String-dominated universe

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A system of cosmic strings may evolve either to a scaling configuration, in which the persistence length scales with the horizon distance, or to a string-dominated universe. The former alternative provides the basis of an attractive theory of galaxy formation. This paper is concerned mainly with the latter alternative. It is shown that, provided the strings are formed later, and hence are lighter, than in a conventional grand unified theory, it is possible that our Universe became string dominated rather recently, at about 10^4 yr. Such a universe would behave very like a matter-dominated one. Strings would constitute the bulk of the dark matter required to make $\Omega=1$, but would be very hard to detect.

I. INTRODUCTION

Phase transitions in the early history of the Universe can generate topological structures of various dimensions—monopoles, strings, and domain walls.[1] Strings, in particular, may yield the density perturbations from which galaxies evolve, as suggested by Zel'dovich[2] and Vilenkin.[3] In Vilenkin's scenario a crucial role is played by the process of formation of small closed loops, which then lose energy by gravitational radiation. The effect of this radiation on the timing of the millisecond pulsar provides a stringent upper limit[4] to the string tension, or mass per unit length μ. The dimensionless parameter $G\mu$ cannot exceed 10^{-5}.

It is obviously important to understand better the process of string evolution and loop formation. In a previous publication[5] I tried to develop a set of equations describing the evolution of the system of strings and loops, and concluded that the system must evolve either toward a scaling solution in which the characteristic persistence length ξ (called L in Ref. 5) increases in proportion to the horizon distance ($\xi=\gamma t$, $\gamma=$constant) or else toward a string-dominated universe.

Here I wish to examine further some aspects of these equations and in particular to discuss in more detail the second alternative. If the phase transition occurs at a typical grand-unification scale, that can almost certainly be ruled out, but if the scale is substantially lower it may be viable. It turns out that contrary to naive expectations a string-dominated universe behaves much like an ordinary matter-dominated universe, with $R\propto t^{2/3}$. Thus it is not impossible that our Universe is dominated by strings, provided that it became so only quite recently, in particular, after nucleosynthesis.

Vilenkin[6] suggested previously the possibility of a string-dominated universe, but based on a very different scenario in which strings are assumed to be unable to exchange partners when they cross (i.e., the "intercommuting probability" p is zero).

It must be emphasized that strings cannot solve both problems. They may provide the seeds for galaxy formation or they may constitute the dark matter that makes $\Omega=1$ (*not* the dark matter in halos), but they certainly cannot do both. The first appears to be the more attractive hypothesis, but the second should not be ignored.

II. EVOLUTION EQUATIONS

Let us begin by recalling the formalism developed earlier.[5] Consider the energy E in a comoving volume R^3 in the form of long strings, and the energy $e(l)dl$ in the form of loops of size between l and $l+dl$. Here "size" is to be interpreted in terms of the total (invariant) length of the loop: a loop of size l has by definition a perimeter $2\pi l$ (if it is at rest) or more generally a total energy in a comoving frame $2\pi\mu l$.

If the system of strings has a random ("Brownian") configuration with persistence length ξ, its energy is given by

$$E=\mu R^3/\xi^2 . \tag{1}$$

The probability that a small segment of string of length l will encounter a long string in a time interval δt is of order $lv\delta t/\xi^2$, where v is a typical (say, rms) transverse string velocity. Consequently, if p is the "intercommuting probability" that strings will exchange partners when they cross, the probability that a loop of size l will survive without reattachment from time t (given that $\xi\propto t$) is approximately $\exp(-plvt/\xi^2)$. It follows that large loops, with $l\gg\xi$, have a very transitory existence. The behavior of a segment of string is quite unaffected by whether it is part of a large loop or of an infinitely long string. Consequently the separation between the energy in large loops and in infinitely long strings is artificial. (This is particularly true for loops that are still outside the horizon, which cannot "know" they are closed.) Thus, as already noted in Ref. 5, it would be reasonable to impose an arbitrary upper cutoff on the loop size at a few times the persistence length, say

$$l_{\max}=x_0\xi,\;\; x_0=\text{const} . \tag{2}$$

In Ref. 5 I suggested that it is unnecessary to choose a specific cutoff because $e(l)$ falls off rapidly for large l. However, as explained below, it does not in fact fall off as rapidly as stated there, and the imposition of a cutoff is essential.

The probability of formation of a loop of size l in a system of strings with persistence length ξ depends essentially on the ratio $x=l/\xi$. Thus the fraction of the energy E lost to loops in the size range l to $l+dl$ within a time interval δt is

$$(pv\delta t/\xi)xa(x)dx \quad (x=l/\xi)\ , \tag{3}$$

where $a(x)$ is an as yet unknown function.

If the Universe were not expanding the system of strings would presumably approach an equilibrium distribution in which the energy loss (3) is exactly balanced by the gain from loops undergoing reattachment. In that case the number density of strings would be given by

$$n_{\rm eq}(l)dl=\frac{1}{2\pi\xi^3}a(x)\frac{dx}{x}\ . \tag{4}$$

This relation can be used to estimate the function $a(x)$ as discussed below.

Vachaspati and Vilenkin[7] have performed a numerical simulation of the string configuration created at a phase transition where a U(1) symmetry is broken. They argue that on scales larger than the persistence length the configuration of loops may be expected to be self-similar. So far as the loop size distribution is concerned, this expectation is confirmed. Since the typical radius of a loop of size l is $(l\xi)^{1/2}$, this implies that the number distribution of loops $n(l)dl$ should be a function only of this variable. Hence, if $n(l)$ may be identified with $n_{\rm eq}(l)$, it follows from (4) that

$$a(x)\propto x^{-3/2} \tag{5}$$

(not x^{-3} as stated in Ref. 5). Vachaspati and Vilenkin also find that only about 20% of the total length of string is in the form of loops, implying that

$$\int a(x)dx\simeq 0.2\ . \tag{6}$$

The simulation cannot determine $a(x)$ for $x<1$ because the lattice spacing used is essentially ξ. In any case, we should expect $a(x)$ to decrease at small x, because it is difficult to create loops of size much less than the persistence length. Consequently we may expect that for $x<1$ it is unknown but decreasing rapidly as $x\to 0$.

Recent work[8] has shown that for the Z_2 strings that appear in non-Abelian gauge theories, the loop distribution is similar, but the proportion of closed loops is only about half as large.

It is of course an assumption that the equilibrium distribution $n_{\rm eq}(l)dl$ should coincide with the initial distribution $n(l)dl$. The rationale behind it is that both distributions presumably correspond to states of minimum free energy, but obviously the assumption may be wrong. Indeed, there is some recent evidence[9] to indicate that in a nonexpanding universe the proportion of loops would increase above its initial value, suggesting that $n_{\rm eq}(l)$ is larger than the $n(l)$ obtained from the simulation.

When we impose the cutoff (2) we must allow for the fact that in any small time interval δt, some loops that have hitherto been included as part of the system of long strings must be reclassified as small loops. This yields an extra term in the equation for E. The number of loops so reclassified, in volume R^3, is

$$R^3 n(x_0\xi)x_0\xi\delta t\ , \tag{7}$$

where $n(l)$ is the current value of the loop distribution function. If we were to assume that $n(l)=n_{\rm eq}(l)$, we could again express (7) in terms of $a(x)$. To make it clear that this equality need not hold but nevertheless emphasize the relationship, it will be convenient to define a_0 by

$$n(x_0\xi)x_0 d\xi=\frac{1}{2\pi\xi^3}a_0\frac{d\xi}{\xi}\ .$$

Then the evolution equations [(19) and (20) of Ref. 5] become

$$\dot E=\frac{\dot R}{R}E(1-2v^2)-E\frac{\dot\xi}{\xi}x_0 a_0 \\ -E\frac{pv}{\xi}\int_0^{x_0} xa(x)dx+pv\int_0^{x_0} e(x\xi)x\,dx \tag{8}$$

and

$$\dot e(x\xi)=E\frac{pv}{\xi^2}xa(x)-\frac{pv}{\xi}x\,e(x\xi)\ . \tag{9}$$

In addition, we must impose an initial condition on e, namely,

$$e(x_0\xi)=Ea_0/\xi\ . \tag{10}$$

It should be noted that the rms velocity v is not strictly constant, but will change slowly with time.

If these equations were strictly correct, it should make no difference where we choose the cutoff x_0. However there are approximations involved, and we must be more careful. The main reason lies in the first term on the right-hand side of (8), which represents the work done on the system of strings by the universal expansion. Since we have not included a corresponding term in (9) it *does* make a difference whether loops of a particular size are included in E or in $e(l)$. The point is that, as discussed in Sec. 3 of Ref. 5, the energy of a small loop is constant, and correspondingly the rms velocity v is $1/\sqrt{2}$, while the energy of a very large loop increases roughly in proportion to R. For consistency we should therefore treat separately only the loops which are small enough to have effectively constant energy; i.e., x_0 should not be much larger than unity.

Another way of looking at this problem is to note that v^2 in the first term on the right-hand side of (8) is an average over the entire string system. The effect of lowering the cutoff and thus including some shorter loops would be to increase the average value, or decrease $1-2v^2$. The corresponding increase in $\dot E/E$ would be compensated by the increased magnitude of x_0a_0 in the second term. (We may expect $x_0a_0\propto x_0{}^{-1/2}$.) As long as the cutoff is low enough so that the loops in question have almost constant energy, its precise value should be irrelevant: small changes in x_0 are compensated by small changes in v^2. Of course v appears elsewhere in the equation, so the compensation can only be partial. Nevertheless, within reasonable limits the choice of x_0 should not matter.

III. QUALITATIVE FEATURES OF THE SOLUTION

In Ref. 5 I argued that the solution of Eqs. (8) and (9) will evolve either toward a scaling solution in which the persistence length scales with the horizon distance, $\xi=\gamma t$, or toward a string-dominated universe in which ξ/t approaches zero.

In the scaling solution the loop-energy distribution function is given by

$$e(x\xi)=Ef(x)/\xi \ , \tag{11}$$

where $f(x)$ is a dimensionless function of $x=l/\xi$, related to $a(x)$ by Eq. (2.7) of Ref. 5. In the radiation-dominated case ($n=\frac{1}{2}$) this relation, modified to incorporate the cutoff at x_0, reads

$$f(x)=\frac{pv}{\gamma}x^{-3/2}e^{pvx/\gamma}\int_x^{x_0} y^{3/2}e^{-pvy/\gamma}a(y)dy + (x_0/x)^{3/2}e^{-pv(x_0-x)/\gamma}a_0 \ . \tag{12}$$

This relation takes a particularly simple and instructive form when (5) holds for the relevant range of values of x. If we set

$$a(x)=\alpha x^{-3/2}$$

and define α_0 similarly by

$$a_0=\alpha_0 x_0^{-3/2} \ ,$$

we then find

$$f(x)=a(x)-(\alpha-\alpha_0)e^{-pv(x_0-x)/\gamma} \ . \tag{13}$$

In particular, if $\alpha=\alpha_0$, then $f(x)=a(x)$, and there is no net contribution from the two integrals in (8). This is an accident of the radiation-dominated case: the expansion of the Universe is exactly matched to the scaling of loops with size, so that to maintain the scaling condition no net production or destruction of loops is required. The entire energy loss from long strings to loops would then be represented by the second term on the right of (8).

The equality $f(x)=a(x)$ depends on the assumption that the initial loop distribution is essentially the same as the equilibrium distribution. If in fact there are *more* loops in equilibrium, then α would be larger than α_0, and hence $f(x)<a(x)$, implying a net negative contribution from the integrals in (8).

In any event the equality $f=a$ cannot extend to small values of x, where $a(x)$ decreases rapidly but f does not. Indeed the whole energy-loss mechanism via gravitational radiation from loops clearly only works because there is an excess over the equilibrium distribution of very small loops. From the region $x \ll 1$ there will be a small net positive contribution from the integrals, because of small loops occasionally becoming reattached.

We are primarily interested in how ξ/t evolves with time. Let us first assume that indeed $f(x)=a(x)$. Then using (1) we easily find

$$\xi/t \propto t^{-k}, \quad k=\frac{1-v^2-x_0a_0}{2-x_0a_0} \ . \tag{14}$$

The parameter x_0a_0 does of course depend on the choice of x_0, though only weakly. From the discussion above we should expect that small changes in x_0 would be compensated by small changes in the effective value of v^2 so it may not be unreasonable to treat x_0a_0 as approximately a constant. This parameter does not have any very direct physical significance, though it is closely related to the proportion of the total length in the form of loops. It may be regarded as a measure of the efficiency of loop formation as a mechanism for transferring energy from long strings to small loops.

If $x_0a_0>\frac{1}{2}$, then for sufficiently small ξ/t, where v^2 approaches $\frac{1}{2}$, k would become negative. This is the case in which the system approaches a scaling solution. On the other hand, if $x_0a_0<\frac{1}{2}$, k would remain positive, ξ/t would continue to decrease toward zero, and string domination would be inevitable.

Compared with the values suggested by the numerical simulations of Vachaspati and Vilenkin,[7] $\frac{1}{2}$ seems a rather large value for x_0a_0: 0.1 would be more likely. On the other hand, the dynamical simulations performed by Albrecht and Turok[10] do suggest a rather rapid evolution of the string systems toward scaling behavior. There is an apparent contradiction here. The most likely explanation appears to be that there is indeed a substantial difference between the equilibrium loop distribution and the initial one. Although initially the proportion of the total length in the form of loops is quite small, it may grow with time and even approach 100%. Indeed this seems to be suggested by the work of Albrecht and Turok. In that event there will be a large positive contribution to the energy loss from the integrals in (8). Recently Bennett[11] has suggested that the origin of the discrepancy lies in the fact that the equations do not explicitly allow for the fragmentation of loops; that process too leads to a change in the effective form of the function $a(x)$. Further numerical studies may well resolve this question.

If indeed a is substantially larger than f, then a scaling solution will be much more likely. Consider, for example, the extreme case in which the last term of (8), representing reconnection of loops, is negligible, and suppose that the integral in the preceding term can be approximated by a constant c. Then one finds in place of (14),

$$\frac{\xi}{t}=\frac{pvc}{1-v^2-x_0a_0}+c't^{-k} \ ,$$

where c' is another constant. This evidently represents a scaling solution at large times.

We have implicitly assumed in this discussion that the loop distribution function is close to its scaling value, so that the problem reduces to a single differential equation for ξ. Once ξ/t is substantially less than unity, this is probably a good approximation. From (9) we see that the time scale on which the loop distribution function approaches its scaling value is roughly ξ/pvx. As long as $\xi \ll t$, this time scale will be short compared to the expansion time (which controls the evolution of ξ/t) except for the smallest loops whose contribution is in any case small. On the other hand, in the interesting region where ξ/t is of order unity, the two time scales are comparable, and we really ought to treat the coupled equations (8) and (9) together.

IV. THE STRING-DOMINATED UNIVERSE

From the discussion of the preceding section it is not possible to decide unambiguously between the two alternatives of scaling and string domination. Only a fuller numerical study is likely to be able to resolve this issue.

Meanwhile, it is interesting to ask what a string-dominated universe would be like.

Once ξ has fallen well below t, it should be a reasonable approximation to equate the loop distribution function to its scaling value, because the relevant time scale is then much shorter than the expansion time. Let us again assume for simplicity that $f(x)=a(x)$, so that the solution during the radiation-dominated era is given by (14). We may set $v^2=\frac{1}{2}$, and hence expect that ξ/t will decrease like $t^{-1/4}$ (for $x_0a_0=0$) or a little slower (for example, $\xi/t \propto t^{-1/6}$ for $x_0a_0=0.2$). The universe will therefore approach string domination rather slowly.

When the energy density of stings exceeds that of radiation, the energy density is $\rho=E/R^3=\mu/\xi^2$. From Einstein's equation it follows that $\xi \propto t$. Constancy of E (corresponding to $x_0a_0=0$) would require

$$\xi \propto R^{3/2} \propto t ,$$

i.e., the Universe behaves exactly as though it were matter dominated. If we allow for some loss of energy to loops, we find that the Universe would expand a little slower. From (8)

$$E \propto \xi^{-x_0a_0} ,$$

or equivalently,

$$R \propto \xi^n \propto t^n, \quad n=(2-x_0a_0)/3 . \tag{15}$$

Since we have assumed that $x_0a_0<\frac{1}{2}$, we have $n>\frac{1}{2}$, i.e., a string-dominated universe always expands faster than a radiation-dominated universe.

In any event, once the strings come to dominate we do arrive at a scaling solution with ξ/t constant at the value

$$\frac{\xi}{t}=\frac{1}{n}\left[\frac{8\pi G\mu}{3}\right]^{1/2} . \tag{16}$$

It is clear that the Universe cannot have been string dominated until very recently. In particular, the success of the nucleosynthesis scenario requires that at that time the Universe was radiation dominated. This implies a stringent limit on $G\mu$.

We can estimate the required value of $G\mu$ by following the evolution of strings from formation onward. The strings are formed at a critical temperature

$$T_c \simeq (G\mu)^{1/2} m_P , \tag{17}$$

where $m_P \simeq 10^{19}$ GeV is the Planck mass. Initially the coherence length ξ is small compared to t. Its early evolution during the period of heavy damping is governed[1,12] by the equation

$$\frac{1}{\xi}\frac{d\xi}{dt} \simeq \frac{t_d}{\xi^2} ,$$

where the damping time is $t_d \simeq \mu/\sigma\rho$. Here $\rho=3/32\pi G t^2$ and σ is the effective linear cross section of a string for particles of momentum $\sim T$, which as Everett[13] has shown is

$$\sigma \simeq \pi^2/T \ln^2(T/T_0) .$$

Neglecting the slow variation of the logarithmic factor, we then find

$$\xi^2 \simeq G\mu m_P{}^{1/2} t^{5/2} .$$

It follows that ξ/t grows to be of order unity when

$$t=t_* \simeq \frac{1}{(G\mu)^2 m_P} , \tag{18}$$

or equivalently at the temperature

$$T_* \simeq G\mu m_P . \tag{19}$$

From this point onward the strings evolve freely, and Eqs. (8) and (9) apply.

We can get at least a rough estimate of the subsequent evolution from (14). If we set $v^2=\frac{1}{2}$ and assume that x_0a_0 lies between 0 and 0.2, we find

$$\tfrac{1}{4} > k > \tfrac{1}{6} .$$

The ratio of the string density to the radiation density is

$$\frac{\rho_s}{\rho}=\frac{32\pi}{3}G\mu\left[\frac{t}{\xi}\right]^2 \simeq 30G\mu\left[\frac{t}{t_*}\right]^{2k} . \tag{20}$$

This is equal to unity when t is

$$t_{\text{eq}} \simeq \frac{t_*}{(30G\mu)^2} \text{ to } \frac{t_*}{(30G\mu)^3} . \tag{21}$$

Let us assume that our present Universe has $\Omega=1$ (as suggested by the inflationary universe scenario) and is string dominated. It must then have become string dominated at about $t_{\text{eq}} \simeq 10^4$ yr, so we require

$$m_P t_{\text{eq}} \simeq 10^{55} .$$

From (19) and (21) we then find

$$G\mu \simeq 10^{-14.5} - 10^{-12} . \tag{22}$$

These values are substantially smaller than those typical of strings in grand unified theories. They correspond to transition temperatures in the range

$$T_c \simeq 10^{12} - 10^{13} \text{ GeV} , \tag{23}$$

and correspondingly

$$T_* \simeq 10^{4.5} - 10^7 \text{ GeV} . \tag{24}$$

The upper limit here is not at all firm. If the parameters were close to the limiting values separating string domination from scaling, the Universe could spend a long time in a near-scaling condition where the effective value of k in (14) would be close to zero. In principle, therefore, one might have, say, $G\mu \simeq 10^{-6}$ and still only reach string dominance at the same epoch t_{eq}. However, this would require rather fine-tuning of the parameters.

Subsequent to t_{eq}, the value of ξ/t has remained constant. Thus its present value would be

$$\left[\frac{\xi}{t}\right]_{\text{now}} = \left[\frac{\xi}{t}\right]_{\text{eq}} = \left[\frac{t_*}{t_{\text{eq}}}\right]^k = (30G\mu)^{1/2} .$$

This corresponds to

$$\xi \simeq 1-30 \text{ kpc} . \tag{25}$$

It is interesting to ask how these strings might be detected. For values of $G\mu$ larger than about 10^{-6} the best hope is to look for the gravitational lensing effects[14] of strings. But with values as small as those in (22) this would be impractical because the maximum value of the angular separation between double images would be only 10^{-11} rad. It is perhaps more likely that one could see characteristic dynamical effects such as the wake left behind a moving string.[15]

The estimate (20) of string density takes no account of the contribution of small loops. One might suppose that as in the case of the scaling solution the loops would contribute a larger amount, proportional to $(G\mu)^{1/2}$ rather than $G\mu$. However, that is not the case. Loops of size around ξ will contribute a similar fraction of the total density as in the scaling solution. However, since the expansion time of the Universe is so long in comparison to their size their chances of survival are exponentially small. Almost all the loops produced will reintersect long strings and become reattached. It is not necessary to assume that every such encounter leads to reattachment. So long as there is some reasonable probability of reattachment, this will almost always happen in the end. The number density of very small loops will be suppressed by an exponential factor of the form $e^{-pv/\gamma}$ with $\gamma \ll 1$.

V. CONCLUSIONS

It is not possible at present to decide unambiguously whether a system of strings would evolve toward a scaling solution or toward string dominance. Both are interesting possibilities cosmologically, though for rather different ranges of the parameter $G\mu$.

For values of $G\mu$ typical of strings in grand unified theories, string dominance is almost certainly incompatible with observation, whereas the scaling solution is the basis for a very attractive theory of galaxy formation. Turok[16] has recently shown that the loop distribution function reproduces remarkably well the observed correlation function of clusters of galaxies.

For smaller values of $G\mu$, both scenarios are in principle viable. In particular with values of $G\mu$ in the range (22), there is a possibility that strings could constitute the dark matter that is required to make $\Omega = 1$. This is not of course a solution to the "dark-matter problem." It would certainly not be easy to identify strings with the dark matter in galactic halos required by observations of rotation curves and the like. Long strings could hardly be bound in galaxies. Of course, we should expect a sizable fraction of the total density to be in the form of loops of size around ξ, and even if their existence on a cosmological time scale is rather transitory they might perhaps live long enough to be at least temporarily bound. Against this, it must be said that they would in general be expected to have rather large velocities. Probably therefore we would have to suppose that the dark matter in halos is of some other form, perhaps baryonic or quark matter.

It is not easy to see how if the Universe were string dominated we should be able to tell, since the strings in question are not heavy enough to yield resolvable double images. There might however be observable, indirect effects. The only immediate prospect of distinguishing the two alternative string scenarios lies in numerical simulations.

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