not bounded below for all unit timelike vectors U^a and that this is also a property of the stress-energy tensor associated with the Casimir effect. These observations are important in view of the fact that Tipler has shown that if T_{ab} is diagonalizable (type I) and if $T_{ab}U^aU^b$ is bounded below, then the weak energy condition is the weakest energy condition that can be defined locally. One might conjecture that the existence of similar (although as yet unknown) quantum processes, in which the weak energy condition is violated locally, could prevent the eventual formation of a singularity in the gravitational collapse of a star. Although we do not present a specific model, it is possible that in such a process the weak energy condition, while violated locally, would still hold on the average. Extending earlier results of Tipler, we show that Penrose's singularity theorem will still hold if the weak energy condition is replaced by a weaker (nonlocal) energy condition and if the null generic condition holds.

I. INTRODUCTION

Some of the most important results of the general theory of relativity in the past two decades have been the development of the singularity theorems, due most notably to Hawking, Penrose, and Geroch. These theorems predict the occurrence of singularities in spacetime under very general and physically reasonable conditions. This is an important prediction since the presence of a singularity indicates the breakdown of the structure of spacetime as we understand it, and hence signals the breakdown of all known physical laws as well.

A central assumption in every singularity theorem is some type of "energy condition." The function of such a condition in a singularity theorem is to ensure focusing of timelike or null geodesics. Once convergence has been initiated in bundles of geodesics, the energy condition guarantees the subsequent formation of conjugate points in the geodesic bundle, a conjugate point being a point where infinitesimally neighboring geodesics intersect. The existence of conjugate points, together with some additional assumptions about the global causal properties of the spacetime, can then be shown to contradict the assumed geodesic completeness of the spacetime, thus implying the presence of a singularity.

The *weak energy condition* requires that $T_{ab}U^aU^b \geq 0$ for every timelike vector U^a. By continuity, this inequality will also hold when U^a is a null vector. This requirement implies that the energy density as measured by any observer is non-negative. It is also a sufficient condition for the focusing of null geodesics. A sufficient condition for the focusing of timelike geodesics is the *strong energy condition* which requires that $(T_{ab}-\frac{1}{2}g_{ab}T)U^aU^b \geq 0$ for every timelike vector U^a. Again, by continuity, the condition will hold when U^a is a null vector. This inequality implies that gravity is always an attractive force. [For further details, see Hawking and Ellis[1] (HE).]

Since the weak and strong energy conditions are sufficient but not necessary requirements for the formation of conjugate points, one can ask whether conjugate points will still develop with milder restrictions on the stress-energy tensor. This question has been extensively investigated by Tipler.[2,3] In particular, Tipler has shown that a complete causal (i.e., timelike or null) geodesic $\gamma(t)$ will have a pair of conjugate points if the following inequality is satisfied:[3]

$$\int_{-\infty}^{\infty} F(t)dt > 0 \tag{1}$$

for

$$F(t)=(1/n)(R_{ab}U^aU^b+2\sigma^2),$$

where U^a is the tangent vector to the geodesic, t is an affine parameter along $\gamma(t)$, $n=2$ for null geodesics and $n=3$ for timelike geodesics, and σ is the shear (see Refs. 1 and 2 for details). The function $2\sigma^2$ is non-negative. For null geodesics $2\sigma^2 \equiv \sigma_{ij}\sigma^{ij}$ where $i,j=1,2$ label the two spacelike directions of a pseudo-orthonormal frame parallel propagated along $\gamma(t)$. For timelike geodesics $2\sigma^2 \equiv \sigma_{kl}\sigma^{kl}$ where $k,l=1,2,3$ label the three spacelike directions of an orthonormal frame parallel propagated along $\gamma(t)$.

The inequality (1) will hold, as Tipler[2] shows, if the following conditions are satisfied: (a) $\int_{-\infty}^{\infty} R_{ab}U^aU^b dt \geq 0$ along every complete causal geodesic $\gamma(t)$, equality holding only if $R_{ab}U^aU^b=0$ over the entire history of $\gamma(t)$;

(b) every causal geodesic contains a point for which $U^c U^d U_{[a} R_{b]cd[e} U_{f]} \neq 0$ (i.e., the "generic" condition holds).

The inequality in condition (a) may be rewritten using the Einstein equations as

$$\int_{-\infty}^{\infty} (T_{ab} - \tfrac{1}{2} g_{ab} T) U^a U^b dt \geq 0 .$$

Therefore condition (a) is equivalent to the statement that the strong energy condition holds on the average, where the average is taken over the entire history of a causal geodesic. Tipler[2] uses this "averaged strong energy condition" to prove several new singularity theorems, most notably the following.

(i) Spacetime $(\mathscr{M}, g)$ is not timelike and null geodesically complete if conditions (a) and (b) hold, there are no closed timelike curves, and there exists a compact achronal set without edge (i.e., the universe is "closed").

(ii) The Hawking-Penrose theorem (the most general of the singularity theorems) will still hold if the strong energy condition is replaced with the assumption that the weak energy condition holds everywhere and the strong energy condition holds on the average.

In considering whether the *weak* energy condition can be violated, Tipler proves the following proposition.

Proposition 4 (Tipler[2]). If $T_{ab} U^a U^b$ is bounded below for all unit timelike vectors U^a in T_p and if T_{ab} is type I, then $T_{ab} K^a K^b \geq 0$ at p for all null vectors K^a in T_p. (T_p is the set of all tangent vectors at a point p in $\mathscr{M}$.)

The assumption that T_{ab} is type I means that at each point p there is an orthonormal frame $\mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ for which the tensor takes the form (HE, p. 89)

$$T_{ab} = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{bmatrix} , \qquad (2)$$

where μ denotes the energy density and the p's denote the principal pressures. The stress-energy tensors of all known classical fields are type I, except for those of zero-rest-mass fields when they represent radiation *all* of which is traveling in a *single* direction (HE-type II, pp. 89 and 90). For a type-I stress tensor, the T_{10} "flux" term can always be made to vanish by an appropriate choice of local Lorentz transformation (i.e., there is always some physical observer who sees no net energy flux in any direction).

Tipler's proposition 4 essentially says that if T_{ab} has the form of most known classical fields and if $T_{ab} U^a U^b$ is bounded below for all unit timelike vectors, then the weak energy condition still holds for all null vectors, even though it may be violated for some timelike vectors. The function of the weak energy condition in singularity theorems is to ensure the focusing of *null* geodesics. Therefore, any singularity theorem which uses the weak energy condition to prove null geodesic incompleteness will still hold if the condition is violated for some timelike vectors, provided that proposition 4 holds. Tipler concludes that the weak energy condition is the weakest energy condition that can be defined locally.

Both the weak and strong energy conditions are local conditions in that they are defined at a point in spacetime. By contrast, the "averaged strong energy condition" is a global condition in the sense that it is defined over the entire length of a complete causal geodesic. The physical implication of the singularity theorems proved by Tipler using this condition is that a small localized violation of the strong energy condition is insufficient to prevent the occurrence of a singularity.

In this paper we extend the results of Tipler by defining an analogous "averaged weak energy condition," and show that Penrose's singularity theorem[1,4] will still hold if the weak energy condition is replaced by the averaged weak energy condition and the null generic condition. Our motivation for considering a weaker energy condition is the fact that there are now known processes, predicted by quantum field theory, which violate the weak energy condition. One such process is the Hawking black-hole evaporation.[5] Calculations of the vacuum expectation value of the stress-energy tensor $\langle T_{ab} \rangle$, representing a massless scalar field, in the vicinity of the horizon[6,7] indicate a "flux" of negative energy across the horizon which accounts for the decrease in the horizon surface area and the (presumed) eventual disappearance of the black hole. Candelas[7] has shown that $\langle T_{ab} \rangle$ is well behaved on the future horizon (in the "Hartle-Hawking" and "Unruh" vacuum states[7]), and from this fact he has computed the area decrease of a four-dimensional Schwarzschild black hole to first order. He found that the area of the black hole decreases at the rate expected from the magnitude of the corresponding positive-energy flux at infinity. Although the Hawking process is negligible for stellar mass holes, it is expected to be very significant for mini black holes. Another, somewhat related, process is the radiation of negative energy from moving mirrors.[8–10] Furthermore, the stress-energy tensor $\langle T_{ab} \rangle$ for each of these processes does not obey *either* of the conditions in Tipler's proposition 4: it is not type I and $\langle T_{ab} \rangle U^a U^b$ is not bounded below for all unit timelike vectors. The latter property is also a feature of the stress-energy tensor associated with the (experimentally verified) Casimir effect.

In light of these observations, it is of interest to see if singularities will still occur with a weaker energy condition than the weak energy condition. (For a slightly different approach to this problem, see Roman and Bergmann.[11]) The paper is organized as follows. In Sec. II we argue that $\langle T_{ab} \rangle$ for a generic spherically symmetric evaporating black hole is type IV (HE, p. 90); i.e., it has no timelike or null eigenvector and it cannot be diagonalized by a local Lorentz transformation. We also show that $\langle T_{ab} \rangle$ for a two-dimensional moving mirror is type II and may violate the weak energy condition under certain circumstances. In Sec. III we prove that for type-IV stress tensors, type-II stress tensors (representing zero-rest-mass fields) which violate the weak energy condition, and the stress tensor associated with the Casimir effect, $T_{ab} U^a U^b$ cannot be bounded below for all unit timelike vectors U^a. This indicates that one should consider a weaker (nonlocal) energy condition.

It is conceivable that there may exist other as yet unknown processes, with properties similar to those discussed in this paper, in which local violations of the weak energy condition could inhibit the formation of a singularity in a collapsing star. Although we do not propose a specific model of such an exotic quantum process, we prove in Sec. IV that the singularities predicted by Penrose's theorem will still occur if there exists a closed trapped surface $\mathcal{T}$ (HE, p. 262) for which the weak energy condition holds on the average along each complete *null* geodesic that generates the boundary of the future of $\mathcal{T}$ and if the "null generic condition" holds (HE, p. 101). Thus, local violations of the weak energy condition would be insufficient to prevent the formation of the singularity, provided that the "averaged" weak energy condition was satisfied in such a hypothetical quantum process.

We will work in units of $G=c=1$ and our metric signature is chosen to be $(-,+,+,+)$. Latin indices range over 0,1,2,3 unless otherwise noted.

II. NONSTANDARD STRESS-TENSOR TYPES

In a paper by Roman and Bergmann[11] (RB), a model representing a collapsing spherical cloud of matter was constructed. This model contained a region of trapped surfaces which formed and subsequently disappeared due to a violation of the weak energy condition (WEC). The metric was taken to be

$$ds^2=-2F(u,v)du\,dv+r^2(u,v)d\Omega^2$$

where $d\Omega^2=d\theta^2+\sin^2\theta\,d\varphi^2$, $r=r(u,v)$ is the luminosity radius, and u,v are null coordinates. Since the goal of RB was to construct a model representing a collapsing "star" which was everywhere singularity-free, the following assumptions were made about the functions F and r. F was taken to be an undetermined function of u,v; the only *a priori* requirements were that $F>0$ and finite everywhere, and that F be C^2. The luminosity radius $r(u,v)$ was taken to be a nonvanishing positive C^2 function of u and v (except at the center where $r=0$).

In Sec. III of RB, it was shown that $T_{vv}<0$ in a neighborhood of the outer boundary of the region containing trapped surfaces (i.e., the apparent horizon). This violation of the WEC along outgoing null rays was unavoidable in the sense that it was shown to be independent of the function F and its derivatives, arising *solely* from the fact that on the apparent horizon the $r=$const curves underwent a transition from spacelike to timelike. (See RB for details.) Therefore, we would expect the same conclusion to hold, i.e., $T_{vv}<0$, in any spherically symmetric spacetime where a transition of this type occurs.

Let us now consider the Hawking black-hole evaporation process. If we assume that a generic spherically symmetric evaporating black hole (SEBH) is correctly represented by a Penrose diagram of the form given in Fig. 1, where the dashed curve $\mathscr{A}$ is the apparent horizon, then the $r=$const curves (indicated by the solid wavy lines) would have the form shown in the figure. The heavy solid line $\mathscr{E}$ is the event horizon; the double solid line represents the singularity at $r=0$.

If the black hole disappears in a finite time (assuming that there is no "Planck mass remnant" left), it is difficult

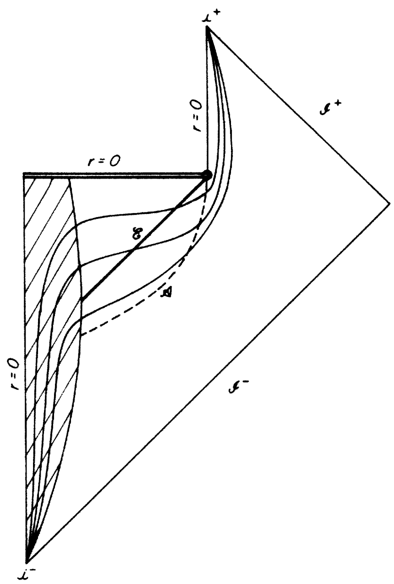

FIG. 1. Penrose diagram for a generic spherically symmetric evaporating black hole.

to see what possible relationship the horizon, the apparent horizon, and the $r=$const curves could have to one another, other than that shown in Fig. 1. Since the $r=$const curves inside the black hole must eventually "join up" at timelike infinity i^+, there must be a region where they undergo a transition from spacelike to timelike. This transition occurs along the apparent horizon $\mathscr{A}$, where the $r=$const curves become null. It seems, therefore, that for a generic SEBH, the apparent horizon always lies outside the event horizon.[12,13] From the results of Sec. III of RB, T_{vv} must be negative in a neighborhood of the apparent horizon of a generic SEBH, provided, of course, that the Penrose diagram resembles that of Fig. 1. (A similar argument was first given by Novikov.[14])

The difference between the SEBH and the RB model is that in the latter, the region of trapped surfaces is located entirely inside the body of the collapsing matter cloud, whereas in the former, this region extends into the vacuum outside the collapsing star. Therefore, in the discussion of the SEBH, when we refer to the component T_{vv} or to T_{ab} in general, we implicitly mean $\langle T_{vv}\rangle$, $\langle T_{ab}\rangle$, i.e., the renormalized vacuum expectation values of these quantities in a suitable vacuum state. $\langle T_{ab}\rangle$ is expected to satisfy the semiclassical Einstein equations

$$G_{ab}=8\pi\langle T_{ab}\rangle\ .$$

In the semiclassical theory, the spacetime geometry is treated classically and an effective stress-energy $\langle T_{ab}\rangle$ is assigned to the created particles which acts as a source of the gravitational field. Unfortunately, no four-dimensional expression for $\langle T_{ab}\rangle$ is known for a nonstatic SEBH, although some progress has been made in two-dimensional calculations.[6,12,13]

We now consider the issue of stress-tensor type. In Sec. IV of RB, it was shown that for a stress tensor of form

$$T_{ab}=\begin{bmatrix} T_{00} & T_{10} & 0 & 0 \\ T_{10} & T_{11} & 0 & 0 \\ 0 & 0 & T_{22} & 0 \\ 0 & 0 & 0 & T_{33} \end{bmatrix}, \tag{3}$$

a necessary (though not sufficient) condition for diagonalization by a local Lorentz transformation is

$$4{T_{10}}^2 \le (T_{00}+T_{11})^2 . \tag{4}$$

An equivalent condition, written in terms of null coordinates is

$$T_{uu}T_{vv} \ge 0 . \tag{5}$$

For the case of the SEBH, since $T_{vv}<0$ (we will hereafter omit the expectation value signs) in a neighborhood of the apparent horizon, the issue of diagonalizability rests on the sign of T_{uu} in this region.

It would seem plausible that outside the collapsing matter in a neighborhood of $\mathscr{A}$, $T_{uu}>0$, if the black hole is seen to produce particles at infinity. This conjecture is supported by two-dimensional calculations of the vacuum expectation value of the stress-tensor operator for a massless scalar field propagating in a Vaidya spacetime. (The two-dimensional Vaidya spacetime represents the simplest nonstatic generalization of the two-dimensional Schwarzschild spacetime. See Balbinot,[15] and Balbinot and Brown,[16] and references therein.) Of course, the main drawback of the two-dimensional calculations is that there are no Einstein equations in two dimensions, so one cannot tie the calculated expectation values of T_{ab} back to the spacetime geometry. However, these results do suggest general properties of T_{ab}, such as those discussed above, that one might expect to hold in four dimensions also.

If $T_{vv}<0$ and $T_{uu}>0$ in a neighborhood of $\mathscr{A}$ then Eq. (5) cannot be satisfied. Classically speaking, there exist no local observers in this region who see zero energy flux. [For the slightly different case of a massless scalar field propagating on a two-dimensional Schwarzschild (nondynamic) background, this property of T_{ab} was first noticed by Fulling.[17]] However, one should be cautious about interpreting $\langle T_{ab}\rangle$ as an actual flow of matter; $\langle T_{ab}\rangle$ represents probabilistic information about the outcomes of certain idealized experiments.[17]

For a stress tensor given by Eq. (3), it was shown in Sec. IV of RB that if $T_{uu}>0$, $T_{vv}<0$ and $T_{00}>0$, $T_{11}>0$ in some local Lorentz frame then T_{ab} is nondiagonalizable and there will also exist local Lorentz frames in which the energy density T_{00} is negative. By continuity, there will also exist local Lorentz frames in which the energy density is zero. In such a frame, the contravariant components of the stress-energy tensor will have the form

$$T^{ab}=\begin{bmatrix} 0 & \nu & 0 & 0 \\ \nu & -\kappa & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix}, \tag{6}$$

where

$$\kappa^2<4\nu^2 . \tag{7}$$

This is the form of stress-energy tensor known as type IV; it has no timelike or null eigenvector and it does not obey the WEC (see HE, p. 90). The argument of RB also applies for the SEBH, so T_{ab} is type IV in that case as well.

To summarize, we have shown that if a generic SEBH is correctly described by a Penrose diagram like Fig. 1, then $T_{vv}<0$ in a neighborhood of the apparent horizon. If, in addition, $T_{uu}>0$ in this region, then T_{ab} is type IV. The conclusion that $T_{vv}<0$ and $T_{uu}>0$ in a neighborhood of the apparent horizon is also supported by two-dimensional calculations of T_{ab} for a massless scalar field in various model BH spacetimes.

We now consider the two-dimensional moving-mirror model. Fulling and Davies[8] have shown that for a mirror traveling along the trajectory

$$x=0,\quad t<0$$
$$=z(t),\quad t>0$$

in Minkowski spacetime with the state of the massless scalar field chosen to correspond to the vacuum in the past ($t<0$), the renormalized vacuum expectation values of T_{ab} are given by

$$\begin{aligned} T_{00}=T_{11}&=-T_{10} \\ &=-T_{01} \\ &=-\frac{1}{12\pi}\frac{(1-\dot z^2)^{1/2}}{(1-\dot z)^2}\frac{d}{d\tau_u}\left[\frac{\ddot z}{(1-\dot z^2)^{3/2}}\right]. \end{aligned} \tag{8}$$

The parameter τ_u is the time coordinate of the mirror when its trajectory intersects the retarded null ray u ($u=t-x$, $v=t+x$). The right-hand side of (8) may be rewritten as

$$-\frac{1}{12\pi}\frac{(1-V^2)^{1/2}}{(1-V)^2}\frac{d\alpha}{d\tau_u}, \tag{9}$$

where $V=\dot z$ and $\alpha=\ddot z/(1-\dot z^2)^{3/2}$ is the acceleration in the instantaneous rest frame of the mirror. The expressions (8) and (9) apply to the right of the mirror. We see that if the acceleration increases with time, to the right, the mirror radiates negative energy—a violation of the WEC. The same will be true if the acceleration decreases to the left.

The stress tensor (8) has the form

$$T_{ab}=\begin{bmatrix} \nu & -\nu \\ -\nu & \nu \end{bmatrix}. \tag{10}$$

This is a two-dimensional example of a type-II stress-energy tensor. The general expression for the contravariant components of a type-II stress-energy tensor is given by

$$T^{ab}=\begin{bmatrix} \nu+\kappa & \nu & 0 & 0 \\ \nu & \nu-\kappa & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix}. \tag{11}$$

(Compare with HE, pp. 89 and 90, and Landau and Lifshitz,[18] pp. 273 and 274.) Equation (10) is the covariant two-dimensional form of (11) for a zero-rest-mass field representing radiation all of which is traveling in the $\mathbf{E}_0+\mathbf{E}_1$ direction. In this case, p_1, p_2, and κ are zero. For (10), ν will be negative when the mirror is emitting negative energy, i.e., when expression (9) is negative.

The question of whether such negative-energy fluxes can violate the second law of thermodynamics has been investigated by Ford and Davies,[9] and by Deutsch, Ottewill, and Sciama.[19] It should also be noted that the existence of negative-energy "acceleration" radiation from moving mirrors is required in the "black-hole mining" process of Unruh and Wald[10] in order to *prevent* a violation of the generalized second law of thermodynamics.

III. IS THE WEC VIOLATION BOUNDED BELOW?

In this section we will show that the condition that $T_{ab}U^aU^b$ be bounded below for all unit timelike vectors cannot be satisfied by type-IV stress tensors, certain kinds of type-II stress tensors which violate the WEC, and the stress tensor associated with the Casimir effect.

Proposition 1. If T_{ab} is type IV, then $T_{ab}U^aU^b$ cannot be bounded below for all unit timelike vectors U^a.

Proof. Since T_{ab} is type IV, then there is an orthonormal frame $\mathbf{E}_0,\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3$ with T_{ab} represented in covariant form by

$$T_{ab}=\begin{bmatrix} 0 & -\nu & 0 & 0 \\ -\nu & -\kappa & 0 & 0 \\ 0 & 0 & p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix}, \tag{12}$$

where $\kappa^2<4\nu^2$. Note that Eq. (7) is the condition that T_{ab} be *nondiagonalizable* by a local Lorentz transformation; i.e., it is the opposite of Eq. (4). If we perform a Lorentz boost in the $\mathbf{E}_1$ direction, we obtain a new unit timelike vector $(\gamma,-\gamma\beta,0,0)$ which satisfies

$$T_{ab}U^aU^b=\gamma^2\beta(2\nu-\beta\kappa)\,, \tag{13}$$

where β is the usual special-relativistic velocity parameter and $\gamma=[1/(1-\beta^2)^{1/2}]$, $\gamma^2\geq 1$, while $1>\beta>-1$. There are two cases: $\beta>0$ and $\beta<0$. For $\beta>0$, Eq. (13) is bounded below only if $2\nu\geq\beta\kappa$ or in the limiting case $2\nu\geq\kappa$ (since β can be arbitrarily close to 1). For $\beta<0$ Eq. (13) will be bounded below only if $2\nu\leq-\kappa$ (since for $\beta<0$, β can be arbitrarily close to -1). Since (13) must be bounded below irrespective of the sign of β, the only way that both conditions can be satisfied is if $-\kappa\geq 2\nu\geq\kappa$, which implies that $\kappa<0$. For T_{ab} to be type IV, the nondiagonalizability condition (7) must also be satisfied. Now we ask: Can $-\kappa\geq 2\nu\geq\kappa$ and $\kappa^2<4\nu^2$ simultaneously hold? Again there are two cases: $\nu>0$, $\nu<0$. For $\nu>0$, $\kappa<0$ then $-\kappa\geq 2\nu$ (since $2\nu\geq\kappa$ always), which implies $|\kappa|\geq|2\nu|$ or $\kappa^2\geq 4\nu^2$. For $\nu<0$, $\kappa<0$, then $2\nu\geq\kappa$ (since $-\kappa\geq 2\nu$ always), which implies $|\kappa|\geq|2\nu|$ or $\kappa^2\geq 4\nu^2$. Thus our conclusion is that if T_{ab} is type IV, then the condition that $T_{ab}U^aU^b$ be bounded below for all unit timelike vectors is incompatible with the nondiagonalizability requirement for type-IV stress tensors.

An explicit (although admittedly naive) example of a type-IV stress tensor[11] is

$$T_{ab}=\begin{bmatrix} 0 & -2\rho_0\beta\gamma^2 \\ -2\rho_0\beta\gamma^2 & 0 \end{bmatrix}. \tag{14}$$

This stress tensor represents two streams of oppositely moving noninteracting particles, with equal and opposite rest-mass densities, in a two-dimensional Minkowski spacetime. The stress tensor of each stream is given by

$$T_{ab}=\pm\rho_0\xi_a\xi_b$$

with $\xi_0=-\gamma$, $\xi_1=\pm\beta\gamma$ with $0<\beta<1$ and $\rho_0>0$. For (14) and the boosted unit timelike vector $(\gamma',-\gamma'\beta')$, Eq. (13) becomes

$$T_{ab}U^aU^b=(\gamma')^2\beta'(2\rho_0\beta\gamma^2)\,. \tag{15}$$

(This is the special case when $\kappa=0$.) If $\beta'>0$, then $\beta\geq 0$ for (15) to be bounded below. Conversely, if $\beta'<0$, then $\beta\leq 0$ for (15) to be bounded below. The only way for both conditions to hold is for $\beta=0$, which implies that T_{ab} vanishes. Therefore (15) is not bounded below, so proposition 1 is satisfied.

There is no noncausal propagation of information in this example, since the particles in each stream move with a velocity less than that of light, although the energy-momentum flux vector $S^a=T^{ab}U_b$ is spacelike ($S^2=4\rho_0{}^2\beta^2\gamma^4>0$). This peculiar state of affairs has arisen because we allowed the particle masses to have either sign. A similar situation occurs in electrodynamics[20] for the case of two streams of oppositely moving charges with equal and opposite charge densities. The four-current density of each stream is given by

$$j^a=\pm\rho_0U^a$$

with $U^0=\gamma$, $U^1=\pm\beta\gamma$ with $0<\beta<1$ and $\rho_0>0$, where here ρ_0 is the absolute value of the charge density in the rest frame of the charges. The total current four-vector J^a of the system is $J^a=(0,2\rho_0\beta\gamma)$, which is spacelike ($J^2=4\rho_0{}^2\beta^2\gamma^2>0$). This is possible because two kinds of charges exist in nature.

We should emphasize that nondiagonalizability of T_{ab} by a Lorentz transformation does not, in itself, imply a violation of the WEC. For example, the stress-energy tensor for electromagnetic radiation, all of which is moving in one direction, is nondiagonalizable, but obviously does not violate the WEC.

The covariant components of T_{ab} for a zero-rest-mass field representing radiation all of which is traveling in the $\mathbf{E}_0+\mathbf{E}_1$ direction (with respect to the orthonormal frame $\mathbf{E}_0,\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3$) and which violates the WEC, can be written in the form

$$T_{ab}=\begin{bmatrix} \nu & -\nu & 0 & 0 \\ -\nu & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ with } \nu<0\,. \tag{16}$$

(Compare with HE, pp. 89 and 90, and Landau and

Lifshitz,[18] pp. 273 and 274.) As we showed in Sec. II, T_{ab} for a moving mirror emitting negative energy is a two-dimensional example of (16).

Proposition 2. If T_{ab} is a type-II stress tensor with form given by (16), then $T_{ab}U^aU^b$ cannot be bounded below for all unit timelike vectors U^a.

Proof. With the stress-energy tensor given by (16) and the boosted unit timelike vector $(\gamma,-\gamma\beta,0,0)$, we obtain

$$T_{ab}U^aU^b=\nu\gamma^2(1+\beta)^2 . \tag{17}$$

Since $\nu<0$ and $\gamma^2\geq 1$ and since $-1<\beta<1$ implies that $(1+\beta)^2>0$, (17) cannot be bounded below.

The renormalized vacuum expectation value of the stress-energy tensor for the electromagnetic field between two neutral infinite parallel plane conductors is given by[21]

$$T^{ab}=\frac{\pi^2}{720a^4}\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -3\end{bmatrix} , \tag{18}$$

where a is the plate separation and the x^3 direction is taken perpendicular to the plates. This is the well-known (and experimentally verified) Casimir effect. Note that the energy density μ and the pressure component p_3 are both negative, so (18) obviously violates the WEC, although in practice the energy density is very small (for a plate separation of 0.5 μm, $T^{00}\sim -0.07$ erg/cm^3). A unit timelike vector which is Lorentz boosted in the α direction ($\alpha=1,2,3$) yields a new unit timelike vector which, for a type-I stress tensor, satisfies

$$T_{ab}U^aU^b=\gamma^2(\mu+\beta^2p_\alpha) . \tag{19}$$

If $\mu<0$, (19) will not be bounded below unless $p_\alpha\geq|\mu|$ (Tipler[2]). Equation (19) applied to the Casimir stress tensor (18) with the boosted unit timelike vector $(\gamma,0,0,-\gamma\beta)$ will be bounded below only if

$$p_3\geq|\mu| . \tag{20}$$

From (18), we see that $p_3=-3|\mu|$, so (20) is not satisfied. Consequently, for the Casimir stress tensor, $T_{ab}U^aU^b$ is not bounded below for all unit timelike vectors, even though T_{ab} is type I.

It therefore would seem that for quantum fields, we should not necessarily expect Tipler's "bounded below" condition to hold, whatever the type of the stress tensor.

Wald[22] has noted that a feature of the expectation value $\langle\psi|\hat{T}_{ab}|\psi\rangle$ of the quantum stress-energy operator $\hat{T}_{ab}$ in a state ψ is that it need not satisfy any of the energy conditions that may be satisfied by the classical stress-energy tensor. He points out that even in flat spacetime one can find states where the expectation value of the normal-ordered Klein-Gordon stress-energy operator has negative-energy density in a region of spacetime, even though the energy density of the classical stress-energy tensor of a Klein-Gordon field is manifestly positive definite everywhere for all field configurations. (However, the total energy $E=\int_\Sigma\langle\hat{T}_{ab}\rangle\xi^an^b$, where ξ^a is a time translation Killing field, is always non-negative for the free Klein-Gordon field in Minkowski spacetime.) Therefore, properties which hold in classical general relativity due to energy conditions satisfied by matter will not necessarily hold for quantum fields.

IV. A "WEAKER" WEAK ENERGY CONDITION?

The results of the last two sections suggest that one should seek a weaker energy condition than the WEC. It seems intuitively clear that such a condition should exist. Consider the following crude but illustrative example: imagine throwing a small "lump" of negative mass energy into a solar-mass black hole (we will ignore the problem of how one obtains such a "lump"). The presence of the lump constitutes a local violation of the WEC. However, it is highly unlikely that this small violation could eradicate the singularity inside the black hole; i.e., we would expect the singularity theorems to remain valid under these circumstances.

Penrose's singularity theorem assumes the existence of a *closed trapped surface,* defined to be a closed spacelike two-surface on which both the ingoing and outgoing null geodesics orthogonal to the surface are converging. The appearance of a trapped surface in the evolution of a collapsing star, for example, signals that gravitational collapse has reached a certain critical stage. If the WEC holds, then a conjugate point will form along every future-directed null geodesic orthogonal to the trapped surface, implying that the boundary of the future of the surface is compact. This situation is topologically incompatible with another assumption of the theorem, namely, that there exists a noncompact Cauchy surface in spacetime (i.e., "the Universe is open"). Hence, one arrives at a contradiction, which implies that spacetime cannot be null geodesically complete.

Singularity theorems such as Penrose's theorem, which use only the WEC, prove *null* geodesic incompleteness of spacetime. It should be possible to reach the same conclusion with a weaker restriction on the stress-energy tensor, provided that this condition still guarantees the existence of a conjugate point along every complete future-directed null geodesic orthogonal to a closed trapped surface. Since it is easily shown that the WEC is locally violated along some null vector in each of the quantum processes described earlier, the indications are that our proposed condition should be of global rather than a local character. We now extend the earlier results of Tipler.

Definition. A point p is said to be *conjugate* to a spacelike two-surface S along a null geodesic $\gamma(\lambda)$ which intersects S orthogonally if there exists along $\gamma(\lambda)$ a function $x(\lambda)$ with $x(p)=0$, and in addition $x(\lambda)$ everywhere satisfies the equation[2,3]

$$\frac{d^2x}{d\lambda^2}+F(\lambda)x=0 \tag{21}$$

and the initial conditions

$$x(0)=1,\ \left.\frac{dx}{d\lambda}\right|_{\lambda=0}=\chi^a_a$$

at the point $\gamma(0)=\gamma(\lambda)\cap S$. χ^a_a is the contraction of the second null fundamental form χ_{ab} of S (see HE, pp. 101 and 102). $F(\lambda)\equiv\frac{1}{2}(R_{ab}K^aK^b+2\sigma^2)$, where K^a is the tangent vector to the null geodesic and σ is the shear. [Equation (21) is essentially just an alternative way of writing the Raychaudhuri equation for null geodesics. See Ref. 2 for details.]

Theorem 1. Let $F(\lambda)$ be continuous on $[0,+\infty)$. If

$$\int_0^\infty F(\lambda)d\lambda>0 \tag{22}$$

along $\gamma(\lambda)$ and the initial conditions

$$x(0)=1,\quad \left.\frac{dx}{d\lambda}\right|_{\lambda=0}=\chi^a_a=-b<0 \tag{23}$$

are satisfied at the point $\gamma(0)=\gamma(\lambda)\cap S$, then there will be a point conjugate to S along $\gamma(\lambda)$ for some value of $\lambda\in[0,+\infty)$. Our theorem will still be true even if the integral in (22) does not converge, provided we regard (22) as a shorthand notation for

$$\liminf_{\lambda'\to+\infty}\int_0^{\lambda'}F(\lambda)d\lambda>0\ .$$

The proof of our theorem 1 is a trivial modification of the proof of theorem 2 of Tipler.[3]

The following theorem shows that if the WEC holds on the average along all the null geodesics which generate the boundary of the future of a closed trapped surface, then the singularities predicted by Penrose's theorem will still occur.

Theorem 2 (modified Penrose theorem). Spacetime $(\mathscr{M},g)$ cannot be null geodesically complete if (1) there is a closed trapped surface $\mathscr{T}$ in $\mathscr{M}$, (2) $\int_0^\infty R_{ab}K^aK^b d\lambda\geq 0$ along every complete null geodesic $\gamma(\lambda)$ orthogonal to $\mathscr{T}$, equality holding only if $R_{ab}K^aK^b\equiv 0$ at every point of $\gamma(\lambda)$ for $\lambda\in[0,+\infty)$ [K^a is the tangent vector to $\gamma(\lambda)$, λ is an affine parameter, and $\gamma(0)\equiv\gamma(\lambda)\cap\mathscr{T}$], (3) every null geodesic orthogonal to $\mathscr{T}$ contains a point $\gamma(\lambda_1)$, with $\lambda_1\in[0,+\infty)$, for which

$$K^cK^dK_{[a}R_{b]cd[e}K_{f]}\neq 0$$

(i.e., the null generic condition holds for every null geodesic in the boundary of the future of $\mathscr{T}$), (4) there is a noncompact Cauchy surface $\mathscr{H}$ in $\mathscr{M}$.

Note. $\int_0^\infty R_{ab}K^aK^b d\lambda\geq 0$ and the Einstein equations,

$$G_{ab}=R_{ab}-\tfrac{1}{2}g_{ab}R=8\pi T_{ab}\ ,$$

imply that $\int_0^\infty T_{ab}K^aK^b d\lambda\geq 0$, provided K^a is a null vector. Therefore, condition (2) implies that the WEC holds on the average along a null geodesic orthogonal to $\mathscr{T}$, where the average is taken over the history of the null geodesic to the future of $\mathscr{T}$.

Proof. Suppose $\mathscr{M}$ were null geodescially complete. Conditions (2) and (3) imply that $\int_0^\infty F(\lambda)d\lambda>0$ along each null geodesic orthogonal to $\mathscr{T}$. (See the proof of theorem 1 of Tipler.[2]) By the definition of a closed trapped surface, ${}_1\chi_{ab}g^{ab}$ and ${}_2\chi_{ab}g^{ab}$, the two second null fundamental forms of $\mathscr{T}$ are negative. Therefore, condition (1) and our theorem 1 imply that there will be a point conjugate to $\mathscr{T}$ along every complete future-directed null geodesic orthogonal to $\mathscr{T}$ for some value of $\lambda\in[0,+\infty)$. By proposition (4.5.14) of HE, points on such a null geodesic beyond the point conjugate to $\mathscr{T}$ would lie in $I^+(\mathscr{T})$. Thus each generating segment of $\dot{J}^+(\mathscr{T})$ would have a future end point at or before the point conjugate to $\mathscr{T}$. Penrose has shown (see Penrose[4] or HE, pp. 263 and 264) that this situation implies $\dot{J}^+(\mathscr{T})$ is compact and that compactness of $\dot{J}^+(\mathscr{T})$ is incompatible with condition (4). Therefore, $\mathscr{M}$ cannot be null geodesically complete.

Comment. The WEC, $T_{ab}U^aU^b\geq 0$ for all timelike vectors U^a, is a *local* condition. By continuity, $T_{ab}K^aK^b\geq 0$ for all null vectors K^a as well. If a global condition such as $\int_0^\infty T_{ab}U^aU^b d\lambda\geq 0$ is satisfied for all timelike geodesics, this does not allow us to conclude, *a priori*, that the condition also holds for all null geodesics. These are separate assumptions. We could say that the stress-energy tensor satisfies the "averaged weak energy condition" if $\int_0^\infty T_{ab}U^aU^b d\lambda\geq 0$ along every timelike or null geodesic $\gamma(\lambda)$, where U^a is the tangent vector to the geodesic and λ is an affine parameter along the geodesic. Note, however, that no restrictions on the behavior of timelike geodesics were required to prove *null* geodesic incompleteness. (In the original Penrose theorem, it is really only necessary to require that the WEC hold for all null vectors.)

The addition of the null generic condition (3) does not represent a drawback. Condition (3) roughly says that the spacetime $\mathscr{M}$ is "general enough" so that every future-directed null geodesic orthogonal to $\mathscr{T}$ encounters some "effective curvature" at least at one point in its history to the future of $\mathscr{T}$. This guarantees that the spacetime $\mathscr{M}$ is not algebraically special (for further details regarding the null generic condition, see HE, p. 101). Although condition (3) tends to fail for the known exact solutions, such as spherically symmetric collapse,[23] one must recall that the original purpose in developing singularity theorems was precisely to prove that singularities are an intrinsic feature of general relativity, i.e., that they will still occur when spacetime possesses no exact symmetries.

V. CONCLUSION

We have shown that the stress-energy tensors of various quantum processes, such as the Hawking black-hole evaporation and radiation by moving mirrors, do not share the usual properties of the stress-energy tensors associated with most known classical matter fields. In particular, we argued that the vacuum expectation value T_{ab} of the stress-energy tensor for a generic spherically symmetric evaporating black hole violates the WEC, and as a result is nondiagonalizable by a local Lorentz transformation, in the vicinity of the apparent horizon, i.e., the stress tensor is type IV in this region. It was also demonstrated that T_{ab} for the massless scalar radiation emitted by a two-dimensional moving mirror is type II and can violate the WEC for certain accelerations of the mirror. These stress tensors, as well as the stress tensor associated with the experimentally verified Casimir effect, were shown to have the peculiar property that $T_{ab}U^aU^b$ is not bounded below for all unit timelike vectors U^a.

Such unusual features are significant because a proposition of Tipler claims that the WEC is the weakest energy condition which can be defined locally provided that T_{ab} is type I and $T_{ab}U^aU^b$ is bounded below for all unit timelike vectors U^a. Tipler proves that if these two conditions are satisfied, then the WEC holds for all null vectors. (All of the quantum stress tensors mentioned earlier violate the WEC along *some* null vectors.) Thus Tipler's assumptions, although reasonable for most classical matter fields, cannot be expected to hold in general for quantum matter fields.

It is therefore important to try to prove singularity theorems using a weaker restriction on the stress-energy tensor than the WEC. Our motivation is the possibility that there could exist as yet unknown quantum processes, whose stress tensors have characteristics similar to the examples discussed in this paper, in which the WEC is locally violated during the gravitational collapse of a star. By extending earlier results of Tipler, we have shown that the singularities predicted by Penrose's theorem (which utilizes only the *weak* energy condition) will still occur if there exists at least one closed trapped surface in spacetime for which the WEC holds only on the average along every null geodesic making up the boundary of the future of that surface. Although we have not presented a specific model for a quantum process exhibiting these properties, the physical implication of our theorem is that once a trapped surface has formed in the gravitational collapse of a star, a small localized violation of the WEC is insufficient to prevent the subsequent formation of a singularity.

Most workers would agree that the prediction of spacetime singularities heralds a breakdown of the field equations of general relativity. Einstein himself was aware of the limitations of his own theory when, in referring to cosmology, he said: "For large densities of field and of matter, the field equations and even the field variables which enter into them will have no real significance. One may not therefore assume the validity of the equations for very high density of field and of matter, and one may not conclude that the 'beginning of the expansion' must mean a singularity in the mathematical sense. All we have to realize is that the equations may not be continued over such regions."[24] Of course, what might result in place of a singularity is still anybody's guess. Perhaps the answer lies in the long-awaited (and as yet unknown) quantum theory of gravity that will one day supercede general relativity.

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