Energy and Regge residues in quantum-mechanical "QCD" sum rules

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(Received 22 November 1985)

It was shown recently by Fishbane, Kaus, and Gasiorowicz that the residues at the poles of quantum-mechanical two-point functions for arbitrary angular momenta l have an incorrect l dependence when calculated by the sum-rule method used for the analogous problem in QCD. Knowledge of the residues is of interest since they are directly related to particle couplings and decay widths. We develop reliable expressions for the energy and Regge residues using semiclassical methods.

I. INTRODUCTION

Sum rules for two-point functions in QCD have been used extensively in recent years to relate such low-energy, nonperturbative quantities as bound-state energies and decay widths in hadronic systems to quantities calculated in perturbative QCD (Refs. 1 and 2). The results of these analyses have been remarkably good. The sum rules give a specific way of implementing the older idea of the duality between bound-state cross sections, e.g., the resonant cross section for $e^+e^-\to q\bar{q}$ bound states, and the cross sections calculated for free quarks in perturbative QCD (Ref. 3).

A number of authors have investigated the theoretical background of the duality relations and QCD sum rules by looking at the analogous sum rules in nonrelativistic quantum mechanics[4–7] or solvable model field theories.[8] For example, the quantum-mechanical energy-moment sum rules for e^+e^- annihilation into heavy confined $q\bar{q}$ states are easily derived starting with the observation that the nonrelativistic photon vacuum-polarization function $\Pi(E)$ is a multiple of the $q\bar{q}$ energy Green's function evaluated at the origin:[6]

$$G(\mathbf{r},\mathbf{r},E)\Big|_{\mathbf{r}'=\mathbf{r}=0}=\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{|R_{n,0}(0)|^2}{E_{n,0}-E}\ , \tag{1}$$

where $R_{n,l}(r)$ is the radial wave function for angular momentum l. The total e^+e^- annihilation cross section at a total center-of-mass energy $W=2m_q+E$ is

$$W^2\sigma_{e^+e^-}(E)=24\pi^2\alpha^2{e_q}^2{m_q}^{-2}\mathrm{Im}G(0,0,E)$$
$$=6\pi^2\alpha^2{e_q}^2{m_q}^{-2}\sum_{n=1}^{\infty}|R_{n,0}(0)|^2\delta(E-E_{n,0})\ . \tag{2}$$

Differentiating Eq. (1) with respect to E and using the relation in Eq. (2), we obtain the moment sum rules:

$$\frac{1}{N!}\left[\frac{d}{dE}\right]^N G(0,0,E)=\frac{1}{\pi}\int_0^{\infty}dE'\frac{\mathrm{Im}G(0,0,E')}{(E'-E)^{N+1}}$$
$$=\frac{{m_q}^2}{24\pi^3\alpha^2{e_q}^2}\int_0^{\infty}dE'\frac{W'^2\sigma_{e^+e^-}(E')}{(E'-E)^{N+1}}$$
$$=\frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{|R_{n,0}(0)|^2}{(E_{n,0}-E)^{N+1}}\ . \tag{3}$$

These relate weighted integrals of $\sigma_{e^+e^-}$ [or equivalently, integrals of $\mathrm{Im}G(0,0,E)$] to the sum of residues and energy denominators. The exponential or "Borel-transformed" sum rules used in many calculations can be obtained by calculating the inverse Laplace transform of $G(0,0,-E)$:

$$\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon}dE\,e^{E\tau}G(0,0,-E)$$
$$=\frac{{m_q}^2}{24\pi^3\alpha^2{e_q}^2}\int_0^{\infty}dE\,e^{-E\tau}W^2\sigma_{e^+e^-}(E)$$
$$=\frac{1}{4\pi}\sum_{n=1}^{\infty}|R_{n,0}(0)|^2e^{-E_{n,l}\tau}\ . \tag{4}$$

The sum rules can be used in either form to estimate the ground-state energy of the (nonrelativistic) $q\bar{q}$ system by following the procedures used in the QCD calculations. For example, if we choose $E=0$ in Eq. (3), $E_{1,0}$ is given exactly by the limit

$$\lim_{N\to\infty}M_N/M_{N+1}=E_{1,0}\ , \tag{5}$$

where

$$M_N=\frac{1}{N!}\left[\frac{d}{dE}\right]^N G(0,0,E)\Big|_{E=0}\ . \tag{6}$$

We can calculate the left-hand side of Eq. (4) approximately by calculating $\sigma_{e^+e^-}$ or $\mathrm{Im}G$ and the M's in low-order perturbation theory, and then estimate $E_{1,0}$ by find-

ing the range of N for which the ratio is "most stable." This technique is easily extended to higher values of l (Ref. 6).

The limitations on sum-rule calculations—and the reasons for their success in correlating data—have been discussed elsewhere.[6] It has been emphasized in particular in Refs. 6 and 7 that it is necessary to know the long-range part of the $q\bar{q}$ interaction explicitly before reliable results can be obtained for the energies. It does not suffice to know only a few terms in the perturbation series for M_N, as is usual in QCD calculations.

The problem of calculating the residues at the poles of $G(0,0,E)$ using sum rules was also investigated recently by Fishbane, Gasiorowicz, and Kaus,[7] who concluded that such plausible methods as using the exact limit

$$\frac{1}{4\pi}|R_{1,0}(0)|^2=\lim_{N\to\infty}E_{1,0}{}^{N+1}M_N,\quad E_{1,0}>0\ , \tag{7}$$

with low-order approximations for the M's are inadequate, especially when applied to large values of l. Since these residues are directly proportional to decay widths of the states, it is clearly desirable to have alternative methods to calculate them, even if the methods go beyond the sum-rule approach. One possibility is provided by the Fermi-Segrè formula[9] generalized to arbitrary l (Ref. 10) which gives the wave function at the origin in terms of the energy spectrum:

$$\lim_{r\to0}|r^{-l}R_{n,l}(r)|^2 =\frac{2m}{\pi}(2mE_{n,l})^{l+1/2}\left[\frac{2^l l!}{(2l+1)!}\right]^2\frac{dE_{n,l}}{dn}\ . \tag{8}$$

In investigating this approach, we found that Eq. (8) is quite inaccurate for large l for confining potentials; e.g., the errors grow exponentially with l for the usually innocuous oscillator potential (see Table I). This observation prompted us to develop several more accurate versions of the arbitrary-l Fermi-Segrè formula which have been reported elsewhere.[11,12] We apply our methods here to the calculation of the energy and Regge residues for confined systems. The results on the latter, and the approach used, are new.

II. CALCULATION OF ENERGY AND REGGE RESIDUES

A. Formulation of the problem

The radial part of the energy Green's function for a problem with a confining interaction can be written for arbitrary l as

$$G_l(r',r,E)=\sum_{n=1}^{\infty}R_{n,l}(r')R_{n,l}(r)/(E_{n,l}-E)\ , \tag{9}$$

where the sum runs over the totally discrete spectrum. The quantity which enters naturally in sum-rule calculations is a two-point function which we will write in the notation of Fishbane, Gasiorowicz, and Kaus[7] as

$$\Pi(\nu,E)=\lim_{\substack{r\to0\\r'\to0}}(r'r)^{-l}(2l+1)G_l(r',r,E) =\sum_{n=1}^{\infty}\frac{2\nu[a_n(\nu)]^2}{E_n(\nu)-E},\quad \nu=l+\tfrac{1}{2}\ , \tag{10}$$

where

$$a_n(\nu)=\lim_{r\to0}r^{-l}R_{n,l}(0),\quad \nu=l+\tfrac{1}{2}\ . \tag{11}$$

Alternatively, as shown in Ref. 7, Π may be written in terms of the "S matrix" $\hat{S}(\nu,E)$ for the confined system,

$$\Pi(\nu,E)=-2m\hat{S}(\nu,E)+\cdots\ , \tag{12}$$

where

$$\hat{S}(\nu,E)=\lim_{r\to\infty}\frac{u(-\nu,E,r)}{u(\nu,E,r)} \tag{13}$$

and we have omitted terms in Eq. (12) which have fixed singularities in ν, but no moving or E-dependent singularities.[13] In the last expression, $u(\nu,E,r)$ and $u(-\nu,E,r)$ are the solutions of the Schrödinger equation ($\hbar=1$):

$$\left[\frac{d^2}{dr^2}+2mE-2mV(r)-\frac{\nu^2-\frac{1}{4}}{r^2}\right]u=0\ , \tag{14}$$

which are, respectively, regular and irregular for $r\to0$, with the normalization

TABLE I. Results for $|r^{-l}R_{n,l}|^2(0)$ for the radial harmonic oscillator calculated using the Fermi-Segrè formula in Eq. (8). The fractional error diverges exponentially for $l\to\infty$:

$$|R_{n,l}^{\text{approx}}/R_{n,l}^{\text{exact}}|^2\sim\pi^{-1/2}(n-1)!2^{-l-1}e^l l^{-n+1/2}e^{2n-1/2},\quad l\gg n\ .$$

n \ l	0	1	2	3	4	5
	$\lvert R_{n,l}^{\text{approx}}/R_{n,l}^{\text{exact}}\rvert^2(0)$					
1	0.9772	1.0513	1.2190	1.4689	1.8119	2.2706
2	0.9951	1.0155	1.0782	1.1823	1.3302	1.5276
3	0.9980	1.0074	1.0405	1.0994	1.1858	1.3023
4	0.9989	1.0043	1.0249	1.0631	1.1206	1.1992

$$u(\pm\nu,E,r) \underset{r\to 0}{\sim} r^{\pm\nu+1/2}, \quad \nu>0 . \tag{15}$$

The Wronskian of $u(\pm\nu,E,r)$ is therefore $W(u^-,u^+)=2\nu$.

One can construct a solution of the Schrödinger equation which vanishes at a large radius R by combining the solutions above:

$$\hat{u}(\nu,E,r)=u(-\nu,E,r)-\frac{u(-\nu,E,R)}{u(\nu,E,R)}u(\nu,E,r)$$

$$\underset{R\to\infty}{\sim} u(-\nu,E,r)-\hat{S}(\nu,E)u(\nu,E,r) . \tag{16}$$

When $\hat{S}$ is infinite, the first term in $\hat{u}$ can be dropped, and $u(\nu,E,r)$ is itself a function regular both for $r\to 0$ and for $r\to\infty$ and, hence, is an eigenfunction of the Schrödinger equation.

The energies at which $\hat{S}(\nu,E)$ is infinite for $\nu=l+\frac{1}{2}$ or $l=$integer are the usual bound-state energies, and give the poles of $G_l(r',r,E)$ or $\Pi(l+\frac{1}{2},E)$, Eqs. (9) and (10). The poles of $\hat{S}$ as a function of ν which move with changing E are the Regge poles of the system. In the next section, we will calculate the residues $2\nu[a_n(\nu)]^2$ at the energy poles of $\Pi(\nu,E)$, and the residues $\beta_n(\nu)$ at the Regge poles of $\hat{S}(\nu,E)$.

B. Exact results for the residues

In Ref. 11 we developed a modified WKB method for the calculation of radial wave functions which is directly relevant to the residue problem. The method uses basis functions which have the correct behavior for $r\to 0$ instead of the usual WKB exponentials. We will use the radial oscillator wave functions as our basis functions in the following discussion since the oscillator potential gives a familiar example of a confining interaction. However, for more realistic $q\bar{q}$ potentials with an attractive Coulomb singularity at the origin, an approximation based on Coulomb wave functions is more appropriate. We will simply state the final results in the Coulomb case.

It is shown in Ref. 11 that the regular radial wave functions for problems with confining potentials and two turning points can be written exactly as

$$u(\nu,E,r)=c^{-\nu}(d\xi/dr)^{-1/2}\xi^{\nu+1/2}e^{-mE\xi^2/2\epsilon} \times\Phi\left[\frac{-\epsilon}{2}+\frac{\nu}{2}+\frac{1}{2},\nu+1,mE\xi^2/\epsilon\right] , \tag{17}$$

where Φ is the confluent hypergeometric function regular for $\xi\to 0$ (Ref. 14) and $\xi(r)$ is a solution of the equation

$$\kappa_0^2(\xi)\left[\frac{d\xi}{dr}\right]^2-\kappa^2(r)=\frac{1}{4r^2}-\frac{1}{4\xi^2}\left[\frac{d\xi}{dr}\right]^2 + \left[\frac{d\xi}{dr}\right]^{1/2}\frac{d^2}{dr^2}\left[\frac{d\xi}{dr}\right]^{-1/2} , \tag{18}$$

such that $\xi(r)\to 0$ for $r\to 0$ and $\xi(r)\to\infty$ for $r\to\infty$. Here

$$\kappa^2(r)=2mE-2mV(r)-\frac{\nu^2}{r^2} \tag{19}$$

and

$$\kappa_0^2(\xi)=2mE_0-\left[\frac{mE_0\xi}{\epsilon}\right]^2-\frac{\nu^2}{r^2} , \tag{20}$$

with E_0 a parameter which we will choose for simplicity to equal E. The function on the right-hand side of Eq. (17) with the factor $(d\xi/dr)^{-1/2}$ omitted, is then a solution of the radial oscillator equation in the variable ξ for energy E, angular momentum $l=\nu-\frac{1}{2}$, and spring constant $\frac{1}{2}m\omega^2=\frac{1}{2}(E/\epsilon)^2$. The parameter $\epsilon=\epsilon(\nu,E)$ is related to ν and E through the requirement that $\xi(r)$ be a smooth function with the limiting behavior given above [see Eq. (29)]. The constant $c=c(\nu,E)$ in Eq. (17) is defined as

$$c(\nu,E)=\lim_{r\to 0}[\xi(r)/r] , \tag{21}$$

and is included so that $u(\nu,E,r)$ has the ν- and E-independent normalization specified in Eq. (15).

The "S matrix" $\hat{S}(\nu,E)$ is determined by the asymptotic form of $u(\pm\nu,E,r)$ for $r\to\infty$:

$$u(\nu,E,r) \underset{r\to\infty}{\sim} c^{-\nu}(d\xi/dr)^{-1/2}\xi^{-\epsilon-1/2}(mE/\epsilon)^{-(\epsilon+\nu+1)/2}e^{mE\xi^2/2\epsilon}\frac{\Gamma(\nu+1)}{\Gamma\left[\frac{-\epsilon}{2}+\frac{\nu}{2}+\frac{1}{2}\right]} . \tag{22}$$

Thus, from Eq. (13),

$$\hat{S}(\nu,E)=c^{2\nu}\left[\frac{mE}{\epsilon}\right]^{\nu}\frac{\Gamma(-\nu+1)\Gamma(-\mu+1)}{\Gamma(\nu+1)\Gamma(1-\mu-\nu)} , \tag{23}$$

where we have introduced a function $\mu(\nu,E)$ defined by

$$\mu(\nu,E)=\tfrac{1}{2}[\epsilon(\nu,E)-\nu+1] . \tag{24}$$

$\hat{S}$ clearly has poles at the points at which $\mu(\nu,E)=n$, $n=1,2,\ldots$:

$$\hat{S}(\nu,E) \underset{\mu\to n}{\sim} c^{2\nu}\left[\frac{mE}{2n+\nu-1}\right]^{\nu}\frac{\nu\Gamma(\nu+n)}{[\Gamma(\nu+1)]^2\Gamma(n)}\frac{1}{\mu-n} . \tag{25}$$

These may be looked at in different ways. If we fix ν at a physical value $\nu=l+\frac{1}{2}$, $l=0,1,\ldots$, the poles in the sequence $\mu(l+\frac{1}{2},E)=n=1,2,\ldots$ are just the energy poles of $\Pi(\nu,E)$ at $E=E_{n,l}$. The residues in Eq. (10) are determined by Eqs. (10), (12), and (25):

$$2\nu[a_n(\nu)]^2\,|_{\nu=l+1/2}=(2l+1)\lim_{r\to 0}[r^{-l}R_{n,l}(r)]^2$$
$$=2m\,\mathrm{Res}_E\hat{S}(\nu,E)\,|_{\nu=l+1/2,\mu=n}$$
$$=c^{2l+1}\left[\frac{mE_{n,l}}{2n+l-\frac{1}{2}}\right]^{l+1/2}\frac{(l+\frac{1}{2})\Gamma(n+l+\frac{1}{2})}{[\Gamma(l+\frac{3}{2})]^2\Gamma(n)}\left[\frac{\partial\mu}{\partial E}\right]_\nu^{-1}\Bigg|_{\nu=l+1/2,\mu=n} \tag{26}$$

Alternatively, if we keep n fixed the poles lie on Regge trajectories $\mu(\nu,E)=n$ or $\nu=\nu_n(E)$ in the ν,E plane. The Regge residues are

$$\beta_n(\nu)=\mathrm{Res}_\nu\hat{S}(\nu,E)\,|_{\mu=n}=c^{2\nu}\left[\frac{mE_n(\nu)}{2n+\nu-1}\right]^\nu\frac{\nu\Gamma(\nu+n)}{[\Gamma(\nu+1)]^2\Gamma(n)}\left[\frac{\partial\mu}{\partial\nu}\right]_E^{-1}\Bigg|_{\mu=n}. \tag{27}$$

$\hat{S}(\nu,E)$ also has fixed poles at $\nu=1,2,\ldots$. These are not of interest here, but introduce sign changes in $\hat{S}$ which guarantee that the residues in Eq. (26) are all positive. (The Regge residues are negative for a completely confining potential, $\partial\mu/\partial\nu<0$.)

C. Approximate results for the residues

The results in Eqs. (26) and (27) are exact, but require knowledge of the function $\xi(r)$ determined by Eq. (18). In Ref. 11 we studied the approximation for $\xi(r)$ [hence for $u(\pm\nu,E,r)$] which results from setting the right-hand side of Eq. (18) to zero and solving the remaining equation,

$$\left[\frac{d\xi}{dr}\right]^2=\frac{\pm\kappa^2(r)}{\pm\kappa_0^2(r)}, \tag{28}$$

subject to the regularity condition that the zeros of the numerator and denominator match. With this constraint, the function $\xi(r)$ is always finite and continues smoothly through both turning points in a two-turning-point problem. The errors in ξ are formally of order $\hbar^2$. For the choice κ_0 in Eq. (20), corresponding to the representation of $u(\nu,E,r)$ in terms of an oscillator wave function as given in Eq. (17), the regularity condition is[11]

$$\pi[\mu(\nu,E)-\tfrac{1}{2}]=\frac{\pi}{2}(\epsilon-\nu)=\int_{r_<}^{r_>}dr\,\kappa(r), \tag{29}$$

where $r_<$ and $r_>$ are the zeros of $\kappa(r)$, defined in Eq. (19). As expected, the wave function in Eq. (17) vanishes for $r\to\infty$ as well as for $r\to 0$ when $\mu(\nu,E)=n=1,2,\ldots$, that is, when $\hat{S}$ has a pole. The condition $\mu(\nu,E)=n$ for $\nu=l+\frac{1}{2}$ is of course just the usual WKB quantization condition:

$$\int_{r_<}^{r_>}dr\left[2mE-2mV(r)-\frac{(l+\frac{1}{2})^2}{r^2}\right]^{1/2}=(n-\tfrac{1}{2})\pi. \tag{30}$$

We can use Eq. (29) to calculate the derivatives which appear in the exact expressions for the residues derived above. Thus, in the present approximation,

$$\left[\frac{\partial\mu}{\partial E}\right]_\nu\Bigg|_{\mu=n,\nu=l+1/2}=\frac{2m}{\pi}\int_{r_<}^{r_>}dr\left[2mE_{n,l}-2mV(r)-\frac{(l+\frac{1}{2})^2}{r^2}\right]^{-1/2}, \tag{31}$$

where $E_{n,l}$ is the value of E which satisfies Eq. (30) for the given values of n and l. Similarly, for n fixed and general ν,

$$\left[\frac{\partial\mu}{\partial\nu}\right]_E\Bigg|_{\mu=n}=-\frac{\nu}{\pi}\int_{r_<}^{r_>}\frac{dr}{r^2}\left[2mE-2mV(r)-\frac{\nu_n^2}{r^2}\right]^{-1/2} \tag{32}$$

where $\nu_n=\nu_n(E)$ lies on the Regge trajectory determined by the condition

$$\int_{r_<}^{r_>}dr\left[2mE-2mV(r)-\frac{\nu_n^2}{r^2}\right]^{1/2}=(n-\tfrac{1}{2})\pi. \tag{33}$$

It is useful to note that $\partial\mu/\partial E$ and $\partial\mu/\partial\nu$ can be related directly (if approximately) to experimental data. If we regard $\mu(\nu,E)$ as a continuous version of the quantum number n, we can identify $\partial\mu/\partial E$ with the observable density of states for a given l:

$$\left[\frac{\partial\mu}{\partial E}\right]_l\to\frac{dn}{dE}(E_{n,l}). \tag{34}$$

The quantity $\partial\mu/\partial\nu$ can be expressed in a similar fashion

$$\left[\frac{\partial\mu}{\partial\nu}\right]_E=-\left[\frac{\partial\mu}{\partial E}\right]_\nu\left[\frac{dE}{d\nu}\right]_\mu\to-\frac{dn}{dE}\left[\frac{d\nu_n}{dE}\right]^{-1}, \tag{35}$$

where $d\nu_n/dE$ is the slope of the nth Regge trajectory at the given value of E.

In contrast with the foregoing, the calculation of the function $c(\nu,E)=\lim_{r\to 0}(\xi/r)$ which appears in $\hat{S}$, Eq. (23), requires the explicit solution of Eq. (28) in the classically forbidden region $r<r_<$. By integrating inward from $r_<$ using the matching condition $\xi_<=\xi(r_<)$ for the zeros of κ_0^2 and κ^2, one can show rather easily that, in the present approximation,[11]

$$[c(\nu,E)]^{2\nu}=\lim_{r\to 0}\exp\left[-2\int_r^{r'_<}dr[-\kappa^2(r)]^{1/2}+2\int_r^{r'_<}dr[-\kappa_0^2(r)]^{1/2}\right] \tag{36a}$$

$$=\lim_{r\to 0}\exp\left\{-2\int_r^{r'_<}dr\left[\frac{\nu^2}{r^2}+2mV(r)-2mE\right]^{1/2}+2\int_r^{r'_<}dr\left[\frac{\nu^2}{r^2}+\left[\frac{mEr}{\epsilon}\right]^2-2mE\right]^{1/2}\right\}. \tag{36b}$$

The right-hand side of this expression is just the ratio of the ordinary WKB barrier penetration factor for the full problem with the confining potential $V(r)$, to the penetration factor for the oscillator problem for an oscillator with the prescribed energy E and $\epsilon=2\mu(\nu,E)+\nu-1$.

Our final results for the residues are obtained by combining the approximate expressions above with the exact formulas in Eqs. (26) and (27). We find after some manipulation that the energy residues are given by the generalized Fermi-Segrè formula:

$$(2l+1)[a_n(l+\tfrac{1}{2})]^2=(2l+1)\frac{2m}{\pi}\left[\frac{2mE}{n+\frac{l}{2}-\frac{1}{4}}\right]^{l+1/2}\frac{\Gamma(n+l+\frac{1}{2})}{(n-1)!}\left[\frac{2^l l!}{(2l+1)!}\right]^2\left[\frac{\partial\mu}{\partial E}\right]_l^{-1}\Bigg|_{E=E_{n,l}}$$

$$\times\lim_{r\to 0}\exp\left[-2\int_r^{r'_<}dr[-\kappa^2(r)]^{1/2}+2\int_r^{r'_<}dr[-\kappa_0^2(r)]^{1/2}\right], \tag{37}$$

in agreement with the results of Ref. 12. The Regge residues are given by a similar expression:

$$\beta_n(\nu)=\left[\frac{mE(\nu)}{2n+\nu-1}\right]^2\frac{\nu\Gamma(\nu+n)}{[\Gamma(\nu+1)]^2(n-1)!}\left[\frac{\partial\mu}{\partial\nu}\right]_E^{-1}\Bigg|_{\nu=\nu_n(E)}$$

$$\times\lim_{r\to 0}\exp\left[-2\int_r^{r'_<}dr[-\kappa^2(r)]^{1/2}+2\int_r^{r'_<}dr[-\kappa_0^2(r)]^{1/2}\right], \tag{38}$$

where $\nu_n(E)$ lies on the trajectory determined by Eq. (33), and $\partial\mu/\partial\nu$ is given by Eq. (32). This result is new.

D. Alternative results for problems with Coulomb interactions

As noted earlier, it is actually more appropriate in quarkonium problems to use Coulomb wave functions instead of oscillator functions in the approximation of $u(\pm\nu,E,r)$, at least for low energies and small ν, since the Coulomb singularity in the $q\bar{q}$ potential is then treated exactly. We will only sketch the main points in this alternative approach, and present the final results. More details may be found elsewhere.[11,12]

We will suppose that the potential $V(r)$ is of the form

$$V(r)=-\frac{e^2}{r}+V_c(r), \tag{39}$$

where $V_c(r)$ is a confining potential, for example, the linear potential $V_c=br$ used in many studies of the $c\bar{c}$ and $b\bar{b}$ systems. We will also assume for simplicity that $V_c(r)$ is finite for $r\to 0$, and that the energy scale has been adjusted so that $V_c(0)=0$. (These assumptions are not necessary.) Then for $E>0$, or more generally, for $E-V_c(r)>0$ for $r\to 0$, we can write $u(\nu,E,r)$ for r inside the outer turning point as

$$u(\nu,E,r)=c^{-\nu}(d\xi/dr)^{-1/2}\xi^{\nu+1/2}e^{-i(2mE)^{1/2}\xi}\Phi(\nu+\tfrac{1}{2}+i\eta,2\nu+1,2i(2mE)^{1/2}\xi), \tag{40}$$

where $\eta=e^2(m/2E)^{1/2}$, while for $E<0$ we can write u for all r as

$$u(\nu,E,R)=c^{\nu}(d\xi/dr)^{1/2}\xi^{\nu+1/2}e^{-(2m|E|^{1/2}\xi)}\Phi(\nu+\tfrac{1}{2}-|\eta|,2\nu+1,2(2m|E|)^{1/2}\xi). \tag{41}$$

In these expressions, Φ is the confluent hypergeometric function regular for $\xi\to 0$, and $c=c(\nu,E)$ is the constant defined in Eq. (21). The function $\xi(r)$ is the solution of Eq. (18) with κ^2 as defined in Eq. (19), and

$$\kappa_0^2(r)=2mE+\frac{2me^2}{r}-\frac{\nu^2}{r^2}. \tag{42}$$

With this definition of κ_0^2, the most divergent terms in κ^2

and κ_0^2 are identical for $r\to 0$ (Ref. 15). The procedure is now to approximate $\xi(r)$ and $u(\nu,E,r)$ by solving Eq. (18) with the right-hand side set to zero.

The approximate wave function constructed for $E<0$ using Eqs. (18) and (41) continues smoothly through both turning points provided

$$\int_{r_<}^{r_>} dr\,\kappa(r)=\pi[\mu(\nu,E)-\tfrac{1}{2}]\,, \tag{43}$$

where

$$\mu(\nu,E)=|\eta|-\nu+\tfrac{1}{2} \tag{44}$$

is the WKB phase function introduced earlier. Using the asymptotic form of $u(\pm\nu,E,r)$ for $r\to\infty,\xi\to\infty$, we find that $\hat{S}$ is given for $E<0$ by

$$\hat{S}(\nu,E)=[2c(2m\,|E|)^{1/2}]^{2\nu}\times\frac{\Gamma(-2\nu+1)\Gamma(1-\mu)}{\Gamma(2\nu+1)\Gamma(1-\mu-2\nu)},\quad E<0\,. \tag{45}$$

$\hat{S}$ has poles for $\mu(\nu,E)=n=1,2,\ldots$:

$$\hat{S}(\nu,E)\underset{\mu\to n}{\sim}[2c(2m\,|E|)^{1/2}]^{2\nu}\times\frac{2\nu\Gamma(n+2\nu)}{[\Gamma(2\nu+1)]^2\Gamma(n)}\frac{1}{\mu-n},\quad E<0\,. \tag{46}$$

The energy residues for $n=1,2,\ldots$ and physical values of ν are[12]

$$(2l+1)[a_n(l+\tfrac{1}{2})]^2=2m\,[2c(2m\,|E|)^{1/2}]^{2l+1}\frac{(2l+1)(n+2l)!}{[(2l+1)!]^2(n-1)!}\left[\frac{\partial\mu}{\partial E}\right]_l^{-1}\Bigg|_{E=E_{n,l}}, \tag{47}$$

where $c^{2\nu}$ is given by the expression in Eq. (36a) with κ_0^2 as given in Eq. (42). The Regge residues are

$$\beta_n(\nu)=\{2c[2m\,|E(\nu)|]^{1/2}\}^{2\nu}\frac{2\nu\Gamma(n+2\nu)}{[\Gamma(2\nu+1)]^2\Gamma(n)}\left[\frac{\partial\mu}{\partial\nu}\right]_E^{-1}\Bigg|_{\nu=\nu_n(E)}, \tag{48}$$

where $\nu_n(E)$ lies on the trajectory determined by Eq. (33).

The case $E>0$ is slightly more complicated. The function in Eq. (40) gives a solution of the Schrödinger equation inside the outer turning point. To construct $\hat{S}$, we need to continue the solution through the turning point using Airy functions, and match the result to a solution in terms of Bessel functions in the outer region.[11] The result is

$$\hat{S}(\nu,E)=\tfrac{1}{2}[2c(2mE)^{1/2}]^{2\nu}\left[\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu+1)}\right]^2 e^{\pi\eta}\left|\frac{\Gamma(\nu+\frac{1}{2}-i\eta)}{\Gamma(\nu+\frac{1}{2})}\right|^2\left[\cot\pi\mu-\frac{\cos2\pi\nu+e^{-2\pi\eta}}{\sin2\pi\nu}\right],\quad E>0\,, \tag{49}$$

where $\mu(\nu,E)$ is defined by Eq. (43). The energy residues given for $\nu=l+\frac{1}{2}$ and $\mu=n=1,2,\ldots$ by[12]

$$(2l+1)[a_n(l+\tfrac{1}{2})]^2=(2l+1)\frac{2m}{\pi}c^{2l+1}(2mE)^{l+1/2}\left[\frac{2^l l!}{(2l+1)!}\right]^2 e^{\pi\eta}\left|\frac{\Gamma(l+1-i\eta)}{\Gamma(l+1)}\right|^2\left[\frac{\partial\mu}{\partial E}\right]_l^{-1}\Bigg|_{E=E_{n,l}}, \tag{50}$$

while the Regge residues are given by

$$\beta_n(\nu)=\frac{1}{2\pi}[2c(2mE)^{1/2}]^{2\nu}e^{\pi\eta}\frac{|\Gamma(\nu+\frac{1}{2}-i\eta)|^2}{[\Gamma(2\nu+1)]^2}\left[\frac{\partial\mu}{\partial\nu}\right]_E^{-1}\Bigg|_{\nu=\nu_n(E)}, \tag{51}$$

where $c(\nu,E)$ is defined by the expression in Eq. (36a) with the form for κ_0^2 given in Eq. (42).

III. COMMENTS

The semiclassical expressions derived above provide useful and accurate[12] approximations for the energy or Regge residues of the quantum-mechanical two-point function $\Pi(\nu,E)$ or $\hat{S}(\nu,E)$ for confined systems. For example, the results for the residues given in Eqs. (37) and (38) are exact for the oscillator potential, in marked contrast with the usual Fermi-Segrè formula for the energy residues, Eq. (8). The error in the latter increases exponentially with l (see Table I). More generally, the errors in our formulas decrease rapidly with increasing l and n, as would be expected from their WKB-like derivation.

The factors $\partial\mu/\partial E$ and $\partial\mu/\partial\nu$ in the residues can be identified through Eqs. (34) and (35) with the density of states of the system, or the density of states divided by the slope of the Regge trajectory, so depend only on local properties of the spectrum. Just this identification has been used in the past to derive duality relations for $q\bar{q}$ annihilation. Analogous formulas have been shown to hold in the relativistic context.[16] It is therefore not necessary, though it would still be interesting, to extract the residues directly from the quantum mechanical or QCD sum rules. It is not clear how this could be done for large l, given the problems noted by Fishbane, Gasiorowicz, and Kaus.[7] The origin of those problems is clear from our results.

First, our expressions for $\hat{S}(\nu,E)$ contain nontrivial ν-dependent factors which are necessary for the residues to maintain the proper signs from pole to pole, see, e.g., Eqs. (23) and (45). These factors do not appear in perturbative calculations of $\hat{S}$ or Π. Second, the barrier penetration factor $[c(\nu,E)]^{2\nu}$ which appears in all of our results depends on the behavior of the potential $V(r)$ in the classically forbidden region near the origin. Information on $V(r)$ in this region is not available locally, but is only attained for a given ν and E by using spectral information from higher energies. This information is also not given by perturbation theory if only a few moments M_N are known. It is therefore important that one have independent information about the short- and long-distance behavior of the potential, as has been emphasized by Fishbane, Gasiorowicz, and Kaus.[7]

If we suppose that the long-distance behavior of the potential *is* known, and separate $V(r)$ into the known confining piece $V_c(r)$ and a short-range piece $V_s(r)$, we can treat the effects of the latter on the high Regge residues as follows. We write the barrier-penetration factor $c^{2\nu}$, Eq. (36a), as

$$[c(\nu,E)]^{2\nu}=\lim_{r\to 0}\exp\left[-2\int_r^{r''_<}dr[-\kappa_1^2(r)]^{1/2}+2\int_r^{r'_<}dr[-\kappa_0^2(r)]^{1/2}\right]$$
$$\times\exp\left[-2\int_r^{r_<}dr[-\kappa^2(r)]^{1/2}+2\int_r^{r'_<}dr[-\kappa_1^2(r)]^{1/2}\right], \tag{52}$$

where κ^2 is defined for the full potential $V=V_c+V_s$ using Eq. (19), κ_1^2 is defined using only V_c, and κ_0^2 is defined for the comparison potential V_0. The first factor in Eq. (52) is just $c_1^{2\nu}$, the barrier-penetration factor for confining potential alone, which we assume is known.

For ν large enough that $r_<$ (the inner turning point for the full potential) is greater than the range of V_s, the Regge trajectories determined by Eq. (33) depend only on V_c, and not on V_s, and $\partial\mu/\partial\nu$, Eq. (32), is unchanged. We can also expand $(-\kappa^2)^{1/2}$ in the exponent of the second factor in Eq. (52) in this regime (large ν,E, small n), and find that

$$[c(\nu,E)]^{2\nu}\approx[c_1(\nu,E)]^{2\nu}\exp\left[-2m\int_0^{r_<}dr\,V_s(r)[-\kappa_1^2(r)]^{-1/2}\right]$$
$$\underset{\substack{\nu\text{ large}\\E\text{ fixed}}}{\sim}[c_1(\nu,E)]^2\exp\left[-\frac{2m}{\nu}\int_0^{\infty}dr\,rV_s(r)\right]. \tag{53}$$

The last expression involves the first radial moment of V_s, but is not directly expressible in terms of the energy moments calculated from a perturbative expansion of $\Pi(\nu,E)$. Extra information is clearly needed for the calculation of the last factor in Eq. (53), hence of the Regge residues.

We remark finally that it would be interesting to see if semiclassical methods could be used directly in QCD to calculate $\Pi(\nu,E)$ or the residues nonperturbatively.

ACKNOWLEDGMENTS

We would like to thank Paul Fishbane, Stephen Gasiorowicz, and Peter Kaus for interesting discussions of their work, especially Peter Kaus for discussions of the subtleties connected with Eq. (12). This work was supported in part by the U.S. Department of Energy under Contract No. DE-AC02-76ER00881.

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