Relativistic wave equations for the dynamics of two interacting particles

H. Sazdjian
Division de Physique Théorique, Institut de Physique Nucléaire, Université Paris XI, 91406 Orsay Cedex, France
(Received 2 July 1985)

We construct relativistic wave equations describing the dynamics of two interacting particle systems involving spin-0 bosons and/or spin-$\frac{1}{2}$ fermions. The method consists in quantizing the manifestly covariant formalism with constraints of classical relativistic Hamiltonian mechanics. In this formalism the two-particle wave function satisfies two independent wave equations, which thus determine in a definite way its relative time evolution. We solve the compatibility condition of the two wave equations and classify the general classes of interaction potentials according to their tensor structure. We outline the relationship of this framework of relativistic quantum mechanics with the Bethe-Salpeter equation and its sector of normal solutions.

I. INTRODUCTION

The use of the manifestly covariant formalism with constraints[1-4] in classical relativistic Hamiltonian mechanics[5-15] has led to substantial progress in the Hamiltonian formulation of relativistic quantum mechanics.[16-21] One of the main features of the Hamiltonian formalism is that it avoids the appearance of unphysical degrees of freedom. In particular the time component of the coordinate four-vector of a particle is assigned to play the role of a parameter and not that of a dynamical variable. On the other hand, the advantage of the constraint formalism is its property of eliminating the redundant variables of the theory without breaking, to some extent, its manifest covariance.

It is then natural to extend this way of constructing relativistic mechanics to the quantum level. Here a system of N spin-0 and/or spin-$\frac{1}{2}$ particles will be described by means of a wave function satisfying N independent wave equations. Each equation is a generalization of the Klein-Gordon or the Dirac equation, according to the spin of the constituent particle of the system to which it refers. These equations are the quantized versions of the N first-class mass-shell constraints of classical mechanics. The interaction appears through potentials, which are functions of the variables of the system. The potentials have to be chosen so as to satisfy the compatibility conditions of the N wave equations. In general these conditions also guarantee the realization of the Poincaré invariance of the theory in its Hamiltonian form.

One consequence of the fact that the dynamics of the N-particle system is described by means of N independent wave equations is the absence of relative time excitations from the energy spectrum of the system.

The purpose of the present work is to study, within the framework of the manifestly covariant formalism with constraints, the structure of the wave equations describing the dynamics of relativistic two-particle systems involving interacting spin-0 and/or spin-$\frac{1}{2}$ particles. This is mainly achieved by solving the compatibility conditions of these equations and exhibiting the corresponding general classes of interaction potentials. We also classify the latter according to their tensor structure, such as scalar, pseudoscalar, vector, etc.

The generality of the results allows one to use these wave equations, by appropriate choices of the interaction potentials, for the phenomenological study of a very wide variety of relativistic two-body problems which can be met in particle physics. In this respect we present in separate papers[22] two applications concerning confining interactions in fermion-antifermion and fermion-boson systems, having straightforward connections with meson and quark phenomenology.

The determination of the wave equations describing the dynamics of relativistic systems represents one part only of the general attempt to construct a relativistic quantum mechanics of interacting particles. The formulation of relativistic quantum mechanics has also to be completed by the construction of the scalar product of physical states. Some features of this problem were examined in Ref. 21 and we intend to present a general analysis of this question in a separate work. (See also the Appendix.)

A particular drawback of the Hamiltonian approach to relativistic quantum mechanics has been consisting for a long time in the lack of explicit connection with field theory. In this respect the quasipotential approach[23-29] provides a first basis for such a relationship, by starting from the Bethe-Salpeter equation[30-32] and reducing it, with appropriate hypothesis, to a three-dimensional equation. Here one uses the Bethe-Salpeter equation in its integral form for off-mass-shell scattering amplitudes. This approach, however, seems to be more suitable for scattering problems, where the basic ingredients are scattering amplitudes, and where the direct connection with the local fields of the underlying field theory becomes less transparent. Wave functions, as well as potentials, are defined after appropriate analytic continuations are utilized and Lippmann-Schwinger-type equations introduced.

In this connection a recent result was obtained by the author,[33] by directly working in configuration space. It was shown, in the case of two equal-mass spinless particles, that the manifestly covariant wave equations of Hamiltonian relativistic quantum mechanics can be de-

rived from the Bethe-Salpeter (BS) equation. This is achieved by transforming the BS equation by algebraic manipulations so as to separate it into two independent equations which have the same structure as the wave equations of relativistic quantum mechanics. The first equation determines the relative time evolution of the system, while the second one yields a three-dimensional eigenvalue equation. The interaction potential and the wave function of relativistic quantum mechanics are thus related in a definite way to the kernel and the wave function of the BS equation, when the "normal" solutions of the latter are considered. (The result is actually generalizable to the unequal-mass case, as well as to fermions.) The relations involve integral operators which can be evaluated, in principle, in perturbation theory. This result establishes the necessary link between Hamiltonian relativistic quantum mechanics and quantum field theory. We shall present a detailed account of the derivation of these relations for the cases of fermions and unequal masses in a separate work.

Because the starting point is the Bethe-Salpeter equation and the transformations which are introduced are of algebraic nature, we think that the latter approach and the quasipotential approach are equivalent, when considered in their general forms. The advantages of each approach depend, however, on the framework or the type of problem which is considered. The connection which we developed in Ref. 33 seems to us more suitable for the quantum-mechanical framework, while the quasipotential approach seems to be more adequate for an S-matrix-type framework.

When a relativistic instantaneous approximation[30,34] is made for the (effective) Bethe-Salpeter kernel, taken in its ladder approximation, then the relationships obtained in Ref. 33 considerably simplify. In particular one can then reduce, in an explicit way, the four-dimensional normalization condition of the Bethe-Salpeter wave function to a three-dimensional one and thus fix the normalization condition of the quantum-mechanical wave function. Throughout this work we shall mainly present these relationships with the above approximation, which is sufficient to cover with enough accuracy the physical problems met at the quantum level.

Finally, the formalism which is presented in this paper concerns systems of particles the total momentum of which is assumed to be timelike. The case of massless bound states necessitates a separate treatment and will be dealt with elsewhere.

The plan of the paper is as follows. In Sec. II we review the main features of classical relativistic Hamiltonian mechanics, as formulated by the manifestly covariant formalism with constraints. Section III presents the method of quantizing a theory of N-particle systems. In Sec. IV we consider two spin-0 particle systems and also outline the relationships with field-theoretic quantities. In Sec. V we display the scalar- and vector-type interactions. Sections VI and VII deal with spin-$\frac{1}{2}$ fermion-antifermion systems and with the classification of the corresponding interactions according to their tensor structure. In Sec. VIII we consider spin-$\frac{1}{2}$—spin-0 particle systems. A summary and concluding remarks follow in Sec. IX. In the Appendix we briefly sketch the construction of the scalar product of the theory.

This paper is a completed version of Orsay Report No. IPNO/TH 84-46, 1984 (unpublished). The contents of Secs. 9 and 10 of that report appear as completed versions, in the following two papers (Ref. 22).

II. CLASSICAL RELATIVISTIC HAMILTONIAN MECHANICS

In the manifestly covariant formalism of classical relativistic Hamiltonian mechanics, describing an N-spinless-particle system, one introduces $8N$ canonical coordinates and momenta $x_{a\mu},p_{a\mu}$ [$a=1,\ldots,N;\mu=0,1,2,3$; we are using the time-like metric $(g_{\mu\nu})=\mathrm{diag}(1,-1,-1,-1)$] satisfying the Poisson-brackets relations:

$$\begin{aligned}&\{x_{a\mu},x_{b\nu}\}=\{p_{a\mu},p_{b\nu}\}=0\quad(a,b=1,\ldots,N)\,,\\&\{p_{a\mu},x_{b\nu}\}=c\delta_{ab}g_{\mu\nu}\end{aligned}\tag{2.1}$$

(c is the velocity of light; henceforth we shall set $c=1$).

The set of canonical variables (x,p) define an $8N$-dimensional phase space Γ_N. Poincaré algebra generators in Γ_N are constructed as the sum of the individual particle contributions:

$$\begin{aligned}P_\mu&=\sum_{a=1}^{N}p_{a\mu}\,,\\M_{\mu\nu}&=\sum_{a=1}^{N}(x_{a\mu}p_{a\nu}-x_{a\nu}p_{a\mu})\,.\end{aligned}\tag{2.2}$$

The N redundant energy variables p_{a0} are eliminated by means of N generalized mass-shell constraints, which also introduce the interaction potentials:

$$H_a\equiv p_a{}^2-m_a{}^2-V_a\approx 0\quad(a\equiv 1,\ldots,N)\,,\tag{2.3}$$

where m_a is the free mass of particle a and V_a is an interaction potential which is a Poincaré-invariant function in Γ_N of the canonical variables. The weak equality sign $\approx$ means that the constraints (2.3) should be used only after the evaluation of the Poisson brackets.

The time parameters of the theory are fixed by imposing the N time constraints:

$$\chi_a\equiv\eta_a(x,p)-t_a\approx 0\quad(a=1,\ldots,N)\,,\tag{2.4}$$

where the η's are functions of the canonical variables. In order that they yield acceptable definitions of time parameters, it is necessary that η_a defines a spacelike hypersurface in the time-position space of particle a and that its Poisson brackets with its "conjugate" mass-shell constraint H_a of (2.3) does not vanish:

$$\{H_a,\chi_a\}\neq 0\quad(a=1,\ldots,N)\,,\tag{2.5}$$

so that the N^2-dimensional matrix $\{\phi,\phi\}$ ($\phi\equiv H$ or χ) of the mutual Poisson brackets of the $2N$ constraints is nonsingular (the time constraints must be independent from each other). In the terminology of the constraint formalism the $2N$ constraints H_a of (2.3) and χ_a of (2.4) form a set of second-class constraints [because of (2.5) and the nonsingularity of the matrix $\{\phi,\phi\}$].

The $2N$ constraints (2.3) and (2.4) reduce the $8N$-

dimensional phase space Γ_N to a $6N$-dimensional phase space Γ_N^*, which constitutes the physical phase space; the dynamics of the system is described within this phase space. The Poincaré invariance of the theory is formulated and realized in Γ_N^*, that is by taking into account the constraints. This is usually done by means of the Dirac brackets,[1,6,35,36] which are modified Poisson brackets preserving the constraints. It can be shown[13] that the necessary and sufficient condition to realize the Poincaré algebra with the Dirac brackets is that the mass-shall constraints (2.3) be first class among themselves, that is, that their Poisson brackets vanish:

$$\{H_a,H_b\}\approx 0 \quad (a,b=1,\ldots,N) . \tag{2.6}$$

These equations impose restrictions on the potentials V_a which must be solutions of them. Furthermore the first-class nature of the mass-shell constraints H_a is an indication that the time constraints (2.4) can be chosen arbitrarily[1,2,6,35] [within the general conditions outlined after Eq. (2.4)]. The physical properties of the system should not depend upon this choice. This freedom in the choice of the time parameters of the theory is usually called "time gauge invariance."

In the two-particle case the problem simplifies considerably, because Eqs. (2.6) reduce to a single one:

$$\{H_1,H_2\}\approx 0 . \tag{2.7}$$

One of the two potentials V_1 and V_2 will then remain completely arbitrary (but Poincaré invariant). One can however simplify the problem a little more. It can be shown (see Ref. 13, Appendix A) that by Γ_2 Poincaré-invariant canonical transformations the two mass-shell constraints (2.3) can be brought into a form where the two potentials V_1 and V_2 are equal:

$$V_1=V_2=V . \tag{2.8}$$

The constraints H_a then become

$$\begin{aligned} H_1 &\equiv p_1^2-m_1^2-V\approx 0 , \\ H_2 &\equiv p_2^2-m_2^2-V\approx 0 , \end{aligned} \tag{2.9}$$

which also means that $(p_1^2-p_2^2)$ is not modified by the interaction:

$$p_1^2-p_2^2\approx m_1^2-m_2^2 . \tag{2.10}$$

It is therefore sufficient, to find the general expressions of the interaction potentials V_1 and V_2, to solve Eq. (2.7) for the cases (2.8) and (2.9) and then to apply on the mass-shell constraints (2.9) arbitrary Γ_2 Poincaré-invariant canonical transformations. Configuration (2.9) is however the simplest one and in the following we shall always use potentials satisfying condition (2.10).

We now search for solutions of Eq. (2.7). [General classes of solutions of Eqs. (2.6) in the N-particle case, satisfying furthermore the separability or the cluster decomposition condition, have been given in Ref. 13.] We shall consider this equation in the strong form, the right-hand side being exactly zero and not involving linear combinations of H_a. (This restriction arises in classical mechanics when searching for the position variables of the particles: each mass-shell constraint H_a plays the role of the generator of an independent time displacement—usually a generalized proper time. See Ref. 13, Sec. 3.1.1.)

Equation (2.7) can be written as

$$\{p_1^2-p_2^2,V\}=0 . \tag{2.11}$$

We introduce the following (vector) notations:

$$\begin{aligned} &p=p_1+p_2, \quad v=\tfrac{1}{2}(p_1-p_2), \quad M=m_1+m_2 , \\ &X=\tfrac{1}{2}(x_1+x_2), \quad x=x_1-x_2 , \\ &r_\mu\equiv x_\mu^T=x_\mu-(\hat{\mathbf{p}}\cdot x)\hat{\mathbf{p}}_\mu , \\ &x^{T^2}=r^2=x^2-(\hat{\mathbf{p}}\cdot x)^2 , \\ &x_\mu^L=(\hat{\mathbf{p}}\cdot x)\hat{\mathbf{p}}_\mu, \quad x_L=\hat{\mathbf{p}}\cdot x , \\ &\hat{\mathbf{p}}_\mu=p_\mu/(p^2)^{1/2} \quad (p^2>0) . \end{aligned} \tag{2.12}$$

For a vector Y we shall often introduce its transverse and longitudinal parts with respect to the total momentum p:

$$\begin{aligned} &Y_\mu^T=Y_\mu-(Y\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}_\mu, \quad Y_\mu^L=(Y\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}_\mu , \\ &Y_L=Y\cdot\hat{\mathbf{p}} . \end{aligned} \tag{2.13}$$

Equation (2.11) becomes

$$\{p\cdot v,V\}=0 . \tag{2.14}$$

Since V is Poincaré invariant (hence independent of X), this means that it can only depend upon the transverse component r of x and eventually upon p and v:

$$V=V(r^2,r\cdot v,p^2,v^2,p\cdot v) . \tag{2.15}$$

It is a general feature of relativistic mechanics that the potentials do not appear as functions of only the relative coordinates x and the relative momenta v. Ignoring explicit dependences of V on the total and relative momenta, one is still left with an implicit dependence upon p through the vector r. It is only in the c.m. frame that r reduces to the momentum-independent vector $(0,\mathbf{x})$. For stationary systems, which have a given value of total momentum p, the latter will be a constant of the motion and its appearance in r will have a kinematic role only. In general one can work covariantly by decomposing all vectors along transverse and longitudinal components as in Eqs. (2.13).

The potential V can also exhibit, as shown in Eq. (2.15), an explicit dependence on p^2. This can arise either from a dependence of the coupling constants on p^2, or from dimensional requirements. The dependence of V upon $p\cdot v$ can be ignored altogether, since the latter can be eliminated by means of the constraint (2.10) $[p\cdot v=(m_1^2-m_2^2)/2]$. Finally the dependences upon $r\cdot v$ and v^2 arise when nonlocal effects are ignored or suitably approximated by local functions, from the tensor structure of the interaction (vector, tensor, etc.) and appear through polynomials.

In order that V yield a finite contribution in the nonrelativistic limit, it must have a maximal c^2 dependence (c is the velocity of light) of degree one. In the limit $c\to\infty$, the Poincaré invariance of the system reduces to the Galileli invariance.

The study of classical mechanics is pursued with the construction of the observables of the system which are essentially the position variables of the particles. However this aspect of the problem is less relevant for the quantization of the theory, where world lines do not play any fundamental role, and we shall therefore not insist on this point any longer. (The position variables are constructed in Refs. 7, 13, and 37.)

III. QUANTIZATION

In order to quantize the classical N-particle system considered in Sec. II in a manifestly covariant form, one begins by replacing the dynamical variables with operators and the Poisson brackets with commutators divided by $i\hbar$. For instance, the third Poisson brackets in Eq. (2.1) becomes

$$[p_{a\mu},x_{bv}]=i\hbar c\delta_{ab}g_{\mu v}\ . \tag{3.1}$$

(Henceforth we shall set $c=1$.)

One then introduces a covariant wave function $\Psi(x_1,\ldots,x_N)$ on which the operators x_a act multiplicatively and p_a as $i\hbar\partial/\partial x_a$. Under the action of the Poincaré group, generated by the operators (2.2), the wave function Ψ transforms covariantly:

$$\Psi(x_1,\ldots,x_N)\rightarrow\Psi(\Lambda^{-1}x_1-a,\ldots,\Lambda^{-1}x_N-a)\ . \tag{3.2}$$

According to Dirac's approach to the quantization problem,[1] the first-class constraints of the classical theory are transformed into wave equations. Equations (2.3) then become

$$H_a\Psi\equiv(-\hbar^2\partial_a{}^2-m_a{}^2-V_a)\Psi=0\quad(a=1,\ldots,N)\ . \tag{3.3}$$

(We shall often use in place of the derivation operators $i\hbar\partial_a$ the corresponding momentum operators p_a.) The potentials V_a must of course be appropriately symmetrized or ordered with respect to their momentum and coordinate dependences. On the other hand, time-reversal invariance demands that every odd expression of the coordinates x in H_a be accompanied by an i factor.

Equations (2.6), which characterize the first-class nature of the constraints H_a, are now transformed into compatibility (or integrability) conditions of the wave equations (3.3):

$$[H_a,H_b]\Psi\approx0\quad(a,b=1,\ldots,N)\ , \tag{3.4}$$

where the weak equality sign means that the commutator of two H's may yield linear combinations of the H's themselves (appearing on the right of the corresponding expressions), which, on account of Eqs. (3.3), give zero.

The N time parameters of the theory are introduced by constraints of the type (2.4):

$$\chi_a\equiv\eta_a(x_a)-t_a\approx0\quad(a=1,\ldots,N)\ , \tag{3.5}$$

where η_a defines a spacelike hypersurface in the four-dimensional space of x_a. Notice that the definition of η_a here is more restricted than in classical mechanics—Eq. (2.4)—where η_a may also depend on the momenta and define generalized proper times. The fact that the momenta p_a are now derivatives excludes such choices in quantum mechanics, where proper times cannot be defined in a kinematic way as parameters.

As in classical mechanics the physical properties of the quantized system should not depend on the particular choices of the time parameters (3.5) of the theory. This would be ensured by the existence in the Hilbert space of states of Poincaré-invariant scalar products constructed from tensor currents of rank N satisfying N conservation laws under the action of the operators $\partial_1^{\mu_1},\ldots,\partial_N^{\mu_N}$, respectively.[21] [We shall show in a separate work that the realization of the compatibility conditions (3.4) ensures at the same time the existence of such currents.]

In order to give a probabilistic meaning to the wave function one has to interpret t_a in (3.5) as the "observation" time of particle a at the "position" x_a (the terms "observation," "particle," and "position" being used here in a heuristic sense). The partial derivative of the wave function with respect to t_a is given by the action of the operator $[(\partial x_a)/(\partial\eta_a)]\cdot p_a$, where $(\partial x_a)/(\partial\eta_a)$ is obtained after expressing x_a^μ with respect to t_a and three other variables parametrizing the surface η_a:

$$i\hbar\frac{\partial}{\partial t_a}\Psi(x_1,\ldots,x_N)=\frac{\partial x_a}{\partial\eta_a}\cdot p_a\Psi(x_1,\ldots,x_N)\qquad(a\text{ fixed})\ . \tag{3.6}$$

The global evolution of Ψ with respect to a common time t is studied by taking the surfaces $\eta_1,\ldots,\eta_N$ of the same form η, and $t_1,\ldots,t_N$ equal:

$$\eta_1(z)=\eta_2(z)=\cdots=\eta_N(z)=\eta(z)\ ,\qquad t_1=t_2=\cdots=t_N=t\ , \tag{3.7}$$

$$i\hbar\frac{\partial}{\partial t}\Psi=\sum_{a=1}^{N}\frac{\partial x_a}{\partial\eta_a}\cdot p_a\Psi\ . \tag{3.8}$$

The simplest choice for η in this case would correspond to hyperplanes defined by a constant timelike unit vector $\mathbf{n}$:

$$\eta_a=\mathbf{n}\cdot x_a\quad(a=1,\ldots,N),\ \mathbf{n}^2=1\ . \tag{3.9}$$

In quantum mechanics the physical phase space Γ^* of classical mechanics is replaced by a physical Hilbert space $\mathscr{H}^*$, spanned by those wave functions which (i) satisfy the constraints (3.3) and (3.5), (ii) have a positive finite norm (also extended to Dirac's distributional sense), and (iii) have a positive mean value for the total energy operator P_0 and, in the two-particle case, positive eigenvalues for each of the longitudinal momenta $\hat{\mathbf{p}}\cdot p_a$ $(a=1,2)$; in the N-particle case $(N>2)$ the last condition is substituted by the positivity of the eigenvalues of generalized longitudinal momenta $\hat{\mathbf{p}}\cdot p_a'$ $(a=1,\ldots,N)$, to be specified below.

Condition (iii) is a consequence of the individual nature of each constituent of the N-particle system and is a covariant generalization of a similar condition adopted in one-particle relativistic quantum mechanics. We should emphasize at this point that, in the presence of interactions, the individual momenta p_a do not transform as four-vectors under Lorentz transformations. This can be checked even at the classical level with the Dirac brackets. This means that the operators $p_a{}^2$ do not define Lorentz-invariant quantities and cannot be assigned, in general, de-

finite signs. In order to remedy this difficulty we have to define or construct individual Lorentz-invariant "energy" operators, the signs of the mean values of which could classify the space of solutions of Eqs. (3.3) into definite subspaces.

In the two-particle case these operators are manifestly provided by the longitudinal momentum operators $\hat{\mathbf{p}}\cdot p_a \equiv p_{aL}$ $(a=1,2)$. Since in this case the potential V [see (2.15)] is independent of the longitudinal coordinate variables, then p_{1L} and p_{2L} commute with the wave equation operators H_1 and H_2 and hence are Lorentz invariant quantities. Furthermore the wave function can be considered as an eigenfunction of both p_{1L} and p_{2L}:

$$\Psi(x_1,x_2)=e^{-i(p_{1L}x_{1L}+p_{2L}x_{2L})}\psi(x^T)\ . \tag{3.10}$$

Equations (3.3) can then be written as

$$({p_{1L}}^2-{m_1}^2+{v^T}^2-V)\psi=0\ , \tag{3.11a}$$

$$({p_{2L}}^2-{m_2}^2+{v^T}^2-V)\psi=0\ . \tag{3.11b}$$

Let us suppose for simplicity that the potential V does not depend, in the c.m. frame, upon the momentum operators p_{1L},p_{2L},p^2. Then if for an appropriate choice of the shape of V, the operator $-({v^T}^2-V)$ has a positive eigenvalue λ^2, Eqs. (3.11) lead to the eigenvalues of ${p_{1L}}^2$ and ${p_{2L}}^2$:

$${p_{1L}}^2={m_1}^2+\lambda^2\ , \tag{3.12a}$$

$${p_{2L}}^2={m_2}^2+\lambda^2\ , \tag{3.12b}$$

which in turn lead to four kinds of eigenvalues for the set (p_{1L},p_{2L}):

$$\begin{aligned} p_{1L}&=\pm({m_1}^2+\lambda^2)^{1/2}\ ,\\ p_{2L}&=\pm({m_2}^2+\lambda^2)^{1/2}\ . \end{aligned} \tag{3.13}$$

The physical Hilbert space $\mathscr{H}^*$ is chosen as corresponding to the positive solutions of each p_{1L} and p_{2L}. This also ensures the positivity of $(p^2)^{1/2}$ and contributes, in the spin-0 case, to the positivity of the norm. (See the Appendix.)

If V depends on p_{1L} and p_{2L}, then Eqs. (3.12) take the form

$${p_{1L}}^2={m_1}^2+\lambda^2(p_{1L},p_{2L})\ , \tag{3.14a}$$

$${p_{2L}}^2={m_2}^2+\lambda^2(p_{1L},p_{2L})\ . \tag{3.14b}$$

We assume that the physically acceptable potentials are those which again lead to two real solutions with opposite signs in each p_{1L} and p_{2L}:

$$p_{1L}=\pm|\lambda_{1\pm}|\ ,\quad p_{2L}=\pm|\lambda_{2\pm}|\ , \tag{3.15}$$

with the physical Hilbert space $\mathscr{H}^*$ corresponding to the subspace of solutions with positive signs. [Notice that the eigenvalues (3.15) are related by the relation ${p_{1L}}^2-{p_{2L}}^2={m_1}^2-{m_2}^2$.]

We can generalize these features to the N-particle case $(N>2)$. Here, it may happen that the N-particle system is composed of several noninteracting subsystems. In this case we have to treat each subsystem independently from the others. Therefore, without loss of generality, we can consider the case where the N-particle system does not contain noninteracting subsystems and is characterized by a total momentum p. Here, however, the problem is more complicated, because in general the potentials exhibit dependences on the longitudinal relative coordinates, mostly when cluster decomposition requirements are imposed.[13] In this case the $\hat{\mathbf{p}}\cdot p_a$'s no longer commute with the first-class constraints H_b and are not Lorentz-invariant quantities (in the sense of the Dirac-brackets formalism or the "star" formalism, see below), neither can they be treated as eigenvalues.

To solve this difficulty we have to construct generalized longitudinal momentum operators $p\cdot p'_a$ $(a=1,\ldots,N)$, which commute with the H's and among themselves, i.e., satisfy the equations

$$[p\cdot p'_a,H_b]=0\quad (a,b=1,\ldots,N)\ , \tag{3.16}$$

$$[p\cdot p'_a,p\cdot p'_b]=0\quad (a,b=1,\ldots,N)\ , \tag{3.17}$$

and thus define Lorentz-invariant quantities. Equations (3.16) and (3.17) define quasilinear partial differential equations, which have solutions, because they are compatible among themselves. The latter fact can be verified by taking the commutators of Eqs. (3.16) and (3.17) with H_c and $p\cdot p'_c$ $(c=1,\ldots,N)$, using Jacobi identity and also assuming Eqs. (3.4) and of course (3.16) and (3.17). $p\cdot p'_a$ can be searched for in the form

$$p\cdot p'_a=p\cdot p_a+p\cdot\tilde{p}_a\quad (a=1,\ldots,N)\ , \tag{3.18}$$

where $\tilde{p}_{a\mu}$ are functions of the relative coordinates and of the momenta of the particles of the system appearing through terms depending on the interaction potentials. The Cauchy conditions of the differential equations (3.16) and (3.17) should be chosen such that when the total interaction is removed then each $p\cdot p'_a$ reduces to $p\cdot p_a$. (The latter condition could even be refined by imposing cluster decomposition requirements.[13]) $p\cdot p'_a$ should also satisfy the obvious relation

$$\sum_{a=1}^{N} p\cdot p'_a=p^2\ , \tag{3.19}$$

which could be trivially realized by considering it as the defining relation of one of the $p\cdot p'_b$'s.

One consequence of Eqs. (3.16) and (3.17) is that the wave function Ψ can be simultaneously considered as an eigenfunction of each $\hat{\mathbf{p}}\cdot p'_a$ and that the latter can be treated as an eigenvalue, as in the two-particle case for $\hat{\mathbf{p}}\cdot p_1$ and $\hat{\mathbf{p}}\cdot p_2$.

We assume that the physically acceptable potentials are those which lead for each $\hat{\mathbf{p}}\cdot p'_a$ to two real eigenvalues with opposite signs, thus splitting the space of solutions of Eqs. (3.3) into 2^N subspaces. The physical Hilbert space $\mathscr{H}^*$ will correspond to the subspace where all eigenvalues $\hat{\mathbf{p}}\cdot p'_a$ have positive signs.

Notice that, as in the two-particle case of Eqs. (3.14), the eigenvalues of $\hat{\mathbf{p}}\cdot p'_a$ will not be independent from each other, and actually will be characterized by a single eigenvalue, essentially determining the eigenvalue of p^2. This is a consequence of the compatibility conditions that Eqs. (3.3) have to satisfy. In other words, if the eigenvalue equation of p^2 is solved, then all the eigenvalues of the in-

dividual "energy" operators $\hat{\mathbf{p}}\cdot p_a'$ $(a=1,\ldots,N)$ will be fixed through Eqs. (3.3). These do not leave any freedom for relative energy excitations.

The Poincaré invariance of the theory in presence of the constraints is realized by means of the "star" operators, the classical analogs of which were introduced by Bergmann and Komar.[38] A star operator A^* is a redefinition of an operator A by means of appropriate Lagrange multipliers such that it preserves the constraints when generating canonical transformations with commutators. In classical mechanics a star variable has the following expression:

$$A^*=A-\sum_{r,s}\{A,\phi_r\}C_{rs}\phi_s\ , \tag{3.20}$$

where the ϕ's form a set of an even number of second-class constraints, such as (2.3) and (2.4), and C is the inverse of the matrix $\{\phi,\phi\}$. The starred version of a constraint ϕ_t is identically zero:

$$\phi_t^*=0\ . \tag{3.21}$$

This means that the physical degrees of freedom of the theory are correctly accounted for. The Poisson brackets of a star variable A^* with any of the constraints ϕ_t is weakly zero:

$$\{A^*,\phi_t\}\approx 0\ . \tag{3.22}$$

In classical mechanics the use of a star variable acting with Poisson brackets is equivalent to that of an ordinary variable acting with Dirac brackets.[1,6,35,36] In quantum mechanics, however, the quantized version of the Dirac brackets with arbitrary operators (in order to preserve manifest covariance) does not satisfy the product law and therefore the use of star operators acting with ordinary commutators seems to be necessary. The quantized version of an arbitrary star operator is more complicated than its classical analog (3.20), because one has to ensure that products and commutators of star operators are also starred and that products of second-class constraints are unambiguously defined. In general it will involve an infinite series of products of the constraints. We shall return to this general question in a separate work, but we note here that for our present purpose, that is for the canonical realization of the Poincaré algebra, these complications do not arise, as we show below.

In the present case the constraints ϕ are composed of N mass-shell constraints H_a, of the type (3.3), and of N time constraints χ_a, of the type (3.5). The $(2N)^2$-dimensional operator matrix $[\phi,\phi]$ reduces to its off-diagonal submatrices $[H,\chi]$ and $[\chi,H]$, because $[H_a,H_b]\approx 0$ and $[\chi_a,\chi_b]=0$ (the χ's are independent of the momenta). Furthermore since the H's are manifestly Poincaré invariant, then $[P_\mu,H]=[M_{\mu\nu},H]=0$ and the star Poincaré group generators take the form

$$A_r^*=A_r-\sum_{a,b=1}^{N}[A_r,\chi_a]C_{ab}H_b\quad(r=1,\ldots,10)\ , \tag{3.23}$$

where A_r designates one of the operators P_μ and $M_{\mu\nu}$ [given by Eqs. (2.2)] and C is the inverse of the operator matrix $[H,\chi]$ (we are assuming that the difficulties of the evaluation of C have been overcome).

It can easily be checked that the star operators (3.23) satisfy (weakly) the Poincaré algebra in the physical Hilbert space $\mathscr{H}^*$ (i.e., when applied on wave functions satisfying the constraint equations). If $\Psi(x_1,\ldots,x_N)$ is a wave function belonging to $\mathscr{H}^*$ then also

$$P_\mu^*\Psi\approx P_\mu\Psi,\quad M_{\mu\nu}^*\Psi\approx M_{\mu\nu}\Psi\ , \tag{3.24}$$

and under the action of the star Poincaré group generators, Ψ still transforms covariantly [as in (3.2)].

The physical interpretation of these results is the following. Let $|\Psi(t_1,\ldots,t_N)\rangle$ be a state belonging to $\mathscr{H}^*$ considered at the observation times $(t_1,\ldots,t_N)$ of particles $(1,\ldots,N)$, respectively, in a reference frame S. In the relativistic Schrödinger picture the star operators P_μ^* and $M_{\mu\nu}^*$ generate canonical transformations (unitary if an appropriate scalar product is constructed), associated with changes of reference frames, which transform $|\Psi(t_1,\ldots,t_N)\rangle$ into $|\Psi'(t_1,\ldots,t_N)\rangle$ representing the state in the transformed reference frame S' at times $(t_1,\ldots,t_N)$ therein. The observation times are the same in both reference frames because by construction the canonical transformations generated by star operators leave invariant the time constraints (this is necessary to give a unified physical interpretation to canonical realizations of relativity groups). Also notice that time evolutions are taken into account as changes of reference frames (global time displacements of the latter). (See Ref. 6, Chap. 16.)

The procedure developed above for constructing quantized systems of interacting spinless particles is also applicable to systems involving spin-$\frac{1}{2}$ particles. The modifications concern the expressions of the various quantities we have met, but not the method itself. Thus the wave function Ψ becomes now a spinor, the Lorentz group generators $M_{\mu\nu}$ of (2.2) contain in addition spin matrices, and some of the mass-shell constraints H_a represent generalizations of the Dirac equation rather than that of the Klein-Gordon equation.

In the remaining part of this work we shall concentrate on two-particle systems and examine more explicitly the corresponding wave equations.

IV. TWO SPIN-0 BOSON SYSTEMS: CONNECTION WITH FIELD THEORY

For two spinless particle systems, the wave equations (3.3) become, after taking into account the particular choice of (2.8) and (2.9)

$$H_a\Psi\equiv(p_a{}^2-m_a{}^2-V)\Psi=0\quad(a=1,2)\ . \tag{4.1}$$

The potential V is, as in Eq. (2.15), a Poincaré-invariant function of r, v, and p, defined in Eqs. (2.12). Since p and x commute, the operator r is well defined. (If Ψ is an eigenfunction of the operator p, one simply replaces the latter in r by its eigenvalue. Unless specified otherwise, we shall not distinguish between both notations.) If V depends on the relative momentum v, one has to symmetrize or order its expression with respect to r and v in such a way that the eigenvalues of p^2 come out to be real, if no

other reason forbids this result. On the other hand, time-reversal invariance demands that every odd expression of r in V be accompanied by a factor i.

One can check that the compatibility condition (3.4) is satisfied in its quantized version, with V given by Eq. (2.15), in the strong sense:

$$[H_1,H_2]=[p\cdot v,V]=0\ . \tag{4.2}$$

We now examine the implications of the two equations (4.1) on the wave functions. Specializing to the case of eigenfunctions of the total momentum p and taking the difference of the two equations (4.1), we get

$$p\cdot v\Psi=\tfrac{1}{2}(m_1{}^2-m_2{}^2)\Psi\ . \tag{4.3}$$

This equation determines the dependence of the wave function on the longitudinal component x_L of x (which is essentially the relative time variable). The solution is

$$\Psi(x_1,x_2)=e^{-ip\cdot X}e^{-i(m_1{}^2-m_2{}^2)p\cdot x/(2p^2)}\psi(x^T)\ , \tag{4.4}$$

and $\psi(x^T)$ defines an "internal" wave function.

Taking the sum of the two equations (4.1), one gets the "eigenvalue" equation:

$$\left[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{1}{4p^2}(m_1{}^2-m_2{}^2)^2+v^{T2}-V\right]\psi(x^T)=0\ , \tag{4.5}$$

which is a three-dimensional Schrödinger-type equation. For V chosen of first order in c^2 (c is the velocity of light), Eq. (4.5) reduces, in the nonrelativistic limit, to a Galilei-invariant Schrödinger equation.

The wave equations (4.1) can also be generalized to include nonlocal potentials (in x^T) without altering any of the results obtained so far. Such equations have the form

$$(p_a{}^2-m_a{}^2)\Psi(X,x)=\int V(x^T,x'^T,p)\Psi(X,x_L,x'^T)\times d^3x'^T\ \ (a=1,2)\ , \tag{4.6}$$

where V is a Poincaré-invariant function of its arguments, may depend on p^2 and also act as a derivative operator in the relative (transverse) coordinates, and x'^T is defined as the transverse part of x' with respect to p, with $x'_L=x_L$. The function Ψ has the same dependence on the longitudinal variable x_L as in Eq. (4.4) and one ends up with a three-dimensional nonlocal Schrödinger-type equation.

The potential V appearing in Eqs. (4.1) or (4.6) is arbitrary in form and has to be chosen according to the physical problem which is dealt with.

Equations (4.6) are the most general Poincaré-invariant wave equations satisfying the compatibility condition (4.2) (up to Poincaré-invariant canonical transformations). Therefore one has to expect that any other Poincaré-invariant description of a two spin-0 particle system, having a nonrelativistic Galilei-invariant limit, is equivalent, by means of some nonsingular transformation, to that obtained with Eqs. (4.6). It appears that this is actually the case for the Bethe-Salpeter equation. It was shown in Ref. 33 that by algebraic transformations the BS equation can be brought into the form of Eqs. (4.6) for the sector of solutions of the former which have nonrelativistic limits (the so-called "normal" solutions), thus establishing a definite connection between the present framework of relativistic quantum mechanics and quantum field theory.

The relationships between the quantum-mechanical wave function Ψ and potential V on the one hand and the BS wave function Φ and kernel D on the other involve an iteration series in terms of the latter through integral operators. If specific approximations are utilized the above relations may still be simplified. In particular, in the ladder approximation they reduce to the following two coupled equations (for the equal-mass case $m_1=m_2=m$):

$$\Psi(X,x)=\Phi(X,x)+\frac{i}{4(p^2)^{1/2}}\int dx'_L D(x'_L,x^T,p_1,p_2)\left[\frac{1}{b(b+a)}(e^{-i(b-a)|x_L-x'_L|}+e^{-i(b+a)|x_L-x'_L|})+\frac{2}{(b^2-a^2)}(e^{-i(b-a)|x_L-x'_L|}-1)\right]\Phi(X,x'_L,x^T)\ , \tag{4.7}$$

$$\left[\frac{p^2}{4}+v^{T2}-m^2\right]\Psi(X,x)=-\frac{i}{2(p^2)^{1/2}}\int dx_L D(x_L,x^T,p_1,p_2)\Phi(X,x_L,xT)\ , \tag{4.8}$$

where

$$a=\tfrac{1}{2}(p^2)^{1/2},\ \ b=(m^2-v^{T2})^{1/2}\ , \tag{4.9}$$

and Ψ satisfies, by construction, Eq. (4.3) for $m_1=m_2$. The momentum dependences of the kernel D here may arise either from the couplings of the mediating field with the external particles or from an effective dependence of the coupling constants on p^2.

In perturbation theory relation (4.7) can be inverted and Φ expressed in terms of Ψ and then replaced in the integral in Eq. (4.8), thus giving, by comparison with Eqs. (4.6), the relationship between V and D. To lowest order in perturbation theory one has

$$\Psi\simeq\Phi\ , \tag{4.10a}$$

$$V(x^T,p_1,p_2)\simeq-\frac{i}{2(p^2)^{1/2}}\int dx_L D(x_L,x^T,p_1,p_2)\ , \tag{4.10b}$$

which is nothing but the covariant form of the usual instantaneous approximation[30,34] of the kernel D.

The contributions of the higher-order terms in the integral relations (4.7) and (4.8) can be taken into account in an approximate but compact form if one uses an appropriate approximation for the action of the kernel D. This consists in replacing the kernel D in both integrals (4.7) and (4.8) by its relativistic instantenous approximation concentrated on the hyperplane $x_L=0$; that is, by making the following replacement:

$$D(x_L,x^T,p_1,p_2)\to\delta(x_L)\int dx'_L D(x'_L,x^T,p_1,p_2)\ . \tag{4.11}$$

The relativistic instantaneous approximation (4.11) is also close in spirit to the method introduced by Blankenbecler and Sugar[24] in the framework of the quasipotential approach to reduce the BS equation to a three-dimensional one. Here one subtracts from and adds to the kernel K of the BS equation, written in its integral form, another kernel $\overline{K}$, appropriately chosen, such that $\overline{K}$, rather than K, serves as the starting point to the physical calculations and reduces at the same time the integral equation to a three-dimensional one. [Although, in our case,[33] the three-dimensional reduction has been obtained independently of approximation (4.11).] The rest of the kernel, K-$\overline{K}$, is then treated as a perturbation in an iteration series. (See also a discussion on this matter in Ref. 29.)

When approximation (4.11) is used, then Eqs. (4.7) and (4.8) can be completely solved for Φ and V in terms of Ψ and D, respectively. One gets

$$\phi(x)=\frac{1}{2b}[(b+a)e^{-i|x_L|(b-a)} -(b-a)e^{-i|x_L|(b+a)}]\psi(x^T)\ , \tag{4.12a}$$

$$\Phi(X,x)=e^{-ip\cdot X}\phi(x)\ . \tag{4.12b}$$

$$V=-\frac{i}{4}\int dx_L D(x_L,x^T,p_1,p_2)\frac{1}{(m^2-v^{T^2})^{1/2}}\ , \tag{4.13}$$

Equation (4.13) shows that the potential V is still nonlocal in x^T. Similarly Eq. (4.12) yields, on the hyperplane $x_L=0$,

$$\phi(x_L=0,x^T)=\frac{(p^2)^{1/2}}{2(m^2-v^{T^2})^{1/2}}\psi(x^T)\ , \tag{4.14}$$

which is also a nonlocal relation in x^T.

In order to get a local expression for the potential V in the variables x^T, it is necessary to approximate the operator $(m^2-v^{T^2})^{-1/2}$ by some local function. The simplest choice is to replace the operator v^{T^2} by its mean value in the state Ψ or even by a common constant value for all the states Ψ:

$$(m^2-v^{T^2})^{-1/2}\simeq(m^2-\langle v^{T^2}\rangle)^{-1/2}\ . \tag{4.15}$$

One thus gets the approximate relations

$$V(x^T,p_1,p_2)\simeq-\frac{i}{4(m^2-\langle v^{T^2}\rangle)^{1/2}} \times\int dx_L D(x_L,x^T,p_1,p_2)\ , \tag{4.16}$$

$$\phi(x_L=0,x^T)\simeq\frac{(p^2)^{1/2}}{2(m^2-\langle v^{T^2}\rangle)^{1/2}}\psi(x^T)\ . \tag{4.17}$$

Relation (4.12) may also be used to fix the normalization condition of the wave function ψ, given that of the BS wave function ϕ. It is known that the latter is normalized through a four-dimensional integration.[32] The use of relations (4.11) and (4.12) permits us to evaluate the x_L integration and to transform the four-dimensional integral into a three-dimensional one. The result in the c.m. frame is

$$\int d^3x^T\left[\psi^*(x^T)\frac{p^2}{4(m^2-v^{T^2})^{1/2}}\psi(x^T)+i\psi^*(x^T)\left[\frac{p^2}{4(m^2-v^{T^2})}\right]^{1/2}\frac{\partial\tilde{D}}{\partial p^2}\left[\frac{p^2}{4(m^2-v^{T^2})}\right]^{1/2}\psi(x^T)\right]=1, \tag{4.18}$$

where

$$\tilde{D}(x^T,p_1,p_2)=\int dx_L D(x_L,x^T,p_1,p_2)\ . \tag{4.19}$$

If the BS kernel D does not explicitly depend on p^2 then the second term in (4.18) will be absent.

Relations of the type (4.7)–(4.18) may also be obtained in the unequal-mass case. We shall present a detailed derivation of the general results in a separate work.

To conclude, we summarize the three successive approximations which lead from the BS kernel to a local potential (in x^T) in quantum mechanics. These are (i) the ladder approximation of the BS kernel, (ii) the relativistic instantaneous approximation (4.11) of the kernel, and (iii) the mean value approximation (4.15) (or an equivalent one) of the nonlocal operator $(m^2-v^{T^2})^{-1/2}$.

V. SCALAR AND VECTOR INTERACTIONS

The connection established for the potential V with field-theoretic quantities allows us to examine its tensor structure in more detail. We shall concentrate here on the local approximation of V, where the latter is not an integral operator in x^T. In this case the dependence of V on the relative momentum v can only arise from the vertices of the mediating field with particles 1 and 2. (We assume that the coupling constants g, as well as effective approximations of the kernel could exhibit complicated p^2 dependences.)

For scalar interactions the momenta p_1 and p_2 do not appear at the vertices and therefore we have

$$V=V(r^2,p^2)\ . \tag{5.1}$$

For vector interactions p_1 and p_2 appear linearly at the vertices and to lowest order of the coupling constants we have

$$V=[p_1^{\mu},[p_2^{\nu},C_{\mu\nu}(r,p)]_+]_+ \tag{5.2}$$

([,]$_+$ is the anticommutator), where $C_{\mu\nu}$ represents the relativistic instantaneous approximation (4.11) of a (effective) propagator of a vector field, and satisfies the properties

$$C_{\mu\nu}(r,p)=C_{\nu\mu}(r,p)=C_{\mu\nu}(-r,p)\ . \tag{5.3}$$

To first order in C, the term (5.2) could also arise from a minimal substitution, in the free equations, of the type

$$\begin{aligned} p_{1\mu}&\to p'_{1\mu}=p_{1\mu}-A_{1\mu}\equiv p_{1\mu}-[p_2^{\nu},C_{\mu\nu}]_+\ , \\ p_{2\mu}&\to p'_{2\mu}=p_{2\mu}-A_{2\mu}\equiv p_{2\mu}-[p_1^{\nu},C_{\mu\nu}]_+\ . \end{aligned} \tag{5.4}$$

This observation allows us to generalize the form of the interaction (5.2) by including higher-order terms in such a way that the total interaction still arises by minimal substitutions of the type (5.4), but with more general expressions, and that the compatibility condition (4.2) and (4.3) still be satisfied.

To this aim we take, for $A_{a\mu}$,

$$\begin{aligned} A_{1\mu}&=[p_2^{\nu},C_{\mu\nu}]_+ +[p_1^{\nu},D_{\mu\nu}]_+ \\ &=\tfrac{1}{2}[v^{\nu},A_{\mu\nu}]_+ +\tfrac{1}{2}p^{\nu}B_{\mu\nu}\ , \\ A_{2\mu}&=[p_1^{\nu},C_{\mu\nu}]_+ +[p_2^{\nu},D_{\mu\nu}]_+ \\ &=-\tfrac{1}{2}[v^{\nu},A_{\mu\nu}]_+ +\tfrac{1}{2}p^{\nu}B_{\mu\nu}\ , \end{aligned} \tag{5.5}$$

where the function D does not contain first-order effects in the coupling constant g^2 when $C_{00}\neq 0$ and in the last expressions we have expressed p_1 and p_2 in terms of v and p [Eq. (2.12)], and C and D by linear combinations of A and B.

Replacing now $p'_1\equiv p_1-A_1$ and $p'_2\equiv p_2-A_2$ in

$$(p_a'^2-m_a^2)\Psi=0\quad (a=1,2)\ , \tag{5.6}$$

and demanding that the constraint (4.3) be satisfied, that is, that

$$(p_1'^2-p_2'^2)\Psi=(p_1^2-p_2^2)\Psi=(m_1^2-m_2^2)\Psi\ , \tag{5.7}$$

which guarantees the validity of the compatibility condition (4.2), one gets the condition

$$(A_{\mu\nu}+B_{\mu\nu}-A_{\nu\alpha}B_{\mu}{}^{\alpha})p^{\mu}=0\ . \tag{5.8}$$

Equations (5.6) then become

$$\begin{aligned} (p_1'^2-m_1^2)\Psi&=(p_2'^2-m_2^2)\Psi \\ &=\Bigg[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1^2+m_2^2)+\frac{(m_1^2-m_2^2)^2}{4p^2}+v^{T^2}-\tfrac{1}{2}B_{\mu\nu}p^{\mu}p^{\nu}-2A_{\mu\nu}v^{\mu}v^{\nu}+\tfrac{1}{4}B_{\mu\alpha}B^{\mu\beta}p^{\alpha}p_{\beta} \\ &\quad +A_{\mu\alpha}A^{\mu\beta}v^{\alpha}v_{\beta}-\tfrac{1}{2}(g_{\mu\nu}-A_{\mu\nu})[v^{\nu},[v_{\alpha},A^{\mu\alpha}]]-[v_{\nu},(2g_{\mu\alpha}-A_{\mu\alpha})A^{\mu\nu}]v^{\alpha} \\ &\quad +\tfrac{1}{4}[v^{\nu},A_{\mu\nu}][v_{\alpha},A^{\mu\alpha}]\Bigg]\Psi=0\ . \end{aligned} \tag{5.9}$$

We can analyze more explicitly Eqs. (5.8) and (5.9) by specifying the tensor structure of the propagator functions A and B. We shall consider three different cases.

Case a:

$$A_{\mu\nu}=g_{\mu\nu}A(r^2,p^2),\quad B_{\mu\nu}=g_{\mu\nu}B(r^2,p^2) \tag{5.10}$$

(analogous to the Feynman gauge).

Condition (5.8) and Eq. (5.9) become, respectively,

$$B=-\frac{A}{1-A}\ , \tag{5.11}$$

$$\begin{aligned} &\Bigg[\tfrac{1}{4}(1-B)^2p^2-\tfrac{1}{2}(m_1^2+m_2^2)+(1-A)^2\frac{(m_1^2-m_2^2)^2}{4p^2}+(1-A)^2v^{T^2}+\frac{\hbar^2}{2}(1-A)(6\dot{A}+4r^2\ddot{A}) \\ &\qquad -4i\hbar(1-A)\dot{A}r\cdot v-\hbar^2r^2\dot{A}^2\Bigg]\Psi=0\ , \end{aligned} \tag{5.12}$$

where we have used the notation

$$\dot{V}=\frac{\partial V}{\partial r^2}\ . \tag{5.13}$$

Case b:

$$A_{\mu\nu}=(g_{\mu\nu}-\hat{\mathbf{p}}_{\mu}\hat{\mathbf{p}}_{\nu})A(r^2,p^2),\quad B_{\mu\nu}=(g_{\mu\nu}-\hat{\mathbf{p}}_{\mu}\hat{\mathbf{p}}_{\nu})B(r^2,p^2) \tag{5.14}$$

(analogous to the temporal gauge).

In this case condition (5.8) is identically satisfied, and Eq. (5.9) becomes

$$\left[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}+(1-A)^2v^{T^2}+\frac{\hbar^2}{2}(1-A)(6\dot{A}+4r^2\ddot{A})-4i\hbar(1-A)\dot{A}r\cdot v-\hbar^2r^2\dot{A}^2\right]\Psi=0\ . \tag{5.15}$$

Case c:

$$A_{\mu\nu}=\left[g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu-\frac{r_\mu r_\nu}{r^2}\right]A(r^2,p^2),\quad B_{\mu\nu}=\left[g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu-\frac{r_\mu r_\nu}{r^2}\right]B(r^2,p^2) \tag{5.16}$$

(analogous to the temporal gauge with the additional condition $r^\mu A_{\mu\nu}=r^\mu B_{\mu\nu}=0$.

Condition (5.8) is identically satisfied and Eq. (5.9) becomes

$$\left[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}+v^{T^2}-(2A-A^2)\frac{1}{r^2}(r^2v^{T^2}-r^\alpha r^\beta v_\alpha^T v_\beta^T-2i\hbar r\cdot v)-\frac{\hbar^2}{r^2}(A-A^2)-2\hbar^2\dot{A}\right]\Psi=0\ . \tag{5.17}$$

On introducing the internal angular momentum of the system by means of the Pauli-Lubanski vector

$$W_\mu=\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}P^\nu M^{\alpha\beta}\quad(\epsilon_{0123}=+1)\ , \tag{5.18}$$

which reduces in the two spin-0 particle case to the relative orbital angular momentum

$$\begin{aligned}&W_\mu=W_{L\mu}\equiv\epsilon_{\mu\nu\alpha\beta}p^\nu r^\alpha v^\beta\ ,\\&W_L{}^2=-p^2(r^2v^{T^2}-r^\alpha r^\beta v_\alpha^T v_\beta^T-2i\hbar r\cdot v)\ ,\end{aligned} \tag{5.19}$$

Eq. (5.17) takes the form

$$\left[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}+v^{T^2}+(2A-A^2)\frac{W_L{}^2}{r^2p^2}-\frac{\hbar^2}{r^2}(A-A^2)-2\hbar^2\dot{A}\right]\Psi=0\ . \tag{5.20}$$

In all the above equations the relative-time (x_L) dependence of the wave function Ψ is given by (4.4) because of the "relative" wave equations (4.3) and (5.7). Equations (5.9), (5.12), (5.15), and (5.20) represent the eigenvalue equation (4.5) where the potential V has been replaced by a more detailed expression taking into account its relative momentum dependence in the vector interaction case. They are most useful when one uses effective expressions for the interaction but maintains its definite tensor properties.

Other interesting expressions for the propagator functions $A(B)$ are

$$A_{\mu\nu}=g_{\mu\nu}A-\frac{r_\mu r_\nu}{(r^2)^2}\int^{r^2}r^2\dot{A}\,dr^2 \tag{5.21}$$

(analogous to the Landau gauge) and

$$A_{\mu\nu}=\left[g_{\mu\nu}+\frac{1}{m^2}\partial_\mu\partial_\nu\right]A \tag{5.22}$$

(analogous to the propagator of a vector particle with mass m).

VI. SPIN-$\frac{1}{2}$ FERMION-ANTIFERMION SYSTEMS

In this section and the following one we study two-particle systems involving spin-$\frac{1}{2}$ particles. We shall concentrate on fermion-antifermion systems, but two-fermion systems can also be treated in an analogous way.

Fermionic systems do not have simple classical analogs and therefore we have to construct from the start quantized systems and wave equations. In this respect, within the quantization framework defined in Sec. III, the results already obtained for two spin-0 boson systems may serve as a guideline for our investigation.

Let us consider a system composed of a fermion with mass m_1 and an antifermion with mass m_2. The covariant wave function of the system is now a 16-component spinor of rank two:

$$\Psi=\Psi_{\alpha_1\alpha_2}(x_1,x_2)\quad(\alpha_1,\alpha_2=1,\ldots,4)\ , \tag{6.1}$$

where α_1 (α_2) refers to the spinor index of particle 1 (2). In the free case the wave function can be expressed in terms of the individual wave functions

$$\Psi^{\text{free}}_{\alpha_1\alpha_2}(x_1,x_2)=\Psi_{1\alpha_1}(x_1)\bar{\Psi}_{2\alpha_2}(x_2)\ , \tag{6.2}$$

and satisfies the two free wave equations

$$H_1\Psi^{\text{free}}\equiv(i\gamma_1\cdot\partial_1-m_1)\Psi^{\text{free}}=0\ , \tag{6.3a}$$

$$H_2\Psi^{\text{free}}\equiv\Psi^{\text{free}}(i\gamma_2\cdot\overleftarrow{\partial}_2+m_2)\ , \tag{6.3b}$$

which are the analogs, in the free case, of Eqs. (3.3). The γ_1 matrix acts on the left-side index α_1 and γ_2 acts on the right-side index α_2, and thus they commute.

The wave function Ψ can be covariantly decomposed on the basis of the Dirac γ matrices:

$$\Psi=S1+V_\mu\gamma^\mu+T_{\mu\nu}\sigma^{\mu\nu}+A_\mu\gamma^\mu\gamma^5+P\gamma^5\ , \tag{6.4}$$

where S,V,T,A,P are functions. It can also be represented in terms of (2×2) matrices involving (2×2) submatrices. To this end we use the following representation of the γ matrices:

$$\gamma_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad \gamma_i=-\begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix},$$
$$\gamma_5=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \gamma_5=\frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\quad(\epsilon_{0123}=1)\,. \tag{6.5}$$

The wave function can then be represented in the form

$$\Psi=\begin{bmatrix}\Psi_{+-} & \Psi_{++}\\ \Psi_{--} & \Psi_{-+}\end{bmatrix}, \tag{6.6}$$

where each Ψ_{rs} ($r,s=+$ or $-$) inside the matrix is itself a (2×2) matrix. The indices, $+,-$ refer to the kind of behavior in the nonrelativistic limit. In this limit it is the component Ψ_{++} which survives, while Ψ_{+-}, Ψ_{-+}, and Ψ_{--} behave as c^{-1}, c^{-1}, and c^{-2}, respectively.

We shall need below to know the behavior of the wave function under charge conjugation. For charge symmetric systems ($m_1=m_2$), we have

$$\Psi(x_1,x_2)\to\Psi^c(x_1,x_2)=C\Psi^t(x_2,x_1)C^{-1}\,,$$
$$C\gamma_\mu C^{-1}=-\gamma_\mu^t \tag{6.7}$$

(t is the transpose). The parity operation gives

$$\Psi(x_1,x_2)\to\Psi^P(x_1,x_2)$$
$$=\gamma_{10}\Psi(x_1^0,-\mathbf{x}_1,x_2^0,-\mathbf{x}_2)\gamma_{20}\,. \tag{6.8}$$

In order to simplify the notation, we shall adopt the following convention. The matrices $\gamma_{1\mu},\sigma_{1\mu\nu},\gamma_{15}$, which act on Ψ from the left (on the left-side index) will be written without the index 1:

$$\gamma_\mu\Psi\equiv\gamma_{1\mu}\Psi,\quad \sigma_{\mu\nu}\Psi\equiv\sigma_{1\mu\nu}\Psi\,,$$
$$\gamma_5\Psi\equiv\gamma_{15}\Psi,\quad \tilde{\gamma}_\mu=\gamma_\mu\gamma_5,\quad \sigma_{\mu\nu}\equiv\frac{1}{2i}[\gamma_\mu,\gamma_\nu]\,. \tag{6.9}$$

The matrices $\gamma_{2\mu},\sigma_{2\mu\nu},\gamma_{25}$ which act on Ψ from the right (on the right-side index) will be represented by matrices $\eta_\mu,\xi_{\mu\nu},\eta_5$ written on the left of Ψ:

$$\eta_\mu\Psi\equiv\Psi\gamma_{2\mu},\quad \eta_\mu\eta_\nu\Psi\equiv\Psi\gamma_{2\nu}\gamma_{2\mu}\,,$$
$$\xi_{\mu\nu}\Psi\equiv\Psi\sigma_{2\nu\mu},\quad \eta_5\Psi\equiv\Psi\gamma_{25}\,,$$
$$\tilde{\eta}_\mu\Psi\equiv\eta_\mu\eta_5\Psi=\Psi\gamma_{25}\gamma_{2\mu}\,, \tag{6.10}$$
$$\xi_{\mu\nu}=\frac{1}{2i}[\eta_\mu,\eta_\nu]\,.$$

The matrices η commute with the previous γ matrices (6.9) and they satisfy the same algebra as the latter. With these notations, Eqs. (6.3) become

$$H_1\Psi^{\text{free}}\equiv(\gamma\cdot p_1-m_1)\Psi^{\text{free}}=0\,, \tag{6.11a}$$
$$H_2\Psi^{\text{free}}\equiv(\eta\cdot p_2+m_2)\Psi^{\text{free}}=0\,. \tag{6.11b}$$

The generators of the Poincaré group in the space of the covariant wave functions are given by formulas analogous to (2.2), with the spin operators being incorporated in $M_{\mu\nu}$:

$$P_\mu=\sum_{a=1}^{2}p_{a\mu}\,,$$
$$M_{\mu\nu}=\sum_{a=1}^{2}(x_{a\mu}p_{a\nu}-x_{a\nu}p_{a\mu})-\frac{\hbar}{2}(\sigma_{\mu\nu}+\xi_{\mu\nu})\,. \tag{6.12}$$

In the physical Hilbert space $\mathscr{H}^*$, where the wave functions are solutions of the wave equations and satisfy the time constraints (3.5), the operators (6.12) have to be replaced by their starred version, as in (3.23). But as far as they act on wave functions belonging to $\mathscr{H}^*$, their action will be equivalent to that of their ordinary counterparts [Eq. (3.24)]. Therefore under the action of the Poincaré group in $\mathscr{H}^*$ the wave functions Ψ still transform covariantly:

$$\Psi(x_1,x_2)\to S(\Lambda)\Psi(\Lambda^{-1}x_1-a,\Lambda^{-1}x_2-a)S^{-1}(\Lambda)\,. \tag{6.13}$$

We turn now to the determination of the structure of the interaction potentials appearing in the mass-shell constraints. By analogy with the spin-0 case, we can introduce the interaction by adding potentials in Eqs. (6.11):

$$H_1\Psi\equiv(\gamma\cdot p_1-m_1-V_1)\Psi=0\,, \tag{6.14a}$$
$$H_2\Psi\equiv(\eta\cdot p_2+m_2+V_2)\Psi=0\,. \tag{6.14b}$$

The potentials V_1 and V_2 which are manifestly Poincaré invariant (including time reversal and parity) contain now γ and η matrices (according to the tensor structure of the interaction) and may therefore involve several scalar, vector, or tensor functions.

A first condition can be imposed on the potentials V_1 and V_2 by demanding that the interaction be charge-conjugation invariant (including the mass exchange $m_1\leftrightarrow m_2$). In this case Eq. (6.14b) must be obtained from Eq. (6.14a) by charge conjugation (and mass exchange $m_1\leftrightarrow m_2$). If the potential V_1 is represented in the form

$$V_1=V_1(1,2;\gamma,\eta) \tag{6.15}$$

(the indices 1 and 2 inside the parentheses representing particle indices in the coordinates, momenta, and masses), then by using transformation (6.7) we get the relation between V_2 and V_1:

$$V_2=V_1(2,1;-\eta,-\gamma)\,. \tag{6.16}$$

The second condition on the potentials comes from the compatibility (or integrability) condition (3.4) of the two equations (6.14):

$$[H_1,H_2]\Psi\approx 0\,. \tag{6.17}$$

As in the spin-0 case this condition also guarantees the Poincaré invariance of the theory in presence of the constraints [wave equations (6.14) and time constraints (3.5); see Sec. III].

Equation (6.17) which is a quasilinear equation for the potentials is still too general. This is related to the fact that one can always apply to Eqs. (6.14) Poincaré- and charge-conjugation-invariant canonical transformations and modify rather arbitrarily the potentials. Therefore

among the general solutions of Eq. (6.17) we have to select a class corresponding to a particular configuration in the space of canonical transformations mentioned above. As in Eq. (4.3) we select among the solutions of Eq. (6.17) those potentials which lead through Eqs. (6.14) to the equation

$$(p_1^2-p_2^2)\Psi=(m_1^2-m_2^2)\Psi\,, \tag{6.18}$$

thus fixing in a definite way the relative time dependence of Ψ [Eq. (4.4)].

To see the additional restrictions imposed by condition (6.18) on the potentials V_1 and V_2, we multiply Eqs. (6.14a) and (6.14b) by $(\gamma\cdot p_1+m_1)$ and $(\eta\cdot p_2-m_2)$, respectively; we get

$$[p_1^2-m_1^2-(\gamma\cdot p_1+m_1)V_1]\Psi=0\,, \tag{6.19a}$$

$$[p_2^2-m_2^2+(\eta\cdot p_2-m_2)V_2]\Psi=0\,. \tag{6.19b}$$

Comparison of these equations with Eq. (6.18) shows that we must have

$$-(\gamma\cdot p_1+m_1)V_1\Psi=(\eta\cdot p_2-m_2)V_2\Psi\,. \tag{6.20}$$

A general class of solutions to Eq. (6.20) is given by the following type of potentials:

$$\begin{aligned} V_1&=(-\eta\cdot p_2+m_2)V\,,\\ V_2&=(\gamma\cdot p_1+m_1)V\,, \end{aligned} \tag{6.21}$$

where V is a symmetric function under the exchanges $1\leftrightarrow 2$ and $\gamma\leftrightarrow-\eta$ [Eqs. (6.15) and (6.16)]:

$$V(1,2;\gamma,\eta)=V(2,1;-\eta,-\gamma)\,. \tag{6.22}$$

Replacing V_1 and V_2 by their expressions (6.21) in Eqs. (6.19) and taking the commutator of the two operators acting on Ψ in Eqs. (6.19a) and (6.19b) we find the compatibility condition

$$[p_1^2-p_2^2,V]\equiv[p\cdot v,V]=0\,, \tag{6.23}$$

which shows that the potential V depends on the coordinate variables through the transverse relative coordinates x^T as in the spin-0 case:

$$V=V(x^T,p_1,p_2,\gamma,\eta)\,. \tag{6.24}$$

(V is a Poincaré-invariant function of its arguments.)

We now replace the potentials V_1 and V_2 by their expressions (6.21) in the wave equations (6.14); we get

$$H_1\Psi\equiv[\gamma\cdot p_1-m_1-(-\eta\cdot p_2+m_2)V]\Psi=0\,, \tag{6.25a}$$

$$H_2\Psi\equiv[\eta\cdot p_2+m_2+(\gamma\cdot p_1+m_1)V]\Psi=0\,. \tag{6.25b}$$

In order to check the compatibility (integrability) condition (6.17) we can use the following method. Instead of H_1 and H_2, we may consider $(\gamma\cdot p_1-m_1)$ and $(\eta\cdot p_2+m_2)$ as the independent operators. Then using $(\eta\cdot p_2+m_2)$ on Eq. (6.25a) and $(\gamma\cdot p_1-m_1)$ on Eq. (6.25b) and taking the difference of the two equations we get

$$[(p_1^2-p_2^2)-(m_1^2-m_2^2)]V\Psi=0\,, \tag{6.26}$$

which is satisfied on account of Eqs. (6.23) and (6.18). (The direct calculation of $[H_1,H_2]$ with the use of Eqs. (6.18) and (6.23) gives $[H_1,H_2]\approx-[\gamma\cdot p_1,V]H_1+[\eta\cdot p_2,V]H_2\approx 0$, which also establishes the above result, the commutators $[\gamma\cdot p_1,V]$ and $[\eta\cdot p_2,V]$ no longer containing the constraints.)

To summarize, the wave equations (6.25) where V satisfies Eqs. (6.22) and (6.23) and is Poincaré invariant, represent a compatible set of equations for Ψ and lead to a Poincaré-invariant formulation of the theory.

Equations (6.25) may also be rewritten differently by bringing the operator $\eta\cdot p_2$ of Eq. (6.25a) on the right of V and then using Eq. (6.25b) to eliminate the quantity $\eta\cdot p_2\Psi$, and similarly for $\gamma\cdot p_1$ in Eq. (6.25b). This procedure "diagonalizes" Eqs. (6.25) with respect to the operators $(\gamma\cdot p_1-m_1)$ and $(\eta\cdot p_2+m_2)$. We shall use it in Sec. VII when dealing with specific types of potentials.

As in the spin-0 case, Eq. (4.6), the potential V in Eqs. (6.25) may also be replaced by a nonlocal function in x^T, without any modification of the results so far obtained. In this case one gets the most general form of the potential V. Then it can be shown, along similar lines as in Ref. 26, that the wave function Ψ and the potential V are connected in a precise way with the wave function and the kernel of the Bethe-Salpeter equation, for the sector of "normal" solutions of the latter. We shall present these relationships in their general forms in a separate work; we present here their simplified expressions when one considers the BS kernel in its ladder approximation and then replaces it in the integral equations by its relativistic instantaneous approximation (4.11). One then gets the following relations (for the equal-mass case $m_1=m_2=m$):

$$\begin{aligned} \phi(x)=&\frac{1}{2b}[(b+a)e^{-i|x_L|(b-a)}-(b-a)e^{-i|x_L|(b+a)}]\psi(x^T)\\ &-\frac{1}{2b}\epsilon(x_L)\left\{\gamma\cdot\hat{\mathbf{p}}\left[\gamma\cdot\left[\frac{p}{2}+v^T\right]-m\right]-\eta\cdot\hat{\mathbf{p}}\left[\eta\cdot\left[\frac{p}{2}-v^T\right]+m\right]\right\}(e^{-i|x_L|(b-a)}-e^{-i|x_L|(b+a)})\psi(x^T)\\ &+\frac{1}{b}[(b-a)e^{-i|x_L|(b-a)}-(b+a)e^{-i|x_L|(b+a)}]\frac{i\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}}{4(p^2)^{1/2}}\int dx_L' D(x_L',x^T,p_1,p_2,\gamma,\eta)\phi(x_L=0,x^T)\,, \end{aligned} \tag{6.27}$$

$$V\psi(x^T)=-\frac{i}{2(p^2)^{1/2}}\int dx_L' D(x_L',x^T)\phi(0,x^T)\,, \tag{6.28}$$

where ϕ and D are the BS wave function and kernel, respectively, and a and b are defined in (4.9). Equations (6.27) and (6.28) are the analogs of Eqs. (4.12) and (4.13) of the spin-0 case. [The origin of the momentum dependence of the kernel

D is explained after Eq. (4.9).]

Equation (6.27) becomes, at $x_L=0$,

$$\phi(x_L=0,x^T)=\frac{a}{b}\psi(x^T)-\frac{i}{4b}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}\tilde{D}(x^T)\phi(0,x^T)\ , \tag{6.29}$$

where we have defined

$$\tilde{D}(x^T)=\int dx_L D(x_L,x^T,p_1,p_2,\gamma,\eta)\ . \tag{6.30}$$

It gives

$$\phi(0,x^T)=\left[1+\frac{i}{4b}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}\tilde{D}(x^T)\right]^{-1}\frac{a}{b}\psi(x^T)\ , \tag{6.31}$$

$$V=-\frac{i}{4}\tilde{D}(x^T)\left[1+\frac{i}{4b}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}\tilde{D}(x^T)\right]^{-1}\frac{1}{b}\ , \tag{6.32}$$

the last equation being also equivalent to

$$\left[1+\frac{i}{4b}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}\tilde{D}(x^T)\right]^{-1}=\frac{1}{b}(1+\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V)b\ . \tag{6.33}$$

Equation (6.32) shows that V is still nonlocal in x^T (mainly because of the operator $1/b$; for the usual types of interaction, D will not depend on the relative momentum v). In order to get for the potential a local expression in x^T, it is necessary to approximate the operator $(m^2-v^{T2})^{-1/2}$ by some local function, for instance as in (4.15).

As in the spin-0 case, relations (6.27)—(6.33) and approximation (4.11) may be used to reduce the normalization condition of the BS wave function to a three-dimensional integral. The result is in the c.m. frame:

$$\int d^3x^T\mathrm{Tr}\left[\bar{\psi}(x^T)\frac{1}{4b}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}\psi(x^T)-\bar{\psi}(x^T)\bar{V}\frac{1}{4b}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V\psi(x^T)\right]$$

$$-i\int d^3x^T\mathrm{Tr}\left[\bar{\psi}(x^T)(1+\bar{V}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}})\frac{a}{b}\frac{\partial\tilde{D}}{\partial p^2}\frac{a}{b}(1+\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V)\psi(x^T)\right]=1\ , \tag{6.34}$$

where $\bar{\psi}=(\gamma_0\eta_0\psi)^\dagger$ and $\bar{V}=\gamma_0\eta_0V^\dagger\gamma_0\eta_0$ (a dagger denotes Hermitian conjugation).

We end this section by observing that two-fermion systems can be studied in a way very similar to that of the fermion-antifermion case. Now the η matrices act on the wave function from the left (on the right-side index) and the charge-conjugation symmetry of the interaction has to be replaced by the exchange symmetry with respect to the two-fermion variables.

VII. SCALAR, PSEUDOSCALAR, AND VECTOR INTERACTIONS

The relationship (6.32) of the interaction potential V with the kernel of the BS equation in its ladder and relativistic instantaneous approximations fixes its expression in terms of field-theoretic quantities. Even if one uses for V an effective form, the above equation may still be used to understand the tensor structure of V in terms of the tensor structure of an effective interaction Lagrangian. In this respect we shall study in this section three types of interaction potential, which are local in x^T [i.e., after approximation (4.15) is made], scalar, pseudoscalar, and vector.

However, before proceeding to this study, we would like to introduce an appropriate transformation for the wave function in order to bring the total energy operator into a form which is manifestly Hermitian in the free norm in the c.m. frame. The fact that the wave function Ψ of Eqs. (6.25) does not satisfy this property is evident from the normalization condition (6.34), where the presence of the potential V in the first large parentheses means that it does not correspond to the free norm [even after the approximation (4.15) is made]. The advantage of the free norm (in the c.m. frame) is that the normalizability properties of the solutions there are more transparent.

It is also evident from Eq. (6.34) that the transformation which we have to use is the following [in the local approximation (4.15) and for Hermitian potentials V not depending explicitly on p^2]:

$$\Psi=[1-(\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V)^2]^{-1/2}\Psi'\ , \tag{7.1}$$

but we shall derive it, independently, from the structure of Eqs. (6.25) without using any connection with field theory.

A. Hermiticity property of the total energy operator

For simplicity we shall work in the local approximation (4.15) where $\langle v^{T2}\rangle$ is now replaced by a common constant

for all the states and V is assumed to be Hermitian (more precisely $V^\dagger\gamma_0\eta_0=\gamma_0\eta_0 V$) and p^2 independent. In this case, in the c.m. frame, the only p^2-dependent pieces in Eqs. (6.25) are the kinematic factors $\gamma\cdot p_1$ and $\eta\cdot p_2$ and thus we can isolate the total energy (or mass) operator. [In the c.m. frame x^T reduces to the vector $(0,\mathbf{x})$.]

Equations (6.25) yield

$$\hat{\mathbf{p}}\cdot p_1\Psi=[-\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}\,\hat{\mathbf{p}}\cdot p_2 V+m_1\gamma\cdot\hat{\mathbf{p}} -\gamma\cdot\hat{\mathbf{p}}\gamma\cdot v^T+\gamma\cdot\hat{\mathbf{p}}(\eta\cdot v^T+m_2)V]\Psi\ , \tag{7.2a}$$

$$\hat{\mathbf{p}}\cdot p_2\Psi=[-\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}\,\hat{\mathbf{p}}\cdot p_1 V-m_2\eta\cdot\hat{\mathbf{p}} +\eta\cdot\hat{\mathbf{p}}\eta\cdot v^T-\eta\cdot\hat{\mathbf{p}}(\gamma\cdot v^T+m_1)V]\Psi\ . \tag{7.2b}$$

The sum of these two equations gives

$$\hat{\mathbf{p}}\cdot p\Psi=(p^2)^{1/2}\Psi=(1+\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V)^{-1}(m_1\gamma\cdot\hat{\mathbf{p}}-m_2\eta\cdot\hat{\mathbf{p}}-\gamma\cdot\hat{\mathbf{p}}\gamma\cdot v^T+\eta\cdot\hat{\mathbf{p}}\eta\cdot v^T)(1-\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V)\Psi\ . \tag{7.3}$$

In the c.m. frame, the terms in the brackets are Hermitian in the scalar product

$$(\chi,\psi)=\int d^3\mathbf{x}\,\mathrm{Tr}(\chi^\dagger\psi)\ , \tag{7.4}$$

but the presence of the two factors on the extreme left and right spoils this property for the total energy operator.

It is evident that transformation (7.1) symmetrizes the expression of $(p^2)^{1/2}$ which becomes Hermitian in the scalar product (7.4) with the new functions χ' and ψ':

$$(p^2)^{1/2}\Psi'=\left[\frac{1-\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V}{1+\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V}\right]^{1/2}(m_1\gamma\cdot\hat{\mathbf{p}}-m_2\eta\cdot\hat{\mathbf{p}}-\gamma\cdot\hat{\mathbf{p}}\gamma\cdot v^T+\eta\cdot\hat{\mathbf{p}}\eta\cdot v^T)\left[\frac{1-\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V}{1+\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V}\right]^{1/2}\Psi'\ . \tag{7.5}$$

According to the relationship with the BS wave function, one has to introduce the constant $\frac{1}{4}(m^2-\langle v^{T^2}\rangle)^{-1/2}$ (for the equal-mass case) in the integral (7.4) in order to normalize the integral over the internal wave functions to unity [Eq. (6.34)].

In the subsequent calculations we shall always use transformation (7.1) but shall omit the prime from the notation of the new wave functions. Then if relationships (6.27)–(6.33) are used to relate the wave function to the BS wave function one has to remember the additional transformation (7.1).

As a last remark, we notice that in order for transformation (7.1) to be finite for finite values of x it is necessary that

$$\tfrac{1}{4}\,\mathrm{Tr}(\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}V)^2<1\ , \tag{7.6}$$

which shows that the potential V must be appropriately bounded and in particular regularized at finite values of x. Confining interactions may occur when the upper bound 1 of (7.6) is reached for some particular value of $|x^{T^2}|$ (in general when $|x^{T^2}|\to\infty$) in which case the wave function Ψ' in (7.1) also vanishes in that region and the two phenomena are balanced to yield a finite value of Ψ which is then generally sufficient for its normalizability, according to formula (6.34).

If condition (7.6) is not satisfied for all values of x^{T^2}, one could still work with the BS norm (6.34). In this case, however, it is necessary to impose appropriate conditions on the coupling constants and on the form of the potentials in order to guarantee the positivity of the norm and of the energy eigenvalues.

B. Scalar interactions

In this and the following two subsections, we shall study in more detail Eqs. (6.25) for given types of tensor structure of the potential V. The relationship (6.32) between V and the BS kernel D shows however that the tensor structures of the two quantities are not identical, except when one neglects second- and higher-order terms in D. Our purpose being rather illustrative here, we prefer to deal with simple expressions for V; therefore we shall directly discuss the structure of V without worrying about the corresponding structure of the kernel D, which in this case will be more complicated than that of V; it is obtained by inverting formula (6.32) or (6.33). Nevertheless, as we pointed out above, both of them coincide to lowest order of the interaction.

In this subsection we consider scalar-type interactions which correspond to potentials V which are independent of the Dirac matrices:

$$V=V(r^2,p^2)\ . \tag{7.7}$$

[As we pointed out in the spin-0 case, after Eq. (4.9) and at the beginning of Sec. V, in local approximations in x^T, the dependence on the relative momentum v can only arise from the couplings of the mediating field to particles 1 and 2.]

Using this expression of V in Eqs. (6.25), we now bring the operators $\eta\cdot p_2$ of Eq. (6.25a) on the right of V and use again the equations of motion to diagonalize the wave equations with respect to $\gamma\cdot p_1$ and $\eta\cdot p_2$. After using transformation (7.1) we find

$$H_1\Psi\equiv\left[\gamma\cdot p_1-m_1-\frac{2}{(1-V^2)}(m_2V+m_1V^2 +i\hbar\dot{V}\eta\cdot r)\right]\Psi=0\ , \tag{7.8a}$$

$$H_2\Psi\equiv\left[\eta\cdot p_2+m_2+\frac{2}{(1-V^2)}(m_1V+m_2V^2+i\hbar\dot{V}\gamma\cdot r)\right]\Psi=0 \quad (7.8b)$$

[$\dot{V}$ defined by Eq. (5.13)]. As can be checked explicitly, the operators H_1 and H_2 in (7.8) satisfy the compatibility condition (6.17) in a weak form.

Taking the "square" of these equations we find the generalized Klein-Gordon-type equation:

$$\tilde{H}\Psi\equiv(H_1+2m_1)H_1\Psi=(H_2-2m_2)H_2\Psi\ , \quad (7.9)$$

$$\tilde{H}\Psi=\left[\tfrac{1}{4}p^2+\frac{(m_1^2-m_2^2)^2}{4p^2}+v^{T^2}-\frac{M^2}{4}\left[\frac{1+V}{1-V}\right]^2-\frac{(m_1-m_2)^2}{4}\left[\frac{1-V}{1+V}\right]^2+4\hbar^2r^2\frac{\dot{V}^2}{(1-V^2)^2}\right.$$
$$+4\hbar^2\left[\frac{\dot{V}}{(1-V^2)}\right]\gamma\cdot r\eta\cdot r-2i\hbar M\frac{\dot{V}}{(1-V)^2}(\gamma\cdot r+\eta\cdot r)$$
$$\left.+2i\hbar(m_1-m_2)\frac{\dot{V}}{(1+V)^2}(\gamma\cdot r-\eta\cdot r)+2\hbar^2\frac{\dot{V}}{(1-V^2)}\gamma^T\cdot\eta^T\right]\Psi=0 \quad (7.10)$$

[γ^T,η^T defined by Eq. (2.13)].

C. Pseudoscalar interactions

Pseudoscalar-type interactions correspond to potentials V which are proportional to the matrices $\gamma_5\eta_5$:

$$V=\gamma_5\eta_5W(r^2,p^2)\ . \quad (7.11)$$

Proceeding as for the scalar interaction case and using transformation (7.1), the wave equations can be rewritten in the form

$$H_1\Psi\equiv(\gamma\cdot p_1-m_1+i\hbar A\gamma_5\tilde{\eta}\cdot r)\Psi=0\ , \quad (7.12a)$$

$$H_2\Psi\equiv(\eta\cdot p_2+m_2-i\hbar A\tilde{\gamma}\cdot r\eta_5)\Psi=0\ , \quad (7.12b)$$

$$A\equiv-\frac{2\dot{W}}{1-W^2}\ . \quad (7.13)$$

[$\dot{W}$ is defined as in Eq. (5.13), and $\tilde{\gamma}$ and $\tilde{\eta}$ in Eqs. (6.9) and (6.10).] Notice that the pseudoscalar interaction acts in Eqs. (7.12) as a purely quantum effect and that the effective potential A determines W up to an arbitrary integration constant.

The operators H_1 and H_2 of (7.12) satisfy the compatibility condition (6.17) in the strong sense:

$$[H_1,H_2]=0\ . \quad (7.14)$$

The "square" of Eqs. (7.12) is

$$\tilde{H}\Psi\equiv\left[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1^2+m_2^2)+\frac{(m_1^2-m_2^2)^2}{4p^2}\right.$$
$$+v^{T^2}+\hbar^2A^2r^2-\hbar^2A\tilde{\gamma}^T\cdot\tilde{\eta}^T$$
$$\left.-2\hbar^2\dot{A}\tilde{\gamma}\cdot r\tilde{\eta}\cdot r\right]\Psi=0 \quad (7.15)$$

[$\tilde{H}$ defined by Eq. (7.9) and $\tilde{\gamma}^T,\tilde{\eta}^T$ by Eqs. (6.9), (6.10), and (2.13)].

The matrices $\tilde{\gamma}^T$ and $\tilde{\eta}^T$ can be expressed in terms of the spin operators. Introducing the Pauli-Lubanski operators

$$W_{1s\alpha}=-\frac{\hbar}{4}\epsilon_{\alpha\beta\mu\nu}P^\beta\sigma^{\mu\nu}\quad(\epsilon_{0123}=+1)\ ,$$
$$W_{2s\alpha}=-\frac{\hbar}{4}\epsilon_{\alpha\beta\mu\nu}P^\beta\xi^{\mu\nu}\ ,$$
$$W_{1s}^2=W_{2s}^2=-\tfrac{3}{4}\hbar^2p^2\ , \quad (7.16)$$
$$W_s=W_{1s}+W_{2s}\ ,$$

which commute with all longitudinal variables and matrices, we get

$$\gamma\cdot pW_{1s\alpha}=\frac{\hbar}{2}p^2\tilde{\gamma}^T_\alpha,\quad \eta\cdot pW_{2s\alpha}=\frac{\hbar}{2}p^2\tilde{\eta}^T_\alpha\ , \quad (7.17)$$

and Eq. (7.15) takes the form

$$\tilde{H}\Psi\equiv\left[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1^2+m_2^2)+\frac{(m_1^2-m_2^2)^2}{4p^2}+v^{T^2}\right.$$
$$+\hbar^2A^2r^2-\frac{4}{p^2}A\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}W_{1s}\cdot W_{2s}$$
$$\left.-\frac{8}{p^2}\dot{A}\gamma\cdot\hat{\mathbf{p}}\eta\cdot\hat{\mathbf{p}}W_{1s}\cdot rW_{2s}\cdot r\right]\Psi=0\ . \quad (7.18)$$

A particular feature of this equation is that the operator $\tilde{H}$ above commutes with the longitudinal matrices $\gamma\cdot\hat{\mathbf{p}}$ and $\eta\cdot\hat{\mathbf{p}}$, and therefore its solutions can be classified according to the eigenvalues of these matrices; the positive-energy solutions (both in $p_1\cdot\hat{\mathbf{p}}$ and $p_2\cdot\hat{\mathbf{p}}$) will correspond to the eigenvalues $+1$ of $\gamma\cdot\hat{\mathbf{p}}$ and -1 of $\eta\cdot\hat{\mathbf{p}}$, that is to the submatrix Ψ_{++} of the decomposition (6.6) in the c.m. frame.

Knowing the solution $\tilde{\Psi}$ of Eq. (7.18) one can then

develop an inverse Foldy-Wouthuysen transformation to get the solutions Ψ of Eqs. (7.12). In the present case the Foldy-Wouthuysen transformation operator can be constructed in a compact form (we shall show this elsewhere). For our present purpose it suffices to observe that if $\tilde{\Psi}$ is an eigenfunction of p_μ, $\gamma\cdot\hat{\mathbf{p}}$, and $\eta\cdot\hat{\mathbf{p}}$, and is a solution of Eq. (7.18), then the solution Ψ of Eqs. (7.12) is given by

$$\Psi=(H_1+2m_1)(H_2-2m_2)\tilde{\Psi}/(-4m_1m_2)\,,$$
$$\tilde{H}\,\tilde{\Psi}=0\,, \tag{7.19}$$
$$\gamma\cdot\hat{\mathbf{p}}\tilde{\Psi}=-\eta\cdot\hat{\mathbf{p}}\tilde{\Psi}=+\tilde{\Psi}\,.$$

This is a direct consequence of the fact that the operators H_1 and H_2 of (7.12) commute strongly [Eq. (7.14] and of the definition of $\tilde{H}$ in (7.9) and (7.18).

The pseudoscalar-type interactions seem to play an important role in the representation of confining interactions and spontaneous breakdown of chiral symmetry, and are further analyzed in Ref. 22 (first paper).

D. Vector interactions

Vector-type interactions correspond to potentials V which are proportional to the matrices $\gamma_\mu\eta_\nu$:

$$V=\gamma_\mu\eta_\nu C^{\mu\nu}(r,p)\,, \tag{7.20}$$
$$C_{\mu\nu}(r,p)=C_{\nu\mu}(-r,p)\,. \tag{7.21}$$

[The latter equation is a consequence of Eq. (6.22).]

The "diagonalization" procedure of Eqs. (6.25), as used for the scalar and pseudoscalar interactions, and transformation (7.1) lead however to rather complicated expressions for the wave equation operators H_1 and H_2. Some of the complicated terms, which contain momentum-dependent tensor functions, are however of third order in C. It can be shown that by adding to the potential V in (7.20) axial vector-type interactions which are of third order in C and appropriately fixed, the resulting expressions of the operators H_1 and H_2 can be considerably simplified. This third-order axial-vector-type interaction can be thought of as representing some effective local approximation of high-order diagrams of the BS kernel. We therefore choose the following expression for V:

$$V=\gamma_\mu\eta_\nu C_0^{\mu\nu}(r,p)+\tilde{\gamma}_\mu\tilde{\eta}_\nu\tilde{C}_0^{\mu\nu}(r,p)\,, \tag{7.22}$$
$$\tilde{C}_0=O(C_0{}^3)\,. \tag{7.23}$$

As in the spinless case, we shall consider three different "gauges" for $C_{0\mu\nu}$.

Case a:

$$C_{0\mu\nu}=g_{\mu\nu}C_0(r^2,p^2)\,,$$
$$\tilde{C}_{0\mu\nu}=g_{\mu\nu}\tilde{C}_0(r^2,p^2)\,. \tag{7.24}$$

(Notice that because of the factor $g_{\mu\nu}$ and the physical degrees of freedom of the vector field of the underlying field theory, C_0 represents minus the scalar propagator; this remark applies to all vector propagator functions met in Secs. V, VII, and VIII.)

$\tilde{C}_0$ satisfies the equation

$$2(C_0-\tilde{C}_0)^2(C_0+\tilde{C}_0)=-\tilde{C}_0\,, \tag{7.25}$$

which means that

$$\tilde{C}_0=-2C_0{}^3+O(C_0{}^4)\,. \tag{7.26}$$

The corresponding wave equations are

$$H_1\Psi\equiv[\gamma\cdot p_1-m_1-\gamma_\mu(\tfrac{1}{2}Bp^\mu+Kv^\mu+\hbar\dot{K}\xi^{\mu\nu}r_\nu + i\hbar\dot{K}r^\mu)]\Psi=0\,, \tag{7.27a}$$

$$H_2\Psi\equiv[\eta\cdot p_2+m_2-\eta_\mu(\tfrac{1}{2}Bp^\mu-Kv^\mu-\hbar\dot{K}\sigma^{\mu\nu}r_\nu - i\hbar\dot{K}r^\mu)]\Psi=0\,, \tag{7.27b}$$

where

$$B=-\frac{2C}{1-C},\quad K=\frac{2C}{1+C}\,,$$
$$C=C_0+O(C_0{}^3)\,. \tag{7.28}$$

Because of condition (7.6), $|C|$ is bounded by $\frac{1}{4}$:

$$|C|<\tfrac{1}{4}\,. \tag{7.29}$$

Equations (7.27) satisfy the compatibility condition (6.17) in its strong form (7.14).

The "square" of these equations is

$$\tilde{H}\Psi\equiv\Bigg[\tfrac{1}{4}(1-B)^2p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}(1-K)^2+(1-K)^2v^{T2}$$
$$+2\hbar(\sigma_{\mu\nu}+\xi_{\mu\nu})\dot{K}(1-K)r^\mu v^{T\nu}-\hbar^2(1-K)(3\dot{K}+2\ddot{K}r^2)+6\hbar^2\dot{K}^2r^2-\hbar^2\sigma_{\mu\alpha}\xi^\alpha_\nu\dot{K}^2(r^2g^{\mu\nu}-2r^\mu r^\nu)$$
$$-\hbar^2\sigma_{\mu\alpha}\xi^\alpha_\nu(1-K)[2\ddot{K}r^\mu r^\nu+\dot{K}(g^{\mu\nu}-\hat{\mathbf{p}}^\mu\hat{\mathbf{p}}^\nu)]-\hbar(\sigma_{\mu\nu}-\xi_{\mu\nu})(1-K)\dot{B}p^\mu r^\nu-\frac{\hbar}{p^2}(m_1{}^2-m_2{}^2)(\sigma_{\mu\nu}+\xi_{\mu\nu})(1-K)\dot{K}p^\mu r^\nu$$
$$-[4i\hbar\dot{K}(1-K)r\cdot v-2\hbar^2(1-K)(3\dot{K}+2r^2\ddot{K})+4\hbar^2r^2\dot{K}^2]\Bigg]\Psi=0\,. \tag{7.30}$$

In this equation the σ and ξ matrices can be expressed in terms of the spin operators W_{1s} and W_{2s} of (7.16) and eventually of γ_5 and η_5. The internal orbital and total angular momenta W_L and W were already introduced in Eqs. (5.18) and (5.19). We list here several useful relations which serve to reexpress Eq. (7.30) in terms of these operators:

$$
\begin{aligned}
&W_L\cdot W_s=\frac{\hbar}{2}p^2(\sigma_{\mu\nu}+\xi_{\mu\nu})r^\mu v^{T\nu}\,,\\
&W_{1s}\cdot W_{2s}=-\frac{\hbar^2}{8}p^2\sigma^{TT}_{\mu\nu}\xi^{TT\mu\nu}\,,\\
&r^2W_{1s}\cdot W_{2s}-(W_{1s}\cdot r)(W_{2s}\cdot r)=-\frac{\hbar^2}{4}p^2\sigma^{TT}_{\mu\alpha}\xi^{TT\mu\beta}r_\beta{}^\alpha r_\beta\,,\\
&W_{1s\mu}\gamma_5=-\frac{i\hbar}{2}\sigma_{\mu\nu}p^\nu,\quad W_{2s\mu}\eta_5=-\frac{i\hbar}{2}p^2\xi_{\mu\nu}p^\nu\,,
\end{aligned}
\tag{7.31}
$$

$$
\begin{aligned}
&[v^2,A(r^2)]=4i\hbar Ar\cdot v-2\hbar(3\dot A+2\ddot A r^2)\,,\\
&[\partial^2A(r^2)]=6\dot A+4r^2\ddot A\,;\\
&\tilde H=\tfrac14(1-B)^2p^2-\tfrac12(m_1{}^2+m_2{}^2)+\frac{1}{4p^2}(m_1{}^2-m_2{}^2)^2(1-K)^2+(1-K)^2v^{T2}+\frac{4}{p^2}W_L\cdot W_s\dot K(1-K)\\
&\quad-\hbar^2(1-K)(3\dot K+2r^2\ddot K)+6\hbar^2\dot K{}^2r^2+\frac{8}{p^2}\dot K{}^2r^2W_{1s}\cdot W_{2s}(1+\gamma_5\eta_5)-\frac{8}{p^2}\dot K{}^2r^2W_{1s}\cdot W_{2s}\\
&\quad+\frac{8}{p^2}\dot K{}^2(W_{1s}\cdot r)(W_{2s}\cdot r)(1-\gamma_5\eta_5)+\frac{8}{p^2}\ddot K(1-K)r^2W_{1s}\cdot W_{2s}\\
&\quad-\frac{8}{p^2}\ddot K(1-K)(W_{1s}\cdot r)(W_{2s}\cdot r)(1-\gamma_5\eta_5)+\frac{4}{p^2}\dot K(1-K)W_{1s}\cdot W_{2s}(2+\gamma_5\eta_5)\\
&\quad+2i\dot B(1-K)(W_{1s}\cdot r\gamma_5-W_{2s}\cdot r\eta_5)+\frac{2i}{p^2}(m_1{}^2-m_2{}^2)\dot K(1-K)(W_{1s}\cdot r\gamma_5+W_{2s}\cdot r\eta_5)\\
&\quad-[4i\hbar\dot K(1-K)r\cdot v-2\hbar^2(1-K)(3\dot K+2r^2\ddot K)+4\hbar^2r^2\dot K{}^2]\,.
\end{aligned}
\tag{7.32}
$$

Case b:

$$C_{0\mu\nu}\equiv(g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu)C_0(r^2,p^2),\quad \tilde C_{0\mu\nu}=(g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu)\tilde C_0(r^2,p^2)\,, \tag{7.33}$$

$$H_1=\gamma\cdot p_1-m_1-\gamma_\mu(2Cv^{T\mu}+2i\hbar\dot Cr^\mu+2\hbar\dot C\xi^{TT\mu\alpha}r_\alpha)\,, \tag{7.34a}$$

$$H_2=\eta\cdot p_2+m_2+\eta_\mu(2Cv^{T\mu}+2i\hbar\dot Cr^\mu+2\hbar\dot C\sigma^{TT\mu\alpha}r_\alpha)\,, \tag{7.34b}$$

with C and $\tilde C_0$ related to C_0 by relations (7.28) and (7.23) and $|C|$ bounded by $\frac13$:

$$|C|<\tfrac13\,. \tag{7.35}$$

These operators satisfy the compatibility condition (6.17) in its strong form (7.14). The "square" operator $\tilde H$ (7.9) is given by

$$
\begin{aligned}
\tilde H=&\tfrac14p^2-\tfrac12(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}+(1-2C)^2v^{T2}-8i\hbar\dot C(1-2C)r\cdot v\\
&+\hbar^2(1-2C)(6\dot C+4r^2\ddot C)+4\hbar^2\dot C{}^2r^2+\frac{8}{p^2}\dot C(1-2C)W_L\cdot W_s+\frac{16}{p^2}\dot C(1-2C)W_{1s}\cdot W_{2s}\\
&+\frac{16}{p^2}\ddot C(1-2C)[r^2W_{1s}\cdot W_{2s}-(W_{1s}\cdot r)(W_{2s}\cdot r)]+\frac{32}{p^2}\dot C{}^2(W_{1s}\cdot r)(W_{2s}\cdot r)\,.
\end{aligned}
\tag{7.36}
$$

It commutes with $\gamma\cdot\hat{\mathbf{p}}$ and $\eta\cdot\hat{\mathbf{p}}$ and therefore the solutions of the equation $\tilde H\Psi=0$ can be classified according to the eigenvalues of the latter. As in the pseudoscalar interaction case, the inverse Foldy-Wouthuysen transformation (7.19) can be applied to get the solutions of the equations $H_1\Psi=H_2\Psi=0$.

Case c:

$$
\begin{aligned}
&C_{0\mu\nu}=[r^2(g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu)-r_\mu r_\nu]C_0(r^2,p^2)\,,\\
&\tilde C_{0\mu\nu}=[r^2(g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu)-r_\mu r_\nu]\tilde C_0(r^2,p^2)\,,
\end{aligned}
\tag{7.37}
$$

$$
\begin{aligned}
H_1=&\gamma\cdot p_1-m_1-\gamma_\mu[2C(r^2v^{T\mu}-r^\mu r\cdot v-i\hbar r^\mu)\\
&+\hbar B\xi^{TT\mu\nu}r_\nu]\,,
\end{aligned}
\tag{7.38a}
$$

$$
\begin{aligned}
H_2=&\eta\cdot p_2+m_2+\eta_\mu[2C(r^2v^{T\mu}-r^\mu r\cdot v-i\hbar r^\mu)\\
&+\hbar B\sigma^{TT\mu\nu}r_\nu]\,,
\end{aligned}
\tag{7.38b}
$$

$$B=\frac{3C+2\dot{C}r^2-2r^2C^2}{(1-2Cr^2)}\ , \tag{7.39}$$

$$|Cr^2|<\tfrac{1}{2}\ , \tag{7.40}$$

C and $\tilde{C}_0$ being related to C_0 by relations (7.28) and (7.23). H_1 and H_2 satisfy the compatibility condition (6.17) in the strong sense of (7.14).

The "square" operator $\tilde{H}$ of (7.9) is given by

$$\begin{aligned}\tilde{H}=&\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1^2+m_2^2)+\frac{(m_1^2-m_2^2)^2}{4p^2}+v^{T2}+\frac{4}{p^2}C(1-Cr^2)W_L^2-2\hbar^2(2\dot{C}r^2+3C-2C^2r^2-B^2r^2)\\&+\frac{4}{p^2}W_L\cdot W_s(2\dot{C}r^2+3C-2C^2r^2)+\frac{8}{p^2}W_{1s}\cdot W_{2s}B(1-2Cr^2+Br^2)\\&+\frac{8}{p^2}[r^2W_{1s}\cdot W_{2s}-(W_{1s}\cdot r)(W_{2s}\cdot r)](\dot{B}+BC-B^2)\ .\end{aligned} \tag{7.41}$$

As in case b above, $\tilde{H}$ commutes with the matrices $\gamma\cdot\hat{\mathbf{p}}$ and $\eta\cdot\hat{\mathbf{p}}$. The angular momentum operators W_L, W_{as}, W_s are defined in formulas (5.18), (5.19), (6.12), and (7.16).

VIII. SPIN-$\frac{1}{2}$–SPIN-0 PARTICLE SYSTEMS

In this section we consider systems composed of one spin-$\frac{1}{2}$ fermion (particle 1) and one spin-0 boson (particle 2). The construction of Poincaré-invariant compatible wave equations parallels, in a simpler way, that of the two spin-$\frac{1}{2}$ particle case, considered in the two preceding sections.

The wave function is now a four-component spinor:

$$\Psi=\Psi_\alpha(x_1,x_2)\quad(\alpha=1,\ldots,4)\ . \tag{8.1}$$

The Lorentz group generators are

$$M_{\mu\nu}=\sum_{a=1}^{2}(x_{a\mu}p_{a\nu}-x_{a\nu}p_{a\mu})-\frac{\hbar}{2}\sigma_{\mu\nu}\ . \tag{8.2}$$

The first wave equation is a generalized Dirac equation, the second one is a generalized Klein-Gordon equation. They have the form

$$H_1\Psi\equiv(\gamma\cdot p_1-m_1-V)\Psi=0\ , \tag{8.3a}$$

$$H_2\Psi\equiv[p_2^2-m_2^2-(\gamma\cdot p_1+m_1)V]\Psi=0\ . \tag{8.3b}$$

The wave function and the potential V satisfy Eqs. (6.18) and (6.23), respectively, and the constraint operators H_1 and H_2 satisfy the compatibility condition (6.17).

The relationships between the wave function Ψ and the potential V on the one hand and the BS wave function Φ and kernel D on the other can be obtained as in the two spin-0 or two spin-$\frac{1}{2}$ particle cases. In the ladder approximation of the kernel and with the relativistic instantaneous approximation (4.11), they read (for the equal-mass case $m_1=m_2=m$):

$$\begin{aligned}\phi(x)=&\frac{1}{2b}[(b+a)e^{-i|x_L|(b-a)}\\&-(b-a)e^{-i|x_L|(b+a)}]\psi(x^T)\\&-\frac{1}{2b}\epsilon(x_L)\gamma\cdot\hat{\mathbf{p}}\left[\gamma\cdot\left[\frac{p}{2}+v^T\right]-m\right]\\&\times(e^{-i|x_L|(b-a)}-e^{-i|x_L|(a+b)})\psi(x^T)\ ,\end{aligned} \tag{8.4}$$

$$\phi(x_L=0,x^T)=\frac{(p^2)^{1/2}}{2(m^2-v^{T2})^{1/2}}\psi(x^T)\ , \tag{8.5}$$

$$V=-\frac{i}{4}\int dx_L D(x_L,x^T,p_1,p_2,\gamma)\frac{1}{(m^2-v^{T2})^{1/2}} \tag{8.6}$$

[a and b defined in (4.9)]. The potential V takes a local form in x^T only when the operator $(m^2-v^{T2})^{-1/2}$ is replaced by a local function, as in (4.15).

Using relation (8.4) and approximation (4.11) in the normalization condition of the BS wave function, one finds that of the wave function ψ (in the c.m. frame):

$$\int d^3x^T\left[\bar{\psi}(x^T)\gamma\cdot\hat{\mathbf{p}}\frac{1}{4}\left[\frac{p^2}{m^2-v^{T2}}\right]^{1/2}\psi(x^T)+i\bar{\psi}(x^T)\left[\frac{p^2}{4(m^2-v^{T2})}\right]^{1/2}\frac{\partial\tilde{D}}{\partial p^2}\left[\frac{p^2}{4(m^2-v^{T2})}\right]^{1/2}\psi(x^T)\right]=1\ , \tag{8.7}$$

$\tilde{D}$ being defined as in (4.19) or (6.30).

We now analyze the tensor structure of the interaction, as we did in Secs. V and VII, in its local approximation (4.15).

Scalar interactions. They correspond to the case in which V does not depend on the γ matrices and the relative momentum v:

$$V=V(r^2,p^2)\ . \tag{8.8}$$

After bringing the operator $\gamma\cdot p_1$ of Eq. (8.3b) on the right of V and using again Eq. (8.3a), the two constraint operators become

$$H_1=\gamma\cdot p_1-m_1-V\ , \tag{8.9a}$$

$$H_2=p_2^2-m_2^2-2m_1V-2i\hbar\gamma\cdot r\dot{V}-V^2\ . \tag{8.9b}$$

The "square" of the operator H_1 defined by

$$\tilde{H}\equiv(H_1+2m_1)H_1 \tag{8.10}$$

is weakly equal to H_2:

$$(\tilde{H}-H_2)\Psi=[(p_1{}^2-m_1{}^2)-(p_2{}^2-m_2{}^2) -2VH_1]\Psi=0\ , \tag{8.11}$$

$$\tilde{H}\Psi=\left[\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}+v^{T^2} -2m_1V-2i\hbar\gamma\cdot r\dot{V}-V^2\right]\Psi=0 \tag{8.12}$$

[$\dot{V}$ defined by (5.13)].

Vector interactions. They correspond to the case in which V is proportional to the γ matrix and the momentum p_2, representing the couplings of the mediating vector field to the fermion and the boson, respectively:

$$V=\gamma_\mu[p_{2\nu},C^{\mu\nu}(r,p)]_+ \tag{8.13}$$

([,]$_+$ is the anticommutator),

$$H_1=\gamma\cdot p_1-m_1-\gamma_\mu[2C^{\mu\nu}p_{2\nu}-i\hbar(\partial_\nu C^{\mu\nu})]\ , \tag{8.13a}$$

$$H_2=p_2{}^2-m_2{}^2-(\gamma\cdot p_1+m_1)\{\gamma_\mu[2C^{\mu\nu}p_{2\nu}-i\hbar(\partial_\nu C^{\mu\nu})]\}\ . \tag{8.13b}$$

Bringing in Eq. (8.13b) $\gamma\cdot p_1$ on the right of C and using Eqs. (8.13a) and (6.18) we get for $\tilde{H}$ in (8.10):

$$\begin{aligned}\tilde{H}\Psi=\Big[&\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}+v^{T^2}-4C^{\mu\nu}p_{1\mu}p_{2\nu}-2i\hbar(\partial_\mu C^{\mu\nu})p_{2\nu}+2i\hbar(\partial_\nu C^{\mu\nu})p_{1\nu}\\&+4C^{\mu\alpha}C_{\mu\beta}p_{2\alpha}p_2^\beta-\hbar^2(\partial_\mu\partial_\nu C^{\mu\nu})-4i\hbar C^{\mu\nu}(\partial_\nu C_{\mu\beta})p_2^\beta-4i\hbar(\partial_\nu C^{\mu\nu})C_{\mu\beta}p_2^\beta\\&-2\hbar^2C^{\mu\alpha}(\partial_\alpha\partial^\beta C_{\mu\beta})-\hbar^2(\partial_\nu C^{\mu\nu})(\partial^\beta C_{\mu\beta})-\hbar\sigma_{\mu\alpha}[2(\partial^\alpha C^{\mu\nu})p_{2\nu}-i\hbar(\partial^\alpha\partial_\nu C^{\mu\nu})\\&-4C^{\mu\nu}(\partial_\nu C^{\alpha\beta})p_{2\beta}+2i\hbar C^{\mu\nu}(\partial_\nu\partial_\beta C^{\alpha\beta})]\Big]\Psi=0\ .\end{aligned} \tag{8.14}$$

We consider for $C_{\mu\nu}$ the three cases studied in boson-boson and fermion-antifermion systems.

Case a:

$$C_{\mu\nu}=g_{\mu\nu}C(r^2,p^2)\ , \tag{8.15}$$

$$H_1=\gamma\cdot p_1-m_1-\gamma_\mu(2Cp_2^\mu-2i\hbar\dot{C}r^\mu)\ , \tag{8.16a}$$

$$\begin{aligned}H_2\approx\tilde{H}\approx&\tfrac{1}{4}p^2(1-2C)^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{1}{4p^2}(m_1{}^2-m_2{}^2)^2(1+2C)^2+(1+2C)^2v^{T^2}\\&+8i\hbar\dot{C}(1+2C)r\cdot v-\hbar^2(1+2C)(\partial^2C)-4\hbar^2r^2\dot{C}^2-\frac{8}{p^2}W_L\cdot W_{1s}\dot{C}(1+2C)\\&-4i\gamma_5W_{1s}\cdot r\dot{C}(1+2C)-2(m_1{}^2-m_2{}^2)\left[C^2-\frac{2i}{p^2}W_{1s}\cdot r\gamma_5\dot{C}(1+2C)\right]\ .\end{aligned} \tag{8.16b}$$

The angular momentum operators were defined in Eqs. (5.18), (5.19), and (7.16) and some of their properties presented in formulas (7.31).

Case b:

$$C_{\mu\nu}=(g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu)C(r^2,p^2)\ , \tag{8.17}$$

$$H_1=\gamma\cdot p_1-m_1+2\gamma_\mu^T(Cv^{T\mu}+i\hbar\dot{C}r^\mu)\ , \tag{8.18a}$$

$$\begin{aligned}H_2\approx\tilde{H}\approx&\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1{}^2+m_2{}^2)+\frac{(m_1{}^2-m_2{}^2)^2}{4p^2}+(1+2C)^2v^{T^2}+8i\hbar\dot{C}(1+2C)r\cdot v\\&-\hbar^2(1+2C)(\partial^2C)-4\hbar^2r^2\dot{C}^2-\frac{8}{p^2}W_L\cdot W_{1s}\dot{C}(1+2C)\ .\end{aligned} \tag{8.18b}$$

Case c:

$$C_{\mu\nu}=[r^2(g_{\mu\nu}-\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu)-r_\mu r_\nu]C(r^2,p^2)\ , \tag{8.19}$$

$$H_1=\gamma\cdot p_1-m_1+2\gamma_\mu C(r^2v^{T\mu}-r^\mu r\cdot v-i\hbar r^\mu)\ , \tag{8.20a}$$

$$H_2 \approx \tilde{H} \approx \tfrac{1}{4}p^2 - \tfrac{1}{2}(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{4p^2} + v^{T^2} + 2\hbar^2(3C + 2r^2\dot{C} + 2r^2C^2)$$

$$-\frac{4}{p^2}C(1+r^2C)W_L^2 - \frac{4}{p^2}W_L\cdot W_{1s}(3C+2r^2\dot{C}+2r^2C^2)\ . \tag{8.20b}$$

Notice that the operator $\tilde{H}$ in cases b and c above commutes with the matrix $\gamma\cdot\hat{\mathbf{p}}$, and therefore, as in the fermion-antifermion case, the Foldy-Wouthuysen transformation can be applied to solve the equation $H_1\Psi=0$. If $\tilde{\Psi}$ is a solution of the equation $\tilde{H}\Psi=0$, being at the same time an eigenvector of $\gamma\cdot\hat{\mathbf{p}}$, then the solution Ψ of the equation $H_1\Psi=0$ will be given by

$$\Psi=(H_1+2m_1)\tilde{\Psi}/(2m_1)\ ,$$
$$\tilde{H}\tilde{\Psi}=0,\quad \gamma\cdot\hat{\mathbf{p}}\tilde{\Psi}=+\tilde{\Psi}\ . \tag{8.21}$$

Also notice that since particles 1 and 2 are of different nature, then in the expressions of $\tilde{H}$ their contributions do not always appear in a symmetric way concerning the terms proportional to the masses [Eqs. (8.12) and (8.16b)].

A further remark concerns the Hermiticity property of the energy operator. In some expressions of $\tilde{H}$—Eq. (8.16b) as well as Eq. (7.32)—the latter does not appear in a manifestly Hermitian form in the free norm in the c.m. frame. This feature does not infirm the Hermiticity property of P_0, since it can be established directly from the Dirac-type equations of H_1 (and eventually of H_2 if particle 2 is also a fermion; see for this Sec. VII A). The Hermiticity of $\tilde{H}$ becomes manifest in norms usually used for spin-0 particles. This phenomenon also exists for the case of a single fermion interacting with an external vector potential.

Axial-vector interactions. They correspond to potentials of the type

$$V=-\hbar\tilde{\gamma}\cdot W_L A(r^2,p^2)$$
$$=-\frac{2}{p^2}\gamma\cdot pW_L\cdot W_{1s}A(r^2,p^2)\ . \tag{8.22}$$

Such potentials cannot arise in the ladder approximation of parity-conserving interactions in renormalizable field theories. However they can arise from a local approximation of fourth-order irreducible diagrams in vector interactions in the Bethe-Salpeter kernel. They correspond to the exchange, between the fermion and the boson, of two vector particles. The vector particles couple to the fermion line at two different vertices with matrices γ_μ and γ_ν, respectively. Furthermore the fermion propagator joining the two vertices is proportional to $(\gamma\cdot p_1'+m_1')$ and one finds, among other terms, the product of three γ matrices which involve the term $i\epsilon_{\mu\nu\alpha\beta}\tilde{\gamma}^\alpha p_1'^\beta$. On the boson line the vertices of the vector particle involve the momenta $p_{2\mu}'$ and $p_{2\nu}''$. In a local approximation, by contracting the two vertices on each line to a single point and then replacing the kernel by an effective function $D(r^2,p^2)$, the above term becomes $i\epsilon_{\mu\nu\alpha\beta}\tilde{\gamma}^\alpha[p_1^\beta,p_2^\mu Dp_2^\nu]_+$, the nonvanishing contribution of which will be proportional to $\hbar\epsilon_{\mu\nu\alpha\beta}\tilde{\gamma}^\mu p^\nu r^\alpha v^\beta\dot{D}$, which is of the form (8.22).

The wave equations are defined by the operators

$$H_1=\gamma\cdot p_1-m_1+\frac{2}{p^2}\gamma\cdot pW_L\cdot W_{1s}A\ , \tag{8.23a}$$

$$H_2\approx\tilde{H}\approx\tfrac{1}{4}p^2-\tfrac{1}{2}(m_1^2+m_2^2)+\frac{(m_1^2-m_2^2)^2}{4p^2}+v^{T^2}+2A\left[1+\frac{(m_1^2-m_2^2)}{p^2}\right]W_L\cdot W_{1s}$$
$$-\hbar^2A^2(W_L^2-2W_L\cdot W_{1s})-4i\gamma_5W_{1s\alpha}[A(r\cdot v+i\hbar)v^{T\alpha}-Ar^\alpha v^{T^2}+i\hbar\dot{A}(r^2v^{T\alpha}-r^\alpha r\cdot v)]\ . \tag{8.23b}$$

IX. SUMMARY AND CONCLUDING REMARKS

We have applied the manifestly covariant formalism with constraints of classical relativistic Hamiltonian mechanics to the construction of relativistic wave equations describing the dynamics of two interacting particle systems. In this formalism the wave function satisfies two independent manifestly covariant wave equations, each of them being, according to the spin value of the corresponding particle, a generalization of the Klein-Gordon or the Dirac equation. The interaction between the two particles is represented by potentials depending on the particle variables alone.

The compatibility condition of the two wave equations sets certain restrictions on the structure of the interaction potentials, guarantees the Poincaré invariance of the theory in the physical Hilbert space and ensures freedom in the choice of the time parameters. The fact that the two-particle wave function satisfies two independent equations determines in a definite way the relative time evolution of the system, thus eliminating from the energy spectrum the possible occurrence of relative energy excitations. The number of degrees of freedom of the system is therefore the same as in nonrelativistic mechanics and in that limit one obtains a Galilei-invariant dynamics, if the interaction potentials have been chosen to have the appropriate behavior in c^2.

The interaction potentials depend on the relative coordinates through their projections on three spacelike axes orthogonal to the total momentum of the system. The po-

tentials may appear either in local or nonlocal forms (i.e., as integral operators with respect to these three spacelike components), the former being a particular case of the latter.

We outlined that the present framework of relativistic quantum mechanics is in direct connection with the Bethe-Salpeter equation and its sector of "normal" solutions. This means that the quantum-mechanical potential and wave function can be expressed, in perturbation theory, in terms of the Bethe-Salpeter kernel and wave function.[33] In particular the normalization condition of the quantum-mechanical wave function can be determined from that of the Beth-Salpeter wave function. When a relativistic instantaneous approximation is made for the Bethe-Salpeter kernel, taken in its ladder approximation, then the above relationships considerably simplify and one may even formally extend them to nonperturbative interactions and effective kernels.

In this respect, the "local" approximation of the interaction, which yields potentials that are local functions of the transverse relative coordinate variables and which was extensively used throughout this paper, may provide a kind of zeroth-order approximation for nonlocal quantities inherent to field theory.

The generality of the results about the interaction potentials makes it possible to apply the above wave equations to the study of a very wide variety of phenomenological problems concerning relativistic two-body systems. We present in separate papers[22] two such applications related to confining interactions in fermion-antifermion and fermion-boson systems.

The formalism used for the construction of two-body wave equations can also be applied to the N-particle ($N>2$) case. Here, however, the technical problems are more difficult to solve than in the two-particle case. This is already evident at the classical level.[13] The requirement of separability (cluster decomposition) imposes severe conditions on the structure of the interaction potentials; furthermore one has to satisfy here $N(N-1)/2$ compatibility conditions of N-independent wave equations. This problem necessitates therefore a separate treatment.

Relativistic quantum mechanics of interacting particle systems provides a simplified relativistic framework to the study of those physical problems, where, to a good approximation, a finite number of degrees of freedom suffices to describe some particular aspects of the dynamics of the system. In this respect, it seems now possible to formulate and construct the theory in a consistent way, being at the same time in close connection with quantum field theory.

ACKNOWLEDGMENT

The Division de Physique Théorique is associated with Centre National de la Recherche Scientifique.

APPENDIX: THE SCALAR PRODUCT

In this appendix we briefly sketch the method of constructing the scalar product of the theory. We shall present some details for the spin-0 case, but shall simply quote the results in the spin-$\frac{1}{2}$ case. We shall devote a separate paper to a detailed presentation of this subject.

The main property which we demand from the scalar product is its Poincaré invariance, which then guarantees, with the covariance of its kernel, the Hermiticity property of the Poincaré group generators and permits a unitary realization of the Poincaré group. Of course, one also has to ensure the positivity of the norm of physical states.

Generalizing a well-known procedure in the one-particle case, we construct the scalar product in the two-particle case from a tensor current of rank two, $j_{\mu\nu}(x_1,x_2)$, built up from two wave functions, satisfying two conservation laws:

$$\partial_1^\mu j_{\mu\nu}(x_1,x_2)=0, \quad \partial_2^\nu j_{\mu\nu}(x_1,x_2)=0 . \tag{A1}$$

The scalar product is then constructed as a double three-dimensional integral of this current over two spacelike hypersurfaces Σ_1 and Σ_2:

$$(\Psi,\Phi)=\int_{\Sigma_1,\Sigma_2} j_{\mu\nu}(x_1,x_2)d\sigma_1^\mu(x_1)d\sigma_2^\nu(x_2) . \tag{A2}$$

The two conservation laws (A1) guarantee the independence of the integral (A2) of the types of surface Σ_1 and Σ_2, and more particularly ensure the Poincaré invariance of the scalar product defined by Eq. (A2).

The case of two spin-0 boson systems was presented in Sec. IV. The wave function satisfies Eqs. (4.1) and (4.5) and we concentrate here on the case of local potentials (2.15). The potential V will be assumed to be superfically Hermitian, in the sense that when p_μ are replaced by real eigenvalues, then V is Hermitian in the usual L_3 norm.

In order to construct the tensor current $j_{\mu\nu}$ in (A1), we first consider the case of the two different eigenvalues p_μ and p'_μ ($p^2\neq p'^2$) of the total momentum operator with corresponding eigenfunctions Ψ_p and $\Psi_{p'}$, respectively; they can be decomposed as in Eq. (4.4). The expression of the norm will be obtained as a limit, with $p'^2\to p^2$. For simplicity we shall also use the notations

$$V=V(x,p,\ldots), \quad V'=V(x,p',\ldots) , \tag{A3}$$

where the transverse variables x^T of (2.12) are calculated with respect to p and p'.

We write $j_{\mu\nu}$ in the form

$$j_{\mu\nu}=j_{\mu\nu}^{(0)}+j_{\mu\nu}^{(1)} , \tag{A4}$$

where $j_{\mu\nu}^{(0)}$ has the expression of the current obtained in the free theory[19,21]

$$j_{\mu\nu}^{(0)}(x_1,x_2)=i^2\Psi_{p'}^*(x_1,x_2)\overleftrightarrow{\partial}_{1\mu}\overleftrightarrow{\partial}_{2\nu}\Psi_p(x_1,x_2) , \tag{A5}$$

and $j_{\mu\nu}^{(1)}$ is a remainder. In general $j_{\mu\nu}^{(0)}$ does not satisfy current conservation (A1) in the presence of an interaction alone. We therefore search for the expression of $j_{\mu\nu}^{(1)}$ to ensure the current conservation (A1) of $j_{\mu\nu}$.

The current $j_{\mu\nu}^{(0)}$ in (A5), satisfies the two equations

$$\partial_1^\mu j_{\mu\nu}^{(0)}\equiv F_{1\nu}=\Psi_{p'}^*\{-(V'-V)\partial_{2\nu}+\overleftarrow{\partial}_{2\nu}(V'-V) + [\partial_{2\nu}(V'+V)]\}\Psi_p , \tag{A6a}$$

$$\partial_{2\mu}^{\nu} j_{\mu\nu}^{(0)} \equiv F_{2\mu} = \Psi_{p'}^{*}\{-(V'-V)\partial_{1\mu} + \overleftarrow{\partial}_{1\mu}(V'-V) + [\partial_{1\mu}(V'+V)]\}\Psi_p \ . \tag{A6b}$$

This means that $j_{\mu\nu}^{(1)}$ in (A4) must be a solution of the equations

$$\partial_{1\mu}^{\mu} j_{\mu\nu}^{(1)}(x_1,x_2) = -F_{1\nu}(x_1,x_2) \ , \tag{A7a}$$

$$\partial_{2\mu}^{\nu} j_{\mu\nu}^{(1)}(x_1,x_2) = -F_{2\mu}(x_1,x_2) \ . \tag{A7b}$$

These two equations are integrable, since F_1 and F_2 satisfy the compatibility equation

$$\partial_{2\nu}^{\nu} F_{1\nu} = \partial_{1\mu}^{\mu} F_{2\mu} \equiv F(x_1,x_2) = \Psi_{p'}^{*}\{2\overleftarrow{\partial}_{\alpha}(\partial^{\alpha}V') + 2(\partial_{\alpha}V)\partial^{\alpha} + (V'-V)(\tfrac{1}{4}p^2 - \partial^2) - (\tfrac{1}{4}p'^2 - \overleftarrow{\partial}^2)(V'-V) + [\partial^2(V'+V)]\}\Psi_p \ , \tag{A8}$$

where $\partial = \frac{1}{2}(\partial_1 - \partial_2)$.

The solution of Eqs. (A7), which vanishes when the interaction is switched off, is

$$j_{\mu\nu}^{(1)} = -i\partial_{1\mu}\int G_A(x_1-x_1')F_{1\nu}(x_1',x_2)d^4x_1' - i\partial_{2\nu}\int G_A(x_2-x_2')F_{2\mu}(x_1,x_2')d^4x_2' - \partial_{1\mu}\partial_{2\nu}\int G_A(x_1-x_1')G_A(x_2-x_2')F(x_1',x_2')d^4x_1'd^4x_2' \ , \tag{A9}$$

where G_A is the advanced Green's function satisfying the equation

$$\partial^2 G_A(x) = -i\delta^4(x) \ . \tag{A10}$$

It can be checked that the kernel of the current $j_{\mu\nu}^{(1)}$, and hence that of $j_{\mu\nu}$ in (A4) is translation invariant and is a pure Lorentz tensor of rank two, as a consequence of the Poincaré-invariance property of the potential V in (2.15). These two features, together with the current conservation (A1), ensure the Hermiticity properties of the Poincaré group generators and therefore the unitary realization of the whole group.

In order to construct the scalar product, we choose, for the surfaces Σ_1 and Σ_2 of formula (A2), two parallel hyperplanes perpendicular to a unit constant timelike vector n, which we take, as usual, parallel to the x^0 axis:

$$n\cdot x_1 = t_1, \quad n\cdot x_2 = t_2, \quad n = (1,\mathbf{0}) \ . \tag{A11}$$

Then the scalar product is

$$(\Psi_{p'},\Psi_p) = \int j_{00}(X,x)d^3X\,d^3x = \int [j_{00}^{(0)}(X,x) + j_{00}^{(1)}(X,x)]d^3X\,d^3x \ . \tag{A12}$$

By using the expressions of $j_{00}^{(0)}$ in (A5) and $j_{00}^{(1)}$ in (A9), the formula

$$\int \partial_0 G_A(x)d^3\mathbf{x} = -i\theta(-x^0) \ , \tag{A13}$$

and integrating by parts the terms of F in (A8), containing derivatives of the potential V, and finally using the equations of motion and some algebraic identities, we end up with the formula

$$(\Psi_{p'},\Psi_p)\big|_{p'^2\neq p^2} = \int d^3X\,d^3x\,\Psi_{p'}^{*}(X,x)\left[i\overrightarrow{\partial}_{10}\overrightarrow{\partial}_{20} - (p_0'+p_0)\left[\frac{V(x,p',\ldots)-V(x,p,\ldots)}{p_0'-p_0}\right]\right]\Psi_p(X,x)$$

$$= (2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\exp[+i(p_0'-p_0)X^0] \times \int d^3x\,\psi_{p'}^{*}(x)\left[i\overrightarrow{\partial}_{10}\overrightarrow{\partial}_{20} - (p_0'+p_0)\left[\frac{V(x,p',\ldots)-V(x,p,\ldots)}{p_0'-p_0}\right]\right]\psi_p(x) \ , \tag{A14}$$

where we have used the notation

$$\psi_p(x) = \exp[-i(m_1^2-m_2^2)p\cdot x/(2p^2)]\psi_p(x^T) \tag{A15}$$

[cf. Eq. (4.4)].

To calculate the norm, we take the limit $p_0' \to p_0$, after using an appropriate ϵ-limiting procedure, where $-i\epsilon$ is an infinitesimal imaginary part which we introduce in the eigenvalues p_0 and p_0'. We obtain

$$(\Psi_{p',a},\Psi_{p,b})\big|_{p'^2=p^2} = (2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\int d^3x\,\psi_a^{*}(x)\left[i\overrightarrow{\partial}_{10}\overrightarrow{\partial}_{20} - 4p_0^2\frac{\partial V}{\partial p^2}\right]\psi_b(x)$$

$$= (2\pi)^3 2p_0\delta^3(\mathbf{p}'-\mathbf{p})\delta_{ab}f_a(p^2) \ , \tag{A16}$$

where the labels a,b distinguish different eigenfunctions with the same mass squared p^2. The origin and the meaning of the additional normalization factor $f_a(p^2)$ in the right-hand side of (A16) will be explained below.

The expressions of the scalar product and of the norm simplify in the c.m. frame, where the operators $i\partial_{10}$ and $i\partial_{20}$ become identical to the operators $\hat{\mathbf{p}}\cdot p_1$ and $\hat{\mathbf{p}}\cdot p_2$ which have well-defined eigenvalues (3.13)–(3.15), with positive signs, or, equivalently,

$$\hat{\mathbf{p}}\cdot p_{1,2}=\frac{(p^2)^{1/2}}{2}\left[1\pm\frac{m_1{}^2-m_2{}^2}{p^2}\right], \qquad p^2>|m_1{}^2-m_2{}^2| \ . \tag{A17}$$

Furthermore in this frame the transverse vector x^T reduces to $(0,\mathbf{x})$ which is independent of p^2. Therefore the only p^2 dependence of the potential V arises from its explicit dependence on this variable in the c.m. frame and no longer from its implicit kinematic dependence through x^T.

We now turn to the interpretation of the normalization factor f in Eq. (A16). If the Hilbert space of physical states is identified with the physical Hilbert space of two-particle relativistic quantum mechanics, then the completeness relation implies that $f=1$. However if the physical states $|p\rangle$ are supposed to belong to a larger space than that of two-particle mechanics, such as in quantum field theory, then the two-particle Hilbert space is only a subspace of the total physical Hilbert space. For this reason it is the global normalization condition of the quantum field theory state $|p\rangle$ which should fix the normalization factor of the two-particle quantum-mechanical wave function. Upon comparing formula (A16), in the c.m. frame, with the normalization condition obtained from the Bethe-Salpeter equation Eq. (4.18) and using the local approximation (4.15) and (4.16) we get for f the expression, in the general unequal-mass case,

$$f=\left[\frac{\hat{\mathbf{p}}\cdot p_1}{2(m_1{}^2-\langle v^{T^2}\rangle)^{1/2}}+\frac{\hat{\mathbf{p}}\cdot p_2}{2(m_2{}^2-\langle v^{T^2}\rangle)^{1/2}}\right]^{-1}, \tag{A18}$$

where $\hat{\mathbf{p}}\cdot p_1$ and $\hat{\mathbf{p}}\cdot p_2$ are given by Eqs. (A17) and the mean value $\langle v^{T^2}\rangle$ is calculated in the c.m. frame, in the L_3 norm for instance. (It is an approximate value.)

The knowledge of the factor f in (A18) is crucial when relating the quantum-mechanical wave function to physical quantities, typical of field theory, such as decay coupling constants.[22]

Finally, we examine the question of the positivity of the norm. Since the norm is Poincaré invariant, it is sufficient to examine this question in the c.m. frame. If, there, the potential V is independent of p^2, then the kernels of the scalar product and of the norm become identical, in the c.m. frame, to those of the free expressions of two-particle relativistic quantum mechanics. In particular, if the eigenvalues of $\hat{\mathbf{p}}\cdot p_1$ and $\hat{\mathbf{p}}\cdot p_2$ have been chosen positive, as in Eqs. (A17), then the norm is positive. This condition was actually imposed as one of the defining conditions of the two-particle physical Hilbert space (cf. item iii) of Sec. III).

If V depends on p^2 in the c.m. frame, then the norm is no longer straightforwardly positive. One must impose additional restrictions on the shape, coupling constant and p^2 dependence of V in order to maintain the positivity of the norm of physical states, characterized by positive eigenvalues of each $\hat{\mathbf{p}}\cdot p_1$ and $\hat{\mathbf{p}}\cdot p_2$.

We now simply quote the results obtained for the norms in the fermionic case. For a fermion-antifermion system we get

$$(\Psi_{p',a},\Psi_{p,b})\,|_{p'^2=p^2}=(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\int d^3x\ \mathrm{Tr}\left[\bar{\psi}_a(x)\left[\gamma_0\eta_0-V\gamma_0\eta_0V+4p_0{}^2\frac{\partial V}{\partial p^2}\right]\psi_b(x)\right]$$

$$=(2\pi)^3 2p_0\delta^3(\mathbf{p}'-\mathbf{p})\delta_{ab}f_a(p^2)\ , \tag{A19}$$

with f_a having the same expression as in (A18); $\bar{\psi}$ is defined after Eq. (6.34) and V has been assumed to be superficially Hermitian ($\gamma_0\eta_0V^\dagger\gamma_0\eta_0=V$, for p_μ real, in the L_3 norm). If V is independent of p^2 in the c.m. frame, then the positivity of the norm requires the inequality (7.6) to be satisfied. In this case the transformation (7.1) brings the norm to its free expression (7.4), were its positivity is obvious.

For a fermion-boson system we get

$$(\Psi_{p',a},\Psi_{p,b})\,|_{p'^2=p^2}=(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})\int d^3x\ \bar{\psi}_a(x)\left[\gamma_0 i\overleftrightarrow{\partial}_{20}-4p_0{}^2\frac{\partial V}{\partial p^2}\right]\psi_b(x)$$

$$=(2\pi)^3 2p_0\delta^3(\mathbf{p}'-\mathbf{p})\delta_{ab}f_a(p^2) \tag{A20}$$

with f_a given by (A18), $\bar{\psi}=\psi^\dagger\gamma_0$ and V superficially Hermitian ($\gamma_0V^\dagger\gamma_0=V$, for p_μ real, in the L_3 norm). If V is independent of p^2 in the c.m. frame, then, there, the norm takes its "free" expression. Furthermore in the c.m. frame the operator $i\partial_{20}$ becomes identical to $\hat{\mathbf{p}}\cdot p_2$. The positivity of the latter then ensures the positivity of the norm.

If V depends on p^2 in the c.m. frame, then the comments made at the end of the case of two spin-0 particle systems hold also in the fermion-antifermion and fermion-boson cases.

[1]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science Monographs Series, Yeshiva University, New York, 1964).
[2]L. D. Faddeev, Teor. Mat. Fiz. **1**, 3 (1969) [Theor. Math. Phys. **1**, 1 (1970)].
[3]I. T. Todorov, Dubna Report No. E2-10125, 1976 (unpublished).
[4]A. Komar, Phys. Rev. D **18**, 1881 (1978); **18**, 1887 (1978); **18**, 3617 (1978).
[5]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[6]E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974).
[7]Ph. Droz-Vincent, Lett. Nuovo Cimento **1**, 839 (1969); Phys. Scr. **2**, 129 (1970); Ann. Inst. Henri Poincaré **27**, 407 (1977); **32A**, 377 (1980).
[8]F. Rohrlich, Ann. Phys. (N.Y.) **117**, 292 (1979); M. King and F. Rohrlich, *ibid.* **130**, 350 (1980); F. Rohrlich, Phys. Rev. D **23**, 1305 (1981); **25**, 2576 (1981).
[9]I. T. Todorov, SISSA report, Trieste, 1980 (unpublished); in *Relativistic Action at a Distance: Classical and Quantum Aspects,* proceedings of the Barcelona Workshop, 1981, edited by J. Llosa (Lecture Notes in Physics, Vol. 162) (Springer, Berlin, 1982), p. 213.
[10]T. Takabayasi, Prog. Theor. Phys. **54**, 563 (1975); Prog. Theor. Phys. Suppl. **67**, 1 (1979); T. Takabayasi and S. Kojima, Prog. Theor. Phys. **57**, 2127 (1977).
[11]A. Barducci, L. Lusanna, and E. Sorace, Nuovo Cimento **46B**, 287 (1978); L. Lusanna, *ibid.* **65B**, 135 (1981).
[12]D. Dominici, J. Gomis, and G. Longhi, Nuovo Cimento **48A**, 257 (1978); **56A**, 263 (1980).
[13]H. Sazdjian, Ann. Phys. (N.Y.) **136**, 136 (1981).
[14]N. Mukunda and E. C. G. Sudarshan, Phys. Rev. D **23**, 2210 (1981); E. C. G. Sudarshan, N. Mukunda, and J. N. Goldberg, *ibid.* **23**, 2218 (1981); J. N. Goldberg, E. C. G. Sudarshan, and N. Mukunda, *ibid.* **23**, 2231 (1981).
[15]*Relativistic Action at a Distance: Classical and Quantum Aspects* (Ref. 9).
[16]H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978); Nucl. Phys. **B133**, 115 (1978); Phys. Lett. **73B**, 75 (1978); J. Jersak, H. Leutwyler, and J. Stern, *ibid.* **77B**, 399 (1979); J. Gasser, H. Leutwyler, J. Jersak, and J. Stern, Helv. Phys. Acta **53**, 77 (1980).
[17]L. P. Horwitz and F. Rohrlich, Phys. Rev. D **24**, 1528 (1981); **26**, 3452 (1982).
[18]Ph. Droz-Vincent, Nuovo Cimento **58A**, 355 (1980); Phys. Rev. D **29**, 687 (1984).
[19]L. Bel, in *Differential Geometry and Relativity,* edited by M. Cahen and M. Flato (Reidel, Dortrecht, 1976), p. 197; Phys. Rev. D **28**, 1308 (1983).
[20]H. Crater and P. Van Alstine, Ann. Phys. (N.Y.) **148**, 1308 (1983); Phys. Rev. Lett. **53**, 1527 (1984); Phys. Rev. D **30**, 2585 (1984).
[21]V. A. Rizov, H. Sazdjian, and I. T. Todorov, Ann. Phys. **165**, 59 (1985).
[22]H. Sazdjian, following papers, Phys. Rev. D **33**, 3425 (1986); **33**, 3435 (1986).
[23]A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963); A. A. Logunov, A. N. Tavkhelidze, I. T. Todorov, and O. A. Khrustalev, *ibid.* **30**, 134 (1963).
[24]R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).
[25]F. Gross, Phys. Rev. **186**, 1448 (1969); Phys. Rev. C **26**, 2203 (1982); **26**, 2226 (1982).
[26]M. H. Partovi and E. L. Lomon, Phys. Rev. D **2**, 1999 (1970).
[27]R. N. Faustov, Teor. Mat. Fiz. **3**, 240 (1970) [Theor. Math. Phys. **3**, 478 (1970)].
[28]I. T. Todorov, Phys. Rev. D **3**, 2351 (1971); in *Properties of Fundamental Interactions,* edited by A. Zichichi (Editrice Compositori, Bologna, 1973), Vol. 9, Part C, p. 951.
[29]A. Klein and T.-S. H. Lee, Phys. Rev. D **10**, 4308 (1974).
[30]E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[31]M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[32]N. Nakanishi, Suppl. Prog. Theor. Phys. **43**, 1 (1969).
[33]H. Sazdjian, Phys. Lett. **156B**, 381 (1985).
[34]E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
[35]A. J. Hanson, T. Regge, and C. Teitlboim, *Constraint Hamiltonian Systems,* Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche Fisiche e loro Applicazioni **22** (Academia Nazionale dei Lincei, Roma, 1976).
[36]K. Sundermeyer, *Constrained Dynamics* (Lecture Notes in Physics, Vol. 169) (Springer, Berlin, 1982).
[37]H. Sazdjian, Nucl. Phys. **B161**, 469 (1979).
[38]P. G. Bergmann and A. B. Komar, Phys. Rev. Lett. **4**, 432 (1960).