Possible evidence for dynamical supersymmetry in the hadron spectrum

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A relativistic supersymmetric quantum mechanics is proposed which uses Osp(1,4)⊃SO(3,2) for the spectrum and constrained Hamiltonian quantum mechanics for the dynamics of one-hadron systems. It combines baryon and meson towers into infinite supermultiplets in a way similar to the use of supersymmetries in nuclear physics. The predicted mass formula is fitted to the masses of baryon and meson resonances.

I. INTRODUCTION

The model which will be discussed in this paper can be considered as a synthesis of two apparently distinct theoretical ideas. Starting from a supersymmetric quantum mechanics[1] one can consider this model as its extension into the relativistic domain. Starting from a new quantization of the relativistic oscillator[2] or relativistic string one can consider this model as its supersymmetric extension. Our investigation was motivated by the apparent success of dynamical supersymmetries in nuclear physics[3,4] which led to the question as to whether similar features exist in the hadron spectrum, and thus to the idea of a relativistic spectrum-generating superalgebra. The mass scale in this model is that given by the slope of the Regge trajectories, which are reproduced as the yrast states of relativistic collective motions. The model therefore applies to a different domain of physics than the usual supersymmetric theories of modern particle physics.

In Sec. II the properties of Osp(1,4) are reviewed in a way which displays the similarity between supersymmetries and spectrum-generating algebras. In Sec. III the infinite-dimensional representations are described and their connections to the spectrum of the three-dimensional oscillator and the hadron spectrum are indicated. In Sec. IV the Hamiltonian is conjectured through comparison with nonrelativistic supersymmetric quantum mechanics. In Sec. V the spectrum predicted by the representation of Sec. III and the Hamiltonian of Sec. IV is compared with the experimental data.

II. Osp(1,4) AS SPECTRUM-GENERATING ALGEBRA

Superalgebras (in relativistic physics) are algebras that connect states with different spins. For this purpose they use Majorana spinor operators $Q_\alpha, \bar{Q}_\alpha$, $\alpha=1,2,\dot{1},\dot{2}$, i.e., operators that transform with respect to the spinor representation of $SO(3,1)_{S_{\mu\nu}}$:

$$[S_{\mu\nu},Q_\alpha]=-\tfrac{1}{2}(\sigma^{(s)}_{\mu\nu})_\alpha{}^\beta Q_\beta \; . \tag{2.1}$$

In here the $S_{\mu\nu}$, $\mu,\nu=0,1,2,3$ which satisfy

$$[S_{\mu\nu},S_{\rho\sigma}]=-i(\eta_{\mu\rho}S_{\nu\sigma}+\eta_{\nu\sigma}S_{\mu\rho}-\eta_{\nu\rho}S_{\mu\sigma}-\eta_{\mu\sigma}S_{\nu\rho}) \; , \tag{2.2}$$

are the generators of $SO(3,1)_{S_{\mu\nu}}$. The numbers $(\sigma^{(s)}_{\mu\nu})_\alpha{}^\beta$ are the matrix elements of the operators $S_{\mu\nu}$ in the $(2+2)$-dimensional spinor representation[5]

$$(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})=(k_0=\tfrac{1}{2},c=\tfrac{3}{2})\oplus(k_0=\tfrac{1}{2},c=-\tfrac{3}{2}) \tag{2.3}$$

whose basis vectors are denoted by $|\alpha\rangle$. Thus,

$$(\sigma^{(s)}_{\mu\nu})_{\alpha\beta}=2\langle\alpha|S_{\mu\nu}|\beta\rangle=\frac{i}{2}[\gamma_\mu,\gamma_\nu]_{\alpha\beta} \; , \tag{2.4}$$

and the choice of the components of the spinor operator depends upon the choice of the basis in this representation space, i.e., upon the choice of the matrix representation for the Dirac γ_μ. As the Q_α are spinor operators they transform between integer-spin and half-integer-spin states.

Spectrum-generating algebras (SGA's) are also algebras that connect states with different spins. Instead of spinor operators the conventional SGA's use vector operators Γ_ρ (or tensor operators $T_{\mu\nu}$), i.e., operators that transform with respect to the vector representation[5] $(j_1=\tfrac{1}{2},j_2=\tfrac{1}{2})=(k_0=0,c=2)$ of SO(3,1):

$$[S_{\mu\nu},\Gamma_\rho]=(\sigma^{(v)}_{\mu\nu})_\rho{}^\sigma\Gamma_\sigma=-i(\eta_{\mu\rho}\Gamma_\nu-\eta_{\nu\rho}\Gamma_\mu) \tag{2.5}$$

with[5]

$$(\sigma^{(v)}_{\mu\nu})_\rho{}^\sigma=\langle e_\rho|S_{\mu\nu}|e_\sigma\rangle=i(\eta_{\nu\rho}g^\sigma_\mu-\eta_{\mu\rho}g^\sigma_\nu) \; . \tag{2.6}$$

As the Γ_μ are vector operators they transform between integer-spin states or between half-integer-spin states only.

The groups connected with the SGA's are called spectrum-generating groups (SGG's) or dynamical groups, or also "dynamical symmetries."[6] The corresponding global structure connected with the superalgebra is the supergroup or supersymmetry (SUSY) (Ref. 3). Dynamical groups have had many successful applications in molecular and atomic physics, in solid-state physics and especially in nuclear physics. The motivation for their introduction was to describe hadron spectra and hadron structure in relativistic physics. Supersymmetries have been very popular in modern particle physics but so far there exists no empirical evidence for these kinds of supersymmetry. However, a substantial amount of evidence for dynamical supersymmetry has been reported in nuclear physics.[4] The low-energy spectrum of an even-even nucleus forms a multiplet of the dynamical group U(6) and combines with the energy levels of the even-odd nucleus—obtained from the even-even nucleus by adding one more fermion—into a supermultiplet of a supersymmetry containing U(6).

In the present paper we want to study whether there is some evidence for supermultiplets of this kind in the spectrum of hadrons.

According to the above discussions the difference between SUSY and a SGG is that the transition operators for the former transform like spinors whereas the transition operators for the latter transform like vectors (or tensors). Using them for the description of the spectrum, i.e., *not* demanding that [energy, Q_α]=0 or [mass, Q_α]=0, there is in principle no difference between dynamical SUSY's and dynamical groups or SGG's.

The set of operators $S_{\mu\nu},\Gamma_\mu,Q_\alpha$ that satisfy the above relations (1), (2), and (5) do not yet have sufficient structure. To make them into the (enveloping) algebra of a group or supergroup one has to postulate commutation and anticommutation relations between the Γ_μ and Q_α.

Osp(1,4) is the least complicated example of a simple superalgebra for the relativistic case and one can hope that it will describe the simplest examples of relativistic physical systems. More complicated systems will probably require larger algebras than Osp(1,4) and its bosonic part SO(3,2). As an alternative for the bosonic part, the SGG SL(4,R) has already been suggested.[7] Its subgroup SL(3,R) describes shape pulsation of a three-dimensional "lump" rather than the simple "relativistic" oscillations of SO(3,2). Its spectrum is more complicated than needed for the present experimental data and most of it has to be constrained away.

SGA's can describe finite or infinite multiplets, depending upon the choice of the commutation relation (CR) between the vector operators (or the tensor operators). If one chooses for Hermitian Γ_μ the CR

$$[\Gamma_\mu,\Gamma_\nu]=-iS_{\mu\nu}\ , \tag{2.7}$$

then the $S_{\mu\nu},\Gamma_\mu$ together form the noncompact SGG $SO(3,2)_{S_{\mu\nu}\Gamma_\mu}$ which has infinite-dimensional representations describing infinite multiplets.

Supersymmetries can also have finite or infinite supermultiplets. If one requires the anticommutation relation (ACR) between the spinor operators

$$Q_\alpha,\bar{Q}_\beta=-(Q^TC)_\beta\ ,$$

$$\{Q_\alpha,\bar{Q}_\beta\}=\tfrac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}S_{\mu\nu}+(\gamma^\mu)_{\alpha\beta}\Gamma_\mu\ , \tag{2.8}$$

and the CR

$$[\Gamma_\mu,Q_\alpha]=-\tfrac{1}{2}(\gamma_\mu)_\alpha{}^\beta Q_\beta \tag{2.9}$$

then the $S_{\mu\nu}$, Γ_ν, and Q_α together form the superalgebra

$$\mathrm{Osp}(1,4)_{S_{\mu\nu}\Gamma_\nu Q_\alpha}\supset \mathrm{SO}(3,2)_{S_{\mu\nu}\Gamma_\nu}$$
$$\supset \mathrm{SO}(3,1)_{S_{\mu\nu}}\ , \tag{2.10}$$

which describes infinite supermultiplets.

III. REPRESENTATIONS OF Osp(1,4)

Recently several infinite-dimensional representations of noncompact superalgebras have been constructed.[8,9] In particular all representations of Osp(1,4) with spectrum $\Gamma_0\geq 0$ are known.[8] These contain a direct sum of up to four irreducible representations (irreps) of SO(3,2).

One obtains an idea of infinite-dimensional irreps of noncompact groups if one considers the weight diagram (or K-type). A weight diagram displays which irreps of the maximal compact subgroup $K=\mathrm{SO}(3)_{Sij}\times \mathrm{SO}(2)_{\Gamma 0}$ occur in an irrep of SO(3,2).

Let

$$\mu=\text{eigenvalue of }\Gamma_0\ ,$$
$$j(j+1)=\text{eigenvalue }(\tfrac{1}{2}S_{ij}S_{ij})\ , \tag{3.1}$$

then the weight diagram is the pattern of dots in a diagram of j vs μ.

A typical weight diagram of an irrep of SO(3,2) is shown in Fig. 1(a). Each dot at the coordinate (μ,j) represents an irrep of SO(3)×SO(2) which is contained in this particular irrep of SO(3,2). The irrep of SO(3,2), which is depicted by the weight diagram of Fig. 1(a), is denoted by $D(1,0)$. From the fact that there is a dot at the coordinate $(\mu=2,j=1)$ we conclude that there is an irrep of SO(3)×SO(2) with $j=1$ and $\mu=2$ in the irrep $D(1,0)$ of SO(3,2).

Figure 1(a) shows more than just the weight diagram of the irrep $D(1,0)$ of SO(3,2). If one draws horizontal lines through the dots then this figure represents the energy diagram of the oscillator in three dimensions with the energy given by

$$E_\nu=\hbar\omega(\nu+\tfrac{3}{2}),\quad \nu\equiv\mu-1=0,1,2,\ldots,$$
$$j=0,2,4,\ldots,\nu\quad\text{for }\nu=\text{even}\ , \tag{3.2}$$
$$j=1,3,5,\ldots,\nu\quad\text{for }\nu=\text{odd}\ .$$

Thus viewed as the energy diagram of the oscillator, each dot represents a state with radial quantum number ν and angular momentum j.

Different mathematical structures can have identical weight diagrams; e.g., a representation of a simple group and its contraction have the same weight diagram. The two interpretations of Fig. 1(a) are a reflection of the fact that $D(1,0)$ of SO(3,2) contracts into the algebra of the oscillator.

We will go one step further, by turning the picture of Fig. 1(a) around and drawing j vs $m^2=m_0{}^2+(1/\alpha')\nu$. Then one obtains the picture of linearly rising Regge trajectories plus some daughters, as shown in Fig. 1(b).

The exploration of this connection to the phenomenology of the Regge trajectories on the one hand and to the oscillator levels on the other, will lead us to the physical interpretation of these weight diagrams. We will discuss this in Sec. IV and refer to Ref. 2 for a detailed discussion of the bosonic substructure. This connection means that the Regge recurrences are relativistic vibrational excitations, which become the usual oscillator levels in the nonrelativistic contraction limit $1/c\to 0$.

The representation $D(1,0)$ is for various reasons not the right representation: it leads to a spinless oscillator in the

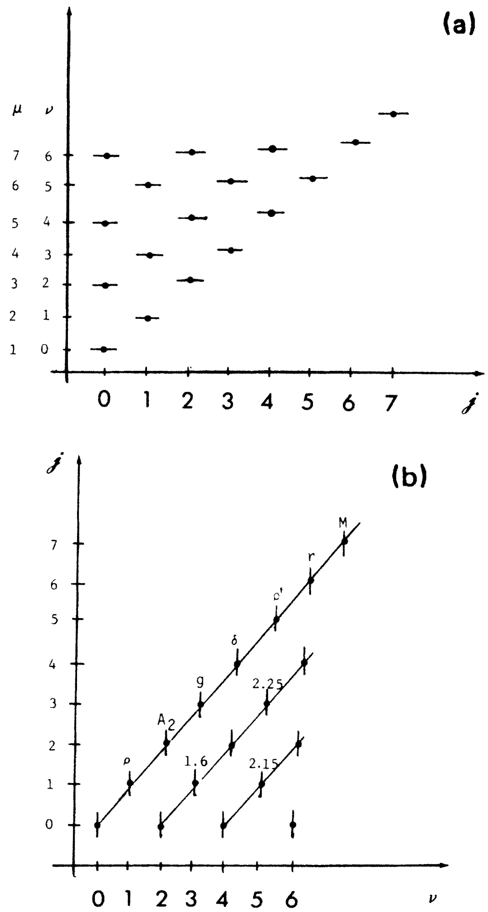

FIG. 1. The weight diagram of the SO(3,2) representation $D(1,0)$ and its physical interpretation. In (a), μ = eigenvalue Γ_0 is drawn vs j. Each level corresponds to the energy of a harmonic oscillator in three dimensions with the well-known angular momentum degeneracy. The oscillator algebra is related to $D(1,0)$ by group contraction. In (b) j is drawn vs μ to depict the linearly rising Regge trajectory. Each level now corresponds to the mass squared of a hadron with spin j and principal quantum number ν. The hadron states are described by an irreducible representation space $[m(\nu,j),j]$ of the Poincaré group, obtained from the representation $(\mu=\nu+1,j)$ of SO(2)×SO(3).

nonrelativistic limit, whereas the quark model requires that the vibrating constituents have spin. Further, fitted to the ρ-meson trajectory, Fig. 1(b) contains a state [the state $(\mu=1,j=0)$] whose phenomenologically determined mass squared is negative. We will therefore choose some other representations of SO(3,2). There are various classes of irreps of SO(3,2) (Ref. 10); we restrict ourselves here to those for which spectrum $\Gamma_0>0$ ("positive-energy representations"), and which are denoted by $D(\mu^{\min},s)$ where $\mu^{\min}$ is a positive real number and s is an integer or half-integer. These irreps are not all multiplicity-free; i.e., in general there may be more than one dot in the weight diagram for a particular value of (μ,j) [or equivalently more than one irrep (μ,j) of K]. We will consider here only multiplicity-free representations (singletons[11] in the notation of Ehrman), because otherwise a quantum number in addition to ν,j would be needed. For these multiplicity-free positive-μ representations

$$\mu^{\min}=\text{smallest eigenvalue of } \Gamma_0\,, \qquad s=j_{\min}=\text{smallest value of } j\,. \tag{3.3}$$

There are again three subclasses of multiplicity-free positive-energy representations. We will consider here only the subclass of irreps $D(s+1,s)$ for which $\mu^{\min}=s+1$ and which are characterized by one number $s=\frac{1}{2},1,\frac{3}{2},2,\ldots$. For $D(s+1,s)$ the eigenvalue of the second-order Casimir operator is

$$C[\mathrm{SO}(3,2)]=-R=-(2-2s^2) \tag{3.4}$$

and the eigenvalue of the fourth-order Casimir operator is[2]

$$P_1=s(s+1)[-R-(s-1)(s+2)] =s(s+1)(s-1)s\,. \tag{3.5}$$

The weight diagram of $D(2,1)$ looks similar to Fig. 1(a) except that the lowest column with $j=0$ is missing[12] and that there are now twice as many states; i.e., there is a dot for every pair (μ,j) of integers up to the line $\nu=\mu-1=j$ (Regge trajectory for the meson). The weight diagram of $D(\frac{3}{2},\frac{1}{2})$ also looks similar, except that the dots are at half-integer values for the pair (μ,j) between the line $j=\frac{1}{2}$ and $\nu-\frac{1}{2}\equiv\mu-1=j$ (nucleon trajectory).[13]

With the knowledge of the representations of SO(3,2) it is easy to describe the irreps of Osp(1,4). There are four types of positive μ representations.[8,9] We are only interested in those which are multiplicity-free, do not contain $j=0$, and have weight diagrams which resemble the energy diagrams of the three-dimensional oscillator. These are the representations which reduce with respect to SO(3,2) into the direct sum

$$D(s+1,s)\oplus D(s+\tfrac{3}{2},s+\tfrac{1}{2}) \tag{3.6}$$

and which are distinguished from each other by the value $s=\frac{1}{2},1,\frac{3}{2},\ldots$. If we want to have $j=\frac{1}{2}$ contained in them we must in particular choose

$$D(\tfrac{3}{2},\tfrac{1}{2})\oplus D(2,1)\,. \tag{3.7}$$

The weight diagram of this representation is shown in Fig. 2. It is just a combination of the two weight diagrams for the irreps $D(\frac{3}{2},\frac{1}{2})$ and $D(2,1)$ of SO(3,2) (Ref. 14).

We denote the representation space of $D(\frac{3}{2},\frac{1}{2})$ by $\mathscr{H}_0^-$ and the representation space of $D(2,1)$ by $\mathscr{H}_0^+$. each $\mathscr{H}_0^{\pm}$ is then a direct sum of irrep spaces $\mathscr{H}^{\mu}(j)$ of the irreps (μ,j) of $\mathrm{SO}(2)_{\Gamma_0}\times\mathrm{SO}(3)_{S_{ij}}$:

$$\mathscr{H}_0^+=\sum_{\substack{\mu=2,3,4,\ldots\\ j=1,2,\ldots,\mu-1}}\oplus\mathscr{H}^{\mu}(j)\,, \qquad \mathscr{H}_0^-=\sum_{\substack{\mu=\frac{3}{2},\frac{5}{2},\frac{7}{2},\ldots\\ j=\frac{1}{2},\frac{3}{2},\ldots,\mu-1}}\oplus\mathscr{H}^{\mu}(j)\,. \tag{3.8}$$

The representation space of the representation (3.7) is then denoted by $\mathscr{H}_0$:

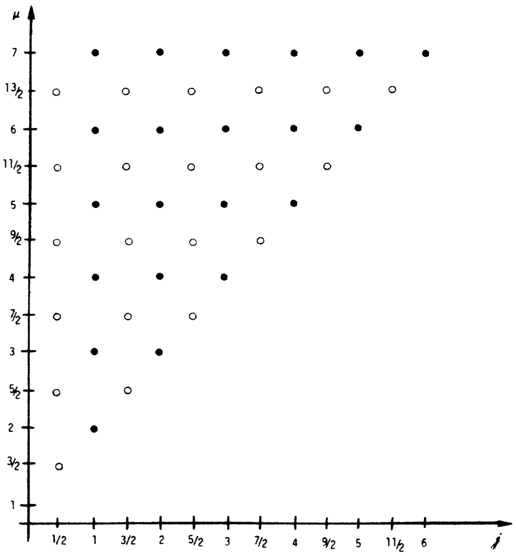

FIG. 2. The weight diagram of the representation $D(\frac{3}{2},\frac{1}{2})\oplus D(2,1)$ of Osp(1,4). The ○ make up the weight diagram of $D(\frac{3}{2},\frac{1}{2})$ and the ● the weight diagram of $D(2,1)$.

$$\mathscr{H}_0=\mathscr{H}_0^-\oplus\mathscr{H}_0^+ \ . \tag{3.9}$$

The basis of eigenvectors of the complete system of commuting operators $\Gamma_0, \frac{1}{2}S_{ij}S_{ij}, S_{12}$ of $\mathscr{H}_0$ we denote by $|0,\mu,j,j_3\rangle$. Thus each $\mathscr{H}^{\mu}(j)$ is spanned by the $(2j+1)$ vectors $|0,\mu,j,j_3\rangle$ with (μ,j) fixed and $j_3=-j,-j+1,\ldots,+j$.

The vector operator Γ_i transform between different $\mathscr{H}^{\mu}(j)$ changing μ and j by integer values. The spinor operators transform between $\mathscr{H}_0^+$ and $\mathscr{H}_0^-$ changing μ and j by half-integer value.

IV. RELATIVISTIC SUPERSYMMETRIC QUANTUM MECHANICS

We want to use Osp(1,4) for a relativistic supersymmetric quantum mechanics. Nonrelativistic supersymmetric quantum mechanics has been studied extensively.[1,15] It uses Hamiltonians of the form

$$H_{\text{int}}=\tfrac{1}{2}\{Q,Q^{\dagger}\} \quad \text{or} \quad H_{\text{int}}=\frac{1}{2n}\sum_{\alpha=1}^{n}\{Q_\alpha,Q_\alpha^{\dagger}\} \ , \tag{4.1}$$

where Q_α are the anticommuting charges. The Hilbert space (space of states) is the direct sum

$$\mathscr{H}=\mathscr{H}^+\oplus\mathscr{H}^- \ ,$$

where $\mathscr{H}^+$ and $\mathscr{H}^-$ are the spaces of bosonic and fermionic states, respectively. Q and Q^+ transform between $\mathscr{H}^+$ and $\mathscr{H}^-$. A popular example has been the supersymmetric oscillator with spin-orbit coupling[15] which uses Osp(1,2) as a spectrum-generating superalgebra. The above Hamiltonian H_{int} describes the intrinsic motion only and the motion of the oscillator as a whole, the center-of-mass (c.m.) motion, is ignored. Including the c.m. motion would require the additional kinetic energy operator $\mathbf{P}^2/2M$ where $\mathbf{P}$ is the c.m. momentum. The total nonrelativistic Hamiltonian would then be

$$H=\frac{\mathbf{P}^2}{2M}+\frac{k}{2}\{Q,Q^{\dagger}\} \ , \tag{4.2}$$

where k is a system constant ($\hbar\omega$) which makes the dimension correct. Thus (4.2) describes the dynamics of an extended vibrating object that also performs c.m. motion.

For the dynamics of a relativistic extended object we will use the quantum version of constrained Hamiltonian mechanics.[16] In analogy to (2) we would postulate the relativistic Hamiltonian of a relativistic model as

$$H=v\left[P_\mu P^\mu-\frac{1}{\alpha'}\frac{1}{4}\sum_{\beta=1}^{4}\{\hat{Q}_\beta,\hat{Q}_\beta^{\dagger}\}-\tilde{m}_0{}^2\right] . \tag{4.3}$$

The first term is the kinetic energy term (c.m. motion); the second describes the intrinsic motion; $\tilde{m}_0$ is an irrelevant additive constant. $1/\alpha'$ is a system constant (which can take different values for different physical systems) which makes the dimension correct. We call it $1/\alpha'$ because it will be empirically related to the slope of the Regge trajectory. v is a Lagrange multiplier (generalized velocity) which is determined from a gauge-fixing constraint that fixes the meaning of the time parameter τ. We choose the constraint such that

$$v=\frac{-1}{2M}, \quad M=\text{mass}=(P_\mu P^\mu)^{1/2} \ ,$$

and τ becomes the proper time of the center of mass.[2]

$\hat{Q}_\beta\equiv Q_\beta(\hat{P})$ are the Lorentz-boosted Osp(1,4) charges. If one had taken for the $\hat{Q}_\beta$ in (4.3) just the Osp(1,4) generators Q_α then the second term in (4.3) would not have been a Lorentz invariant because

$$\sum_\alpha \{Q_\alpha,Q_\alpha^{\dagger}\}=\sum_\alpha \{Q_\alpha,(\bar{Q}\gamma^0)_\alpha\}$$

is the zero component of a vector.

Let $L(\hat{P})$ denote the operator matrix

$$L(\hat{P})^{\mu}_{\ \nu}=\begin{bmatrix}\hat{P}_0 & \hat{P}_m \\ -\hat{P}^m & \delta^m_n-\hat{P}^m\hat{P}_n(1+P_0)^{-1}\end{bmatrix} ,$$

$$\mu,\nu=0,1,2,3, \quad m,n=1,2,3 \tag{4.4}$$

which depends upon the center-of-mass velocity $\hat{P}_\mu=P_\mu M^{-1}$; then obviously

$$L(\hat{P})^{\mu}_{\ \nu}\hat{P}^{\nu}=\eta^{\mu 0}1 \ . \tag{4.5}$$

For any generator A of Osp(1,4) one can define the boosted operator[17]

$$A(\hat{P})\equiv U^{-1}(L(\hat{P}))AU(L(\hat{P})) \ , \tag{4.6}$$

where $U(L(\hat{P}))$ is the representation in the space of states $\mathscr{H}$ of the Lorentz transformation L. For example, for the Lorentz vector operator Γ^μ one has

$$\Gamma^\mu(\hat{P})\equiv U^{-1}(L)\Gamma^\mu U(L)=L^{\mu}_{\ \nu}\Gamma^{\nu} \ . \tag{4.7}$$

In particular, as one can immediately see using (4.4)

$$\Gamma^0(\hat{P})=L(\hat{P})^0_{\ \mu}\Gamma^\mu=\hat{P}_\mu\Gamma^\mu \ . \tag{4.8}$$

For the spinor operators Q_α one defines the boosted operators

$$\hat{Q}_\alpha \equiv Q_\alpha(\hat{P}) \equiv U^{-1}(L) Q_\alpha U(L) = D(\hat{P})_{\alpha\beta} Q_\beta \ , \quad (4.9)$$

where $D(\hat{P})$ is the representation matrix of the Lorentz transformation $L(\hat{P})$ in the spinor representation $(\frac{1}{2},\frac{3}{2})\oplus(\frac{1}{2},-\frac{3}{2})$ (Ref. 18).

As a consequence of (2.8) one obtains

$$\sum_{\beta=1}^{4} \{Q_\beta, Q_\beta^\dagger\} = 4\Gamma_0 \quad (4.10)$$

and therefore, using (4.7)—(4.9),

$$\sum_{\beta=1}^{4} \{Q_\beta(\hat{P}), Q_\beta^\dagger(\hat{P})\} = 4\hat{P}_\mu \Gamma^\mu \ . \quad (4.11)$$

The relativistic Hamiltonian (3) can thus also be written in the form

$$H = v \left[P_\mu P^\mu - \frac{1}{\alpha'} \hat{P}_\mu \Gamma^\mu - \tilde{m}_0{}^2 \right] \quad (4.12)$$

and the constraint relation that follows from it is

$$P_\mu P^\mu - \frac{1}{\alpha'} \hat{P}_\mu \Gamma^\mu - \tilde{m}_0{}^2 \underset{\text{con}}{=} 0 \ . \quad (4.13)$$

We want to emphasize here the difference between our interpretation of Osp(1,4) as a relativistic spectrum-generating superalgebra and the interpretation of Osp(1,4) used previously.[9] In the previous interpretations SO(3,2) is the group of motions in the curved universe; $SO(3,1)_{S_{\mu\nu}}$ is the Lorentz group and $\Gamma_\mu m$ is the momentum; in particular, Γ_0 is identified as the energy operator. In our usage $SO(3,2)_{\Gamma_\mu, S_{\mu\nu}}$ is something like the $SO(3,2)\gamma_\mu\sigma_{\mu\nu}$ of Dirac γ matrices except that the generators are operators with an infinite-dimensional matrix representation. The energy is the Poincaré group generator P_0 and only through the constraint relation (13), which at rest reads

$$P_0 \underset{\text{con}}{=} \frac{1}{\alpha'} \Gamma_0 + \tilde{m}_0{}^2 \ ,$$

is the energy related to Γ_0. The angular momentum generating the physical Lorentz group is $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ which is the infinite-dimensional generalization of a well-known relation[19] valid on the spinor basis.

The form (4.12) of the Hamiltonian for our physical system agrees with the Hamiltonian of the quantum relativistic oscillator[2] (QRO) for which SO(3,2) was the relativistic spectrum-generating group. This means that the restriction H_+ (H_-) of the operator H to the bosonic (fermionic) subspace $\mathcal{H}^+$ ($\mathcal{H}^-$) is identical with the Hamiltonian of the QRO. This, together with the nonrelativistic analogy[15] upon which our present model is built, suggests that the physical system described by the Osp(1,4) spectrum-generating supergroup and the Hamiltonian (4.12)=(4.3) should be something like a relativistic oscillator with "spin-orbit coupling."

That (4.3) or (4.12) is a sensible choice of a relativistic Hamiltonian (for a physical system that is defined by it) can be seen from the following facts: For $1/\alpha' = 0$ (or for the trivial representation $\Gamma^\mu = 0$), it gives the Hamiltonians of a relativistic mass point; for $\Gamma_\mu = \Gamma_\mu$(Dirac) (i.e., $\langle \ | \Gamma_\mu | \ \rangle = \frac{1}{2}\gamma_\mu$), it gives the Hamiltonian of the Dirac electron; for $\Gamma_\mu = \Gamma_\mu(D(1,0))$ [where $D(1,0)$ is the particular representation of SO(3,2), discussed in Sec. III], it leads in the nonrelativistic contraction limit to the energy operator of the nonrelativistic spinless three-dimensional oscillator.[2]

The Hamiltonian (4.12) does not describe mass dependence upon the hadron spin j, that is upon the angular momentum j in the c.m. rest frame [$j(j+1)$=eigenvalue $\hat{W}$ where $\hat{W}$ is given in (4.16)]. There is also another quantum number s defined by (3.3) and (3.5) which will not occur in the mass formula that follows from the Hamiltonian (4.12). This quantum number s is something like an angular momentum quantum number, as one can see from the factor $s(s+1)$ in (3.5). But $s(s+1)$ becomes the eigenvalue of the square of an angular momentum operator only in the nonrelativistic contraction limit.[2,20] Then the extended relativistic object can be viewed as a vibrating and rotating diquark (or triquark) and s can be interpreted as the sum of the spins of the constituents. In the nonrelativistic limit, the total "constituent spin" s combines with the intrinsic orbital angular momentum of the constituents (which is also defined only in the nonrelativistic limit) to give the total angular momentum j, the spin of the extended object. Although this picture of a vibrating and rotating diquark makes only sense in the nonrelativistic limit,[2] the "constituent spin" quantum number s is defined already relativistically by the eigenvalue of the Casimir operators of SO(3,2).

To incorporate splitting due to this "constituent spin" and j dependence, one has to add further terms to the Hamiltonian (4.12). This can be done without problems as (4.12) is not the most general form for the Hamiltonian by the rules of the spectrum generating dynamical group approach.

Corresponding to the dynamical subgroup chain (at rest):

$$\text{Osp}(1,4) \supset SO(3,2)_{\Gamma_\mu S_{\mu\nu}} \supset SO(3)_{S_{ij}} \times SO(2)_{\Gamma_0} \ , \quad (4.14)$$

the Hamiltonian can have the form

$$H = v \left[P_\mu P^\mu - \frac{1}{\alpha'} \hat{P}_\mu \Gamma^\mu - \lambda^2 \hat{W} + \beta \hat{C}(SO(3,2)) - \tilde{m}_0{}^2 \right] . \quad (4.15)$$

In here

$$\hat{W} = -\hat{W}_\mu \hat{W}^\mu$$

with (4.16)

$$\hat{W}_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} \hat{P}^\nu J^{\rho\sigma} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} \hat{P}^\nu S^{\rho\sigma}$$

is at rest the Casimir operator of SO(3); $\hat{P}_\mu \Gamma^\mu$ is, at rest, equal to Γ_0, i.e., the Casimir operator of SO(2), and C is the Casimir operator of SO(3,2).

$1/\alpha$, λ^2, and β are empirical system parameters, like the spring constant of the oscillator or the moment of inertia of the rotator (which they are actually related to in the nonrelativistic limit).

There are two independent Casimir operators of SO(3,2). However, in the representations we shall use they are related to each other; their eigenvalues are functions of one additional quantum number, the before-mentioned s. Phenomenologically one will not be able to distinguish between using the second-order Casimir operator C_2, the fourth-order Casimir operator C_4, or a function of both (e.g., C_4/C_2) for C in (4.15). We will use[20]

$$C(\mathrm{SO}(3,2)) = -R = \Gamma_\mu \Gamma^\mu + \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} \tag{4.17}$$

and the Hamiltonian (4.15) for our comparison with the experimental hadron spectrum. Our theoretical discussions will be based on the "unperturbed" oscillator Hamiltonian (4.3).

The space $\mathscr{H}$ in which all these operators act is obtained by inducing from the space $\mathscr{H}_0$ of (3.9). This construction is possible whenever SGG$\supset$SO(3,1) and SGG commutes with the c.m. velocity operator $\hat{P}_\mu$ (Ref. 21). Let $\hat{p}_\mu$ denote the eigenvalue of the c.m. velocity operator with $\hat{p}_\mu \hat{p}^\mu = 1$, $\hat{p}_0 > 1$ and let $U(L^{-1}(\hat{p}))$ be the representation of the boost $L^{-1}(\hat{p})$ where L is the matrix of (4.4). Then one obtains the basis that spans $\mathscr{H}$ from the basis vectors $|0,\mu,j,j_3\rangle$ of Sec. III by

$$|\hat{p},\mu,j,j_3\rangle = U(L^{-1}(\hat{p}))\,|0,\mu,j,j_3\rangle \ . \tag{4.18}$$

These $|\hat{p},\mu,j,j_3\rangle$ are now eigenvectors of $\hat{P}_\mu \Gamma^\mu$ and $\hat{W}$ with eigenvalues μ and $j(j+1)$, respectively, due to (4.7) and (4.8) and similar relations for $\hat{W}$ of (4.16), which can also be written as

$$\hat{W} = \tfrac{1}{2} \Sigma_{\mu\nu} \Sigma^{\mu\nu} \ , \tag{4.19}$$

where

$$\Sigma_{\mu\nu} = \hat{g}_\mu{}^\rho \hat{g}_\nu{}^\sigma S_{\rho\sigma}, \quad \hat{g}^\mu{}_\rho = \eta^\mu{}_\rho - \hat{P}^\mu \hat{P}_\rho \ . \tag{4.20}$$

$\Sigma_{\mu\nu}$ is the operator of the spin tensor. The operators $\Gamma^\mu(\hat{P})$ of (4.7) and $Q_\alpha(\hat{P})$ of (4.9) act on the basis vectors $|\hat{p},\mu,j,j_3\rangle$ of $\mathscr{H}$ in the same way as the operators Γ^μ and Q_α act on the vectors $|0,\mu,j,j_3\rangle$ of Sec. III. From (3.9) and (3.8) follows that

$$\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^- = \sum_{\substack{\mu=2,3,\ldots,\\ j=1,2,\ldots,\mu-1}} \oplus \mathscr{H}^\mu(m(j,\mu),j) \oplus \sum_{\substack{\mu=3/2,\ldots,\\ j=1/2,\ldots,\mu-1}} \oplus \mathscr{H}^\mu(m(j,\mu),j) \ . \tag{4.21}$$

One has therewith in $\mathscr{H}$ a representation space of the Poincaré group $\mathscr{P}_{P_\mu, J_{\mu\nu}}$ and of Osp(1,4).

The spaces $\mathscr{H}^\mu(m,j)$ are irrep spaces of $\mathscr{P}$ with spin j and mass $m = m(j,\mu)$ which are in addition labeled by the principal quantum number μ=eigenvalue $\hat{P}\cdot\Gamma$. The value of the mass follows from the constraint relation (4.13) if one uses the Hamiltonian (4.3):

$$m^2(\mu,j) = \frac{1}{\alpha'}\mu + \tilde{m}_0{}^2 \ . \tag{4.22}$$

For the states with highest j for a given μ (i.e., the states with $j=\mu-1\equiv\nu$ for $\mathscr{H}^+$ and $j=\mu-\frac{1}{2}\equiv\nu$ for $\mathscr{H}^-$) this leads to

$$m^2(j) = \frac{1}{\alpha'} j + m_0{}^2(\pm) \ ,$$

where

$$m_0{}^2(\pm) = \tilde{m}_0{}^2 + \frac{1}{\alpha'} \begin{cases} 1 \text{ for } \mathscr{H}^+ \ , \\ \frac{1}{2} \text{ for } \mathscr{H}^- \ . \end{cases} \tag{4.22a}$$

The states with $j<\nu$ for a given value of ν (daughters) have the same mass. If one treats $1/\alpha$, $m_0{}^2(+)$, and $m_0{}^2(-)$ as free parameters, one obtains already a fairly accurate description of the mass spectrum.

If one uses the Hamiltonian (4.15), then the mass formula, which follows from the constraint relation, is

$$m^2 = \frac{1}{\alpha'}\mu + \lambda^2 j(j+1) + \beta(2-2s^2) + \tilde{m}_0{}^2 \ , \tag{4.23}$$

where (3.4) has been used. This is the mass formula which we shall compare with the experimental data in Sec. V.

Before we compare the spectrum predicted by Osp(1,4) and the Hamiltonian with the experimental hadron spectrum, we want to mention how this relativistic model differs from the usual relativistic generalizations of the oscillator[22] and the usual quantization of the relativistic string,[23] in particular from the one-mode version of the quantum relativistic string.[24] Usually one takes for the relativistic intrinsic position ξ_μ and momenta π_μ the (3 + 1) or d-dimensional Heisenberg commutation relations

$$[\xi_\mu,\xi_\nu] = 0 = [\pi_\mu,\pi_\nu], \quad [\pi_\mu,\xi_\nu] = i\eta_{\mu\nu} \ , \tag{4.24}$$

and

$$S_{\mu\nu} = \xi_\mu \wedge \pi_\nu + \tilde{S}_{\mu\nu} \quad (\tilde{S}_{\mu\nu} = 0 \text{ for spinless case}) \ . \tag{4.25}$$

In the present model based on Osp(1,4)$\supset$SO(3,2) one defines the intrinsic position by[25]

$$\xi_\mu^{\mathrm{rel}} = -S_{\mu\nu} \frac{P^\nu}{(cM)^2} \ . \tag{4.26}$$

Using the Hamiltonian (4.3) one can calculate $\dot{\xi}_\mu^{\mathrm{rel}}$ and define

$$\pi_\mu^{\mathrm{rel}} \equiv 2Mc\dot{\xi}_\mu^{\mathrm{rel}} = 2Mc\frac{1}{i}[\xi_\mu^{\mathrm{rel}},H] = -\frac{1}{\alpha'}\frac{1}{cM}\hat{g}_\mu{}^\sigma \Gamma_\sigma \ . \tag{4.27}$$

$\hat{g}^\mu{}_\nu$ is the operator defined in (4.20), it projects into the plane perpendicular to the direction $\hat{P}_\mu$. The spin tensor of (4.19) can also be written as

$$\Sigma_{\mu\nu} = S_{\mu\nu} + \xi_\mu^{\mathrm{rel}} \wedge P_\nu \ . \tag{4.28}$$

Often (e.g., third and fourth references in Ref. 16) ξ_μ^{rel} is constrained to zero so that $\Sigma_{\mu\nu} = S_{\mu\nu}$, which is already not satisfied for the Dirac electron. We do not constrain it to zero, use the Hamiltonian (4.3) to calculate the equation of motion for $\xi_\mu^{\rm rel}$, and find that it performs a harmonic motion (*Zitterbewegung*) around the direction of the center-of-mass momentum P_μ. It is thus the natural candidate for an intrinsic position (dipole operator).

On c.m. rest states $\xi_\mu^{\rm rel}$ and $\pi_\mu^{\rm rel}$ have the form

$$\xi_i^{\rm rel}=\frac{1}{cM}S_{0i},\quad \pi_i^{\rm rel}=\frac{1}{\alpha' cM}\Gamma_i,\quad i=1,2,3\ , \tag{4.29}$$

so they are essentially defined by the generators of SO(3,2). From (4.26) and (4.27) using the CR of SO(3,2) one calculates

$$[\xi_\mu^{\rm rel},\xi_\nu^{\rm rel}]=-\frac{i}{c^2M^2}\Sigma_{\mu\nu}\ , \tag{4.30a}$$

$$[\pi_\mu^{\rm rel},\pi_\nu^{\rm rel}]=-\frac{i}{(\alpha' cM)^2}\Sigma_{\mu\nu}\ , \tag{4.30b}$$

$$[\xi_\mu^{\rm rel},\pi_\nu^{\rm rel}]\underset{\rm con}{=}-i\hat{g}_{\mu\nu}\ , \tag{4.30c}$$

where the con in the last CR of (4.30) indicates that the constraint (4.13) has been used. Thus the new intrinsic position and momentum do not satisfy the usual relativistic Heisenberg CR (4.24). However, in the Inonu-Wigner group contraction limit $1/c\rightarrow 0$, when $\mathscr{P}\rightarrow$Galilei group,

$$\xi_\mu^{\rm rel}\rightarrow(0,\xi_i)\quad \pi_\mu^{\rm rel}\rightarrow(0,\pi_i),\quad i=1,2,3\ , \tag{4.31}$$

where the three ξ_i and π_i satisfy the usual three-dimensional Heisenberg CR. Thus (4.30) is as valid a relativistic generalization of the usual three-dimensional CR as (4.24). For the spinless case (4.30) is just a different quantization of the (lowest mode of the) relativistic string (in which the Dirac bracket obtained from the center of mass gauge constraint, and not the Poisson brackets, are replaced by the commutators).[8] Although (4.3) is a different Hamiltonian than the Hamiltonian for the string, it leads to the same mass formula. But, whereas it is impossible to construct the representation space for the four-dimensional string because there is no consistent way in which the constraint relations can be accounted for (except for the lowest mode approximation when the c-number term in the Virasoro algebra vanishes), our model is simple enough to derive the representation spaces. This simplicity is achieved by avoiding unobservable variables, like the position on the string, and the resulting troublesome constraints. Still our model describes more than the one-mode approximation for which the constraints eliminate all but the yrast states of "rigid" rotation. Our group operators do represent collective dynamical variables which describe collective rotational *and* vibrational degrees of freedom. However, their relation to the intrinsic coordinates is not as transparent as in the usual nonrelativistic picture, but is given by group contraction.

With the representations of SO(3,2) and Osp(1,4) at hand, it is simple to include any spin s. One has just to choose the appropriate representations $D(s+1,s)$ and combine them into the right supermultiplets if one wants a unified description of mesons and baryons.

V. COMPARISON WITH THE EXPERIMENTAL HADRON SPECTRUM

The spectrum described by our model is given by the weight diagram, Fig. 2, and the mass formula (4.23). We want to test it on the best-known class of meson resonances (those with normal j^P and positive C_nP) and the nucleon resonances. These are the diquark mesons with total quark spin $s=1$ and the triquark nucleons with total quark spin $s=\frac{1}{2}$. According to the picture that emerges from our model in the nonrelativistic limit, these are the objects that should be described by the representation (3.7).

We have used in our fit, all meson resonances of this kind listed in the meson table,[26] except for the following resonances with $j^P=0^+$: $S(975)$, $\epsilon(1300)$, $S(1730)$, the latter of which may not exist. And we have used all the nucleon resonances with $j^P=\frac{1}{2}^+,\frac{3}{2}^-,\frac{5}{2}^+,\ldots$ (Ref. 27). One can make various fits with different assignments of the principal (vibrational) quantum number ν to the experimental resonances. After numerous attempts with many different assignments for this new quantum number ν, we have concluded that the best fit is obtained if one assigns the resonances on the ρ, ω, and nucleon Regge trajectory to the states with $\nu=j$ ("yrast" states). In addition, there are the daughters with $j=\nu-1,\nu-2,\ldots$. This assignment of the masses[26] to the levels of the Osp(1,4) weight diagram is shown in Fig. 3. As the $I=0$ and $I=1$ resonances of the same kind (e.g., ρ and ω) are almost degenerate in mass, we have assigned them to the same level; each meson state is therefore degenerate in isospin. [It appears that this degeneracy cannot be explained by going to Osp(N,4).]

We have first fitted the meson resonances and the nucleon resonances separately to (4.23) determining two sets of values for the parameters $(1/\alpha',\lambda^2,\beta,\tilde{m}_0{}^2)$. It turned out that

$$\frac{1}{\alpha'}(\text{meson})\sim\frac{1}{\alpha'}(\text{nucleon})\ ,\qquad \lambda^2(\text{meson})\sim\lambda^2(\text{nucleon})\ . \tag{5.1}$$

The first equality was to be expected from the slope of the two Regge trajectories.

In analogy to supersymmetry in nuclear physics, where different nuclei (even-even and even-odd) have the same level spacing after the ground-state levels have been adjusted, the equality (5.1) is the empirical evidence for dynamical supersymmetry for hadrons.

A joint fit to all resonances in the ω,ρ (with states in $\mathscr{H}^+$) and the nucleon tower (with states in $\mathscr{H}^-$) gives

$$\frac{1}{\alpha'}=(1.03\pm0.04)\ \text{GeV}^2\ ,\qquad \lambda^2=(0.015\pm0.008)\ \text{GeV}^2\ ,\qquad \beta=(0.53\pm0.03)\ \text{GeV}\ \text{ or }\beta\approx\frac{1}{2}\frac{1}{\alpha'}\ . \tag{5.2}$$

Figure 3 illustrates the evidence for the supersymmetry. There we have drawn $m^2-\tilde{m}_0{}^2-\beta(2-2s^2)$ for the y coordinate of the levels using Eq. (4.23) with the empirical parameters of Eq. (5.2). In addition to the resonances

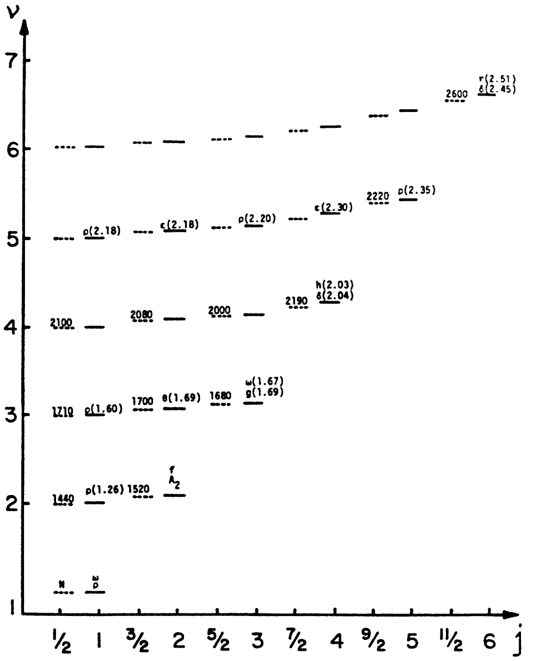

FIG. 3. The mass level diagram as obtained from a fit of the nucleon— and of the ($Y=0, CP=+1, j^P$=normal) meson—resonances to the mass formula (4.23). On the horizontal axis is plotted the spin j of the resonance. Vertically is plotted $m^2-\tilde{m}_0{}^2-\beta(2-s^2)$, where m^2 is the value calculated from (4.23) with the parameters (5.2), so that the baryon and meson ground-state levels coincide. The values of the parameters (5.2) have been obtained in a χ^2 fit of the experimental data for the resonances which are shown [by particle symbol and mass (in MeV for nucleons and GeV for mesons)] near the levels to which they have been assigned. These are only the resonances whose existence and whose spin are fairly well established. Predictions of (4.23) and their comparison with mass and spin values reported in the literature are discussed in the text. In addition to the nucleon resonances with $\frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, \ldots$ shown there are listed in the Particle Data Group table partners with opposite parity and almost the same mass. They are not shown here but have been mentioned in Ref. 2.

shown in Fig. 3, there is the $M(2.75)$ resonance with $j^P=7^-$ whose predicted mass is 2.76 GeV, and the $j^P=\frac{13}{2}^+$ nucleon N(2700) whose predicted mass is 2791 MeV. Also not shown in Fig. 3 are resonances that have been reported in the literature but for which the evidence is very weak and which have therefore not been used in the fit. For example, there has been reported a $j^P=1^-$ at 1920 MeV with a width of 190 MeV, a $j^P=2^+$ at 2020 MeV and width 160 MeV, a 3^- around 2080—2110 MeV (Ref. 28), which would fill the gaps at the $\nu=4$ levels. In addition, there have been reported[28] a $j^P=4^+$ around 2260 MeV, a $j^P=5^-$ around 2500 MeV, and a $j^P=6^+$ around 2710 which could fill the levels ($\nu=5, j=4$), ($\nu=6, j=5$), and ($\nu=7, j=6$), respectively.

It may appear curious that the value of β turned out to be $1/2\alpha'$. But at least so far there is no significance to it and it does not constitute a prediction of the meson-baryon mass difference. There is one free parameter β to fit one experimental value, e.g., $m^2(\rho)-m^2(\omega)$.

We have also made fits to other classes of resonances containing s, c, or b quarks, from which we conclude that the parameter $1/\alpha$ is flavor dependent. These fits give further phenomenological evidence for the bosonic sector only [SO(3,2) multiplets]. The only other indication for supermultiplets comes from the strange mesons and baryons, but there the data are still scarcer than for the non-strange tower.

The fit by which the values of the parameters (5.2) were obtained has a $\chi^2/n_D=10/28$ which is good; but the experimental errors for the mass and the widths are large, in particular for the high-spin resonances. There is no expermentally well-established resonance (with $j>0$) which cannot be accommodated in the supermultiplet of Fig. 3, and there are no predictions of the model which contradict experimental facts.[29] But then the value of the new quantum number ν (or μ) can be arbitrarily assigned to the resonances and have been assigned such that (4.23) gives the best fit. Thus the experimental evidence for this supermultiplet of hadrons is good but not overwhelming. But this model reproduces the linear Regge trajectories and explains them as vibrational excitations, by a theory which is nothing more than relativistic quantum mechanics. It predicts "daughters" as radial excitations, unites the meson and baryon resonances and has a very well accepted nonrelativistic limit.

We were led to the present model from a relativistic quantum mechanics for a rotating and vibrating extended object (Ref. 2 and references therein). Our fits to the predicted mass formulas showed that the mass splittings between the meson and baryon levels are approximately the same which we took as an indication for supersymmetry, in analogy to the dynamical supersymmetries in nuclear physics. Our phenomenological fits reproduced the almost linear Regge trajectories as the "yrast" states of the relativistic collective motions. Going in the opposite direction and starting from the fact that the slope of the Regge trajectories has the same value for baryons and mesons, supersymmetric theories in hadron physics have been arrived at before.[30] These theories use relativistic Bargmann-Wigner equations in a first-order approach to effective Lagrangians.

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[1]E. Witten, Nucl. Phys. **B188**, 513 (1981); J. Diff. Geometry **17**, 661 (1982); P. Salomonson and J. W. Van Holten, Nucl. Phys. **B196**, 509 (1982); F. Cooper and B. Freedman, Ann. Phys. (N.Y.) **146**, 262 (1983); M. de Combrugghe and V. Rittenberg, *ibid.* **151**, 99 (1983); E. Gozzi, Phys. Lett. **129B**, 432 (1983); R. Akhoury and A. Comtet, Nucl. Phys. **B246**, 253 (1984); M. Bernstein and L. S. Brown, Phys. Rev. Lett. **52**, 1933 (1984); T. Curtright and G. Ghandour, Phys. Lett. **136B**, 50 (1984); E. D'Hoker and L. Vinet, *ibid.* **137B**, 72 (1984); S. Fubini and E. Rabinovici, Nucl. Phys. **B245**, 17 (1984); M. M. Nieto, Phys. Lett. **146B**, 208 (1984); A. Khare and J. Maharana, Nucl. Phys. **B244**, 409 (1984); A. Jevicki and J. P. Rodrigues, Phys. Lett. **146B**, 55 (1984); D. Lancaster, Nuovo Cimento **79A**, 28 (1984); H. Ui and G. Takeda, Prog. Theor. Phys. (to be published); J. C. D'Olivo, L. F. Urrutia, and F. Zertuche, UNAM report, Mexico, 1985 (unpublished).

[2]A. Bohm, M. Loewe, and P. Magnollay, Phys. Rev. D **31**, 2304 (1985); **32**, 791 (1984); A. Bohm, M. Loewe, P. Magnollay, M. Tarlini, R. R. Aldinger, L. C. Biedenharn, and H. van Dam, *ibid.* **32**, 2828 (1985).

[3]*Supersymmetry in Physics,* edited by V. A. Kostelecký and D. K. Campbell (North-Holland, Amsterdam, 1985), and references therein.

[4]A. B. Balantekin, I. Bars, and F. Iachello, Phys. Rev. Lett. **47**, 19 (1981); H.-Z. Sun, M. Vallieres, D. H. Feng, R. Gilmore, and R. F. Casten, Phys. Rev. C **29**, 352 (1984). A similar kind of evidence for supersymmetry in atomic physics has recently been reported by V. A. Kostelecký and M. M. Nieto, Phys. Rev. Lett. **53**, 2285 (1984).

[5](k_0,c), where k_0 is an integer or half-integer and c is a complex number, label all irreducible linear representations (Harish-Chandra modules) of SO(3,1). For $c=\pm(k_0+n)$, $n=$natural number, the representation is finite dimensional; for $c=$pure imaginary the representation is unitary (principal series). The representation $(k_0,-c)$ is the conjugate (adjoint inverse) representation of (k_0,c). The four-dimensional vector representation in this notation is $(k_0=0,c=2)=(k_0=0,c=-2)$. Its basis, used in (2.6), is

$$e_{\rho=0}=if^{j=0,c=2}_{j_3=0}\,,\quad e_{\rho=3}=f^{j=1,c=2}_{j_3=0}\,,$$

$$e_{\rho=1}=-\frac{1}{\sqrt{2}}(f^{j=1,c=2}_{j_3=+1}-f^{j=1,c=2}_{j_3=-1})\,,$$

$$e_{\rho=2}=\frac{i}{\sqrt{2}}(f^{j=1,c=2}_{j_3=+1}+f^{j=1,c=2}_{j_3=-1})\,,$$

where $f^{jc}_{j_3}$ is the basis used in M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964). The finite-dimensional representations are often denoted by (j_1,j_2) which are connected to (k_0,c) by $k_0=|j_1-j_2|, c=(j_1+j_2+1)\,\mathrm{sgn}\,(j_1-j_2)$. k_0 expresses the lowest angular momentum value j of the SO(3) subgroup and $|c|-1\equiv j^{\max}$ expresses (for finite-dimensional representations) the highest value of j that is contained in the irrep(k_0,c) of SO(3,1). The left-hand side of Eq. (2.3) uses the notation (j_1,j_2) for the irreps of SO(3,1).

[6]A. Bohm and Y. Ne'eman, University of Texas report, 1986 (unpublished), and references therein.

[7]Y. Ne'eman and Dj. Sijacki, Phys. Lett. **157B**, 267 (1985); G. A. Goldin and D. H. Sharp, Commun. Math. Phys. **92**, 217 (1983); Phys. Rev. Lett. **52**, 2246 (1983).

[8]C. Fronsdal, Phys. Rev. D **26**, 1988 (1982); W. Heidenreich, Phys. Lett. **110B**, 461 (1982); M. Günaydin, P. van Nieuwenhuizen, and N. P. Warner, Nucl. Phys. **B255**, 63 (1985); M. Günaydin and N. P. Warner, Caltech Report No. CALT-68-1202 (unpublished).

[9]C. Fronsdal, Phys. Rev. D **12**, 3819 (1975); M. Flato and C. Fronsdal, Phys. Lett. **97B**, 236 (1980); D. Z. Freedman and H. Nicolai, Nucl. Phys. **B237**, 342 (1984); C. Fronsdal, Report No. UCLA/85/TEP/8 (unpublished); C. J. C. Burges, D. Z. Freedman, S. Davis, and G. W. Gibbons, MIT Report No. CTP1259, 1985 (unpublished).

[10]J. B. Ehrman, thesis, Princeton, 1954, Sec. VII.i.3; L. Jaffe, J. Math. Phys. **12**, 882 (1972); E. Angelopoulos, in *Quantum Theory, Groups, Fields and Particles,* edited by A. O. Barut (Reidel, Boston, 1983), p. 101.

[11]In the later literature, following Fronsdal (Ref. 8), the name singleton has been used for the very special representations $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$ which were also called Majorana representations. $D(\frac{1}{2},0)\oplus D(1,\frac{1}{2})$ is called the Dirac singleton of Osp(1,4).

[12]In particular, this representation does not contain the $j=0$ state which would have a negative mass squared for the ρ and ω trajectory.

[13]The other "positive energy" representations have similar weight diagrams; however, they are in general not multiplicity-free, i.e., each dot at (μ,j) represents more than one irrep of SO(3)$\times$SO(2). In addition there is the representation $D(2,0)$ whose weight diagram fills the empty spaces in Fig. 1. Further, the most degenerate representations $D(\frac{1}{2},0)$ and $D(1,\frac{1}{2})$ (Ref. 11) have the simplest weight diagrams with dots appearing only along the diagonal $\mu=j+\frac{1}{2}$ with $j=0,1,2,\ldots$ and $j=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$, respectively, i.e., with states only along the Regge trajectory. With every positive-μ representation $D(\mu^{\min},s)$ there exists also a negative-μ representation $D(\mu^{\max},s)$ whose weight diagram is obtained from that of $D(\mu^{\min},s)$ by taking μ into $-\mu$. In addition to the "positive μ" and "negative μ" multiplicity-free representations of SO(3,2) there are other multiplicity-free representations which have the remarkable property that they reduce into a *discrete* direct sum of irreps of the noncompact subgroup SO(3,1). Then there is the huge class of irreps which are not multiplicity-free (Ref. 10).

[14]The representations $D(s+1,s)$, $s\geq\frac{1}{2}$ [and also the representation $D(1,0)\oplus D(2,0)$] of SO(3,2) have a very special property, they are actually of the very special class of representations of SO(4,2)=SU(2,2) which have been derived in A. O. Barut and A. Bohm, J. Math. Phys. **11**, 2938 (1970). These representations of SO(4,2) have been applied many years ago by A. O. Barut and collaborators to the Coulomb ($s=0$) and dyon ($s\neq0$) problems, e.g., A. O. Barut and J. L. Bornzin, *ibid.* **12**, 841 (1971), and references therein. They have recently resurfaced again in connection with the spectrum supersymmetries of particles in a Coulomb potential; E. D'Hoker and Luc Vinet, Nucl. Phys. **B260**, 79 (1985) and Phys. Rev. Lett. **55**, 1043 (1985). In these applications s represents the magnetic charge g, $(eg)=s$, whereas for our relativistic oscillator s represents the total quark spin (Ref. 2). Although a connection between quark spin in the nonrelativistic limit (Ref. 2) and charge of a magnetic monopole would be intriguing, we do not see it. Also the SO(3,2) generators $\Gamma_\mu, S_{\mu\nu}$ of our relativistic oscillator are relativistic quantities (generalizations of the Dirac γ_μ and $\sigma_{\mu\nu}$) whereas the SO(3,2)

generators for the dyon problem are functions of the nonrelativistic position and momentum. Barut also gives a passage to a relativistic theory for hadrons [A. O. Barut, *Dynamical Groups and Generalized Symmetries in Quantum Theory* (University of Canterbury Press, New Zealand, 1972), Chap. VI] or *Lectures in Theoretical Physics* (Gordon and Breach, New York, 1968), Vol. X-B, which we do not understand and are therefore unable to compare with our relativistic oscillator model.

[15]A. B. Balantekin, Ann. Phys. (N.Y.) **164**, 277 (1985); V. A. Kostelecký, M. M. Nieto, and D. R. Traux, Phys. Rev. D **32**, 2627 (1985).

[16]P. A. M. Dirac, Proc. R. Soc. London **A328**, 1 (1972); *Lectures on Quantum Mechanics* (Yeshiva University Press, New York, 1964); Can. J. Math. **2**, 129 (1950); A. J. Hanson and T. Regge, Ann. Phys. (N.Y.) **87**, 498 (1974); A. J. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale del Lincei, Roma, 1976); N. Mukunda, H. van Dam, and L. C. Biedenharn, *Relativistic Models of Extended Hadrons Obeying a Mass-Spin Trajectory Constraint* (Springer, New York, 1982).

[17]H. van Dam and L. C. Biedenharn, Phys. Rev. D **14**, 405 (1976); W. A. Bardeen, I. Bars, A. J. Hanson, and R. D. Peccei, *ibid.* **14**, 2193 (1976); Y. S. Kim and M. E. Noz, Am. J. Phys. **46**, 480 (1978); D. J. Almond, Ann. Inst. Henri Poincaré, A **19**, 105 (1973); T. Takabayashi, Prog. Theor. Phys. Suppl. **67**, 1 (1979).

[18]If one arranges the components of the Majorana spinor operator into the matrix

$$Q_\alpha=\begin{bmatrix} Q_\sigma \\ \epsilon_{\sigma\sigma'}Q^\dagger_\sigma \end{bmatrix}, \quad \epsilon_{\sigma\sigma'}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where Q_σ, $\sigma=1,2$ transforms according to the representation $(\frac{1}{2},\frac{3}{2})$ of SO(3,1), then

$$Q_\sigma(\hat{p})\equiv D^{(1/2,3/2)}_{\sigma\sigma'}(\hat{p})Q_{\sigma'}=c\,(1+\hat{p}_0-\hat{p}^{\,m}\sigma^m)_{\sigma\sigma'}Q_{\sigma'}\ ,$$

where $c=[2(1+\hat{p}_0)]^{-1/2}$ and σ^m are the Pauli matrices. Therefore

$$Q_\alpha(\hat{p})=c\begin{bmatrix} (1+\hat{p}_0-\hat{p}^{\,m}\sigma^m)_{\sigma\sigma'} & 0 \\ 0 & (1+\hat{p}_0+\hat{p}^{\,m}\sigma^m)_{\sigma\sigma'} \end{bmatrix}\begin{bmatrix} Q_{\sigma'} \\ (\epsilon Q^\dagger)_{\sigma'} \end{bmatrix}$$

$$=c\,(1+\hat{p}_\mu\gamma^\mu\gamma^0)_{\alpha\alpha'}Q_\alpha\ .$$

The last expression is independent of the choice of the basis for the spinor operator and for the γ matrices and therefore valid for any γ-matrix representation.

[19]For example, Eq. (2.25), in P. A. Carruthers, *Spin and Isospin in Particle Physics* (Gordon and Breach, New York, 1971).

[20]In the nonrelativistic contraction limit $1/c\to 0$ the fourth-order Casimir operator C_4 has the following limit:

$$\left[\frac{1}{(Mc)^2}\frac{1}{\alpha'}\right]^2 C_4\to(\mathbf{S}-\xi^0\wedge\pi^0)^2\equiv\widetilde{\mathbf{S}}^{\,2}\ ,$$

where ξ^0 and π^0 are the intrinsic coordinates and momenta after contraction (Ref. 2). Thus $\widetilde{\mathbf{S}}$, which is only defined after contraction, is the difference between the intrinsic total angular momentum **S** and the intrinsic orbital angular momentum $\xi^0\wedge\pi^0$, i.e., something like the sum of the spins of the constitutents. The eigenvalue of $\widetilde{S}^2$ is $s(s+1)$ where s is already defined before contraction. Because of this property of C_4, it may be theoretically more appealing to use instead of (4.17) $C(\mathrm{SO}(3,2))=C_4$ in (4.15) if one wants to establish the connection with the nonrelativistic supersymmetric oscillator (Ref. 15).

[21]The construction is also possible if SGG commutes with the c.m. momentum P_μ but then one would obtain a trivial mass spectrum, i.e., SGG would not be a spectrum generating (super) group. For the special representation $D(\frac{1}{2},0)$ this construction has been described in all details in L. C. Biedenharn *et al.*, Phys. Rev. D **28**, 3032 (1983): One starts with the reduction chain (2.10) and a basis of $\mathscr{H}_0$ in which this chain is diagonal. The direct product of this basis with the basis of the spin-zero representation of the Poincaré group leads to the spinor basis of $\mathscr{H}$. Then one performs a basis transformation to the Wigner basis. On the Wigner basis at rest the subgroup chain (4.14) is diagonal. This transformation is the infinite-dimensional analogue of the transformation from the spinor basis to the Wigner basis for the Dirac case $\Gamma_\mu=\frac{1}{2}\gamma_\mu$, cf., e.g., Eq. (4.62) of S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966).

[22]T. Takabayashi, Prog. Theor. Phys. Suppl. **67**, 1 (1979).

[23]Y. Nambu, in *Symmetries and Quark Models,* proceedings of the International Conference, Detroit, Michigan, 1969, edited by R. Chand (Gordon and Breach, New York, 1970), p. 269; J. Scherk, Rev. Mod. Phys. **47**, 123 (1975); John M. Schwarz, Phys. Rep. **89**, 223 (1982); D. J. Almond, J. Phys. G **9**, 1309 (1983); F. Rohrlich, Nuovo Cimento **37**, 242 (1977).

[24]V. I. Borodulin, O. L. Zorin, G. P. Pronko, V. A. Razumov, and L. D. Soloviev, Report No. IHEP 84-202, Serpukohv, 1984 (unpublished); L. D. Soloviev, in *Proceedings of the XII International Conference on High Energy Physics, Leipzig, 1984,* edited by A. Meyer and E. Wieczorek (Academie der Wissenschaften der DDR, Zeuthen, DDR, 1984).

[25]c is the velocity of light which we do not want to set equal to one here in order to see what happens in the nonrelativistic limit $1/c\to 0$.

[26]Particle Data Group, Rev. Mod. Phys. **56**, S1 (1984).

[27]The Particle Data Group table lists for every one of these resonances (except the ground state) partners with opposite parity: $\frac{1}{2}^-, \frac{3}{2}^+, \ldots$. In most cases, the masses of partners with the same spin and opposite parity are almost degenerate, which is reminiscent of the l-type doubling in molecular physics. With this degeneracy all existing low-lying $I=\frac{1}{2}$ nucleon resonances are accommodated in our fit.

[28]C. Evangelista *et al.*, Nucl. Phys. **B153**, 256 (1979); M. Rosanka *et al.*, *ibid.* **B162**, 505 (1980).

[29]In the fit of the bosonic sector in the $c\bar{c}$ tower, however, there is a discrepancy for the $(\nu=2, j=1)$ state.

[30]S. Catto and F. Gürsey, Lett. Nuovo Cimento **35**, 241 (1982); I. Bars and H. C. Tze (private communication).