# Class of stationary axisymmetric solutions of Einstein's equations in empty space

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We present a stationary axisymmetric solution of the Einstein vacuum equations which could represent the exterior gravitational field of a rotating mass. The solution possesses all multipole moments, the Zipoy-Voorhees parameter, and two arbitrary parameters resulting from two rank-zero Hoenselaers-Kinnersley-Xanthopoulos transformations.

## I. INTRODUCTION

In the last years, one of the most usual methods for finding exact solutions of Einstein's equations has been the generation of new solutions from those already known. We will use here the rank-zero transformations of Hoenselaers, Kinnersley, and Xanthopoulos[1] (HKX) to obtain a stationary axisymmetric solution with all multipole moments from a generalization of the static axisymmetric Erez-Rosen metric[2] by means of two rank-zero HKX transformations. The choice of the seed metric is based on the following physical considerations: (i) It contains an infinite set of parameters which represent, as we will see below, the mass multipole moments of an axisymmetric distribution of mass in the Newtonian approximation; (ii) it reduces to the exterior Schwarzschild metric in the limiting case that all the moments higher than the monopole moment vanish. These two conditions on the seed metric are very important in order to obtain a physically realistic solution. Dietz and Hoenselaers[3] found the general form of a stationary axisymmetric solution derived from a static one with the Ernst potential

$$E_0=\exp\left[2\sum_{n=0}^{\infty}M_n r^{-n-1}P_n(\cos\theta)\right], \tag{1}$$

where $r$ and $\theta$ are spherical coordinates, $M_n$ are constants, and $P_n$ are the Legendre polynomials. This seed metric does not satisfy condition (ii) given above since for $M_n=0$, $n\neq0$, Eq. (1) is the Ernst potential of the Chazy-Curzon solution. However, at spatial infinity the asymptotic behavior of the seed metric presented below agrees with that of Dietz and Hoenselaers.[3]

## II. SOLUTION

Let us consider the general static axisymmetric line element[4] in prolate spheroidal coordinates $(X,Y,\phi)$,

$$ds^2=\sigma^2\exp(-2\Psi)\left[\exp(2\gamma)(X^2-Y^2)\times\left[\frac{dX^2}{X^2-1}+\frac{dY^2}{1-Y^2}\right]+(X^2-1)(1-Y^2)d\phi^2\right]-\exp(2\Psi)dt^2\ , \tag{2}$$

where $\sigma=$const, and $\Psi$ and $\gamma$ are functions of the coordinates $X$ and $Y$ which are related to the canonical Weyl coordinates $(\rho,Z,\phi)$ by

$$\rho=\sigma(X^2-1)^{1/2}(1-Y^2)^{1/2},\quad Z=\sigma XY\ . \tag{3}$$

The field equations for (2) turn out to be

$$[(X^2-1)\Psi_X]_X+[(1-Y^2)\Psi_Y]_Y=0\ , \tag{4}$$

$$\gamma_X=\frac{1-Y^2}{X^2-Y^2}[X(X^2-1)\Psi_X{}^2-X(1-Y^2)\Psi_Y{}^2-2Y(X^2-1)\Psi_X\Psi_Y]\ , \tag{5}$$

and

$$\gamma_Y=\frac{X^2-1}{X^2-Y^2}[Y(X^2-1)\Psi_X{}^2-Y(1-Y^2)\Psi_Y{}^2+2X(1-Y^2)\Psi_X\Psi_Y]\ , \tag{6}$$

where $\Psi_X$ represents $\partial\Psi/\partial X$, etc. Solving Eq. (4) by separation of variables,[2,5] we obtain

$$\Psi=\sum_{n=0}^{\infty}q_nP_n(Y)Q_n(X)\ , \tag{7}$$

where $q_n$ are constants, and $Q_n$ are the associated Legendre functions of the second kind for which we use the definition given in Ref. 6. Taking $q_0=1$, $q_1=0$, $q_2=q$, and $q_k=0$ $(k>2)$, we get from (7) the Erez-Rosen metric[2] after the coordinate change $X\rightarrow -X$. The corresponding $\gamma$ function can be calculated from Eqs. (5) and (6) by demanding asymptotic flatness.

Note that if a solution $(\Psi,\gamma)$ of Eqs. (4)–(6) is known then $(\delta\Psi,\delta^2\gamma)$ with $\delta=$const is also a solution (see Refs. 7 and 8). We call $\delta$ the Zipoy-Voorhees parameter.

Let us now consider the seed metric (2) whose Ernst potential $E_0=\exp(2\Psi)$ we choose in the form

$$E_0=\exp\left[2\delta\sum_{n=0}^{\infty}q_nP_n(Y)Q_n(X)\right], \tag{8}$$

where $q_0=1$, $q_k$, $k>0$ are constants and we let $X\rightarrow -X$. The interpretation of this static solution becomes very simple when Schwarzschild-type coordinates are introduced: $X=r/m-1$ and $Y=\cos\theta$. The asymptotic expansion of the metric function $g_{tt}$ gives *exactly* the gravitational potential of an axisymmetric distribution of mass in the Newtonian approximation. The correspondence with the work of Dietz and Hoenselaers[3] is thus established since for $r \gg m$, $Q_n(-X) \simeq a_n(m/r)^{n+1}$, with $a_n = (-1)^{n+1} n!/(2n+1)!!$, and hence for $\delta = 1$, the Newtonian multipole moments are $M_n = q_n a_n m^{n+1}$. In the Newtonian approximation, the constant $q_1$ can be made to vanish by choosing the coordinate frame so that its origin coincides with the center of mass of the body. However, we shall leave $q_1$ arbitrary here for the sake of generality.

Applying two rank-zero HKX transformations to the Ernst potential (8) and performing the coordinate change $X \to -X$, one arrives at the new Ernst potential $E$ given by (see Refs. 3 and 9)

$$E = \exp(2\delta\hat{\Psi}) d_- / d_+ \ , \tag{9}$$

where

$$\hat{\Psi} = \sum_{n=1}^{\infty} (-1)^{n+1} q_n P_n(Y) Q_n(X) \tag{10}$$

and

$$d_\pm = (X \pm 1)^{\delta-1} [X(1-\lambda\mu) + iY(\lambda+\mu) \pm (1+\lambda\mu) \mp i(\lambda-\mu)] \ , \tag{11}$$

with

$$\lambda = \alpha_1 (X^2-1)^{1-\delta} (X+Y)^{2\delta-2} \times \exp\left[2\delta \sum_{n=1}^{\infty} (-1)^n q_n \beta_{n-}\right] \tag{12}$$

and

$$\mu = \alpha_2 (X^2-1)^{1-\delta} (X-Y)^{2\delta-2} \times \exp\left[2\delta \sum_{n=1}^{\infty} (-1)^n q_n \beta_{n+}\right] . \tag{13}$$

Here $\alpha_1$ and $\alpha_2$ are the two arbitrary parameters introduced by the two rank-zero HKX transformations, and $\beta_{n\pm}$ is the solution of the differential equations

$$(X \mp Y)(\beta_{n\pm})_X = (1 \mp XY) P_n(Y) [Q_n(X)]_X \mp (1-Y^2) Q_n(X) [P_n(Y)]_Y \tag{14}$$

and

$$(X \mp Y)(\beta_{n\pm})_Y = (1 \mp XY) Q_n(X) [P_n(Y)]_Y \pm (X^2-1) P_n(Y) [Q_n(X)]_X \ , \tag{15}$$

for $n \geq 0$. Using recurrence relations for the Legendre functions, the general solution of Eqs. (14) and (15) can be written in the form

$$\beta_{n\pm} = (\pm 1)^n \beta_{0\pm} - (\pm 1)^n Q_1(X) + P_n(Y) Q_{n-1}(X) - \sum_{k=1}^{n-1} (\pm 1)^k P_{n-k}(Y) [Q_{n-k+1}(X) - Q_{n-k-1}(X)] \quad (n \geq 1) \ , \tag{16}$$

with

$$\beta_{0\pm} = \frac{1}{2} \ln \frac{(X \mp Y)^2}{X^2-1} \ . \tag{17}$$

The Ernst potential (9) corresponds to the stationary axisymmetric line element

$$ds^2 = \sigma^2 f^{-1} \left[ \exp(2\gamma)(X^2-Y^2) \left[ \frac{dX^2}{X^2-1} + \frac{dY^2}{1-Y^2} \right] + (X^2-1)(1-Y^2) d\phi^2 \right] - f(dt - \omega\, d\phi)^2 \ , \tag{18}$$

where $f$, $\gamma$, and $\omega$ are functions of $X$ and $Y$ which can be determined from the Ernst potential (9). To carry out the calculations of the metric functions we use the algebraic method of Yamazaki,[10] Cosgrove,[11] and Dietz and Hoenselaers.[3] The resulting metric functions are (for this calculation we use the algebraic computer language[12] REDUCE 3.0)

$$f = 2R \left[ (1+\cos\tau) L_+ \left[ \frac{X-1}{X+1} \right]^{1-\delta} \exp(-2\delta\hat{\Psi}) + (1-\cos\tau) L_- \left[ \frac{X-1}{X+1} \right]^{\delta-1} \exp(2\delta\hat{\Psi}) + 4\sin\tau (XN_- + YN_+) \right]^{-1} , \tag{19}$$

$$\omega = K_1 + \sigma \sin\tau [\delta\hat{p} + 2Y(1-\delta)] - \frac{\sigma}{R} \left[ (1+\cos\tau) M_+ \left[ \frac{X-1}{X+1} \right]^{1-\delta} \exp(-2\delta\hat{\Psi}) + (1-\cos\tau) M_- \left[ \frac{X-1}{X+1} \right]^{\delta-1} \exp(2\delta\hat{\Psi}) + 2\sin\tau [X(\lambda^2-\mu^2)(1-Y^2) + Y(1-\lambda^2\mu^2)(X^2-1)] \right] , \tag{20}$$

and

$$\exp(2\gamma) = K_2 \exp(2\delta^2 \hat{\gamma}) R / (X^2-1) \ , \tag{21}$$

with

$$R = (X^2-1)(1-\lambda\mu)^2 - (1-Y^2)(\lambda+\mu)^2 \ , \tag{22}$$

$$L_\pm = (1-\lambda\mu)[(X \pm 1)^2 - \lambda\mu(X \mp 1)^2] + (\lambda+\mu)[\lambda(1 \mp Y)^2 + \mu(1 \pm Y)^2] \ , \tag{23}$$

$$M_\pm = (X^2-1)(1-\lambda\mu)[\lambda+\mu \mp Y(\lambda-\mu)] + (1-Y^2)(\lambda+\mu)[1-\lambda\mu \pm X(1+\lambda\mu)] \ , \tag{24}$$

$$N_{\pm}=(\lambda+\mu)(1\pm\lambda\mu) . \quad (25)$$

Here $K_1$, $K_2$, and $\tau$, which have to be chosen so that the metric is asymptotically flat, are constant parameters determined by $\alpha_1$ and $\alpha_2$. The function $\hat{p}$ is defined by

$$\hat{p}=\sum_{n=1}^{\infty}(-1)^n q_n p_n , \quad (26)$$

where $p_n$ is the solution of the equations

$$(p_n)_X=-2(1-Y^2)Q_n(X)[P_n(Y)]_Y \quad (27)$$

and

$$(p_n)_Y=2(X^2-1)P_n(Y)[Q_n(X)]_X . \quad (28)$$

Using recurrence relations for the Legendre functions, we obtain

$$p_n=-\frac{2}{2n+1}(1-Y^2)[P_n(Y)]_Y[Q_{n+1}(X)-Q_{n-1}(X)]. \quad (29)$$

Finally, the function $\hat{\gamma}$ of Eq. (21) is the solution of the differential equations (5) and (6), where $\Psi$ is given by Eq. (7) with $X$ replaced by $-X$ and the condition

$$\lim_{X\to\infty}\hat{\gamma}(X,Y)=0 ,$$

which ensures the asymptotic flatness of the seed metric, is satisfied. The structure and properties of the general $\hat{\gamma}$ function will be presented in a forthcoming paper.

The metric (18)–(21) possesses the following parameters: $q_n$ which can be interpreted as parameters determining the Newtonian mass multipole moments, $\alpha_1$ and $\alpha_2$ which follow from the two rank-zero HKX transformations, and the Zipoy-Voorhees parameter $\delta$, which generalizes each solution to a one-parameter class of solutions by taking different real values. Performing the coordinate transformations

$$X=(r-\sigma)/\sigma, \quad Y=\cos\theta$$

and giving special values to the parameters of the metric (18)–(21), we obtain the following already-known metrics.

Schwarzschild ($m$=mass):

$$\alpha_1=\alpha_2=\tau=0, \quad \delta=1, \quad q_n=0 \quad (n>0) ,$$

$$K_1=0, \quad K_2=1, \quad \sigma=m .$$

Zipoy-Voorhees:[7,8]

$$\alpha_1=\alpha_2=\tau=0, \quad q_n=0 \quad (n>0) ,$$

$$K_1=0, \quad K_2=1 .$$

Erez-Rosen:[2]

$$\alpha_1=\alpha_2=\tau=0, \quad \delta=1, \quad q_1=0, \quad q_2=q ,$$

$$q_k=0 \quad (k>2), \quad K_1=0, \quad K_2=1, \quad \sigma=m .$$

Hoenselaers-Kinnersley-Xanthopoulos:[1]

$$q_n=0 \quad (n>0) .$$

Kerr-NUT[13] (Newman-Unti-Tamburino) ($a$=Kerr parameter, $b$=NUT parameter):

$$\alpha_1=a-\alpha_2, \quad \alpha_2=\tfrac{1}{2}[a\pm(a^2-8\sigma^2)^{1/2}] ,$$

$$q_n=0 \quad (n>0), \quad \delta=1, \quad K_1=-2a ,$$

$$K_2=1, \quad \sigma^2=m^2-a^2+b^2 ,$$

$$\cos\tau=\frac{\pm b(a^2-8\sigma^2)^{1/2}-3m\sigma}{m^2+b^2} ,$$

$$\sin\tau=\pm\frac{3\sigma\cos\tau+m}{m^2+b^2} ,$$

for special values of the parameters (cf. Ref. 9).

Quevedo-Mashhoon:[5]

$$\delta=1, \quad q_1=0, \quad q_2=q, \quad q_k=0 \quad (k>2) .$$

In conclusion, the metric given here can be interpreted as a nonlinear superposition of the static seed solution with all Newtonian mass multipole moments and a Kerr-NUT solution.

Let us now explain briefly some physical properties of the static seed solution which lead us to consider it as describing the *exterior* gravitational field of a deformed mass. The solution is asymptotically flat and corresponds, in the Newtonian limit, to the gravitational potential of an axisymmetric matter distribution with all multipole moments. A preliminary investigation of curvature scalars shows that the spacetime region exterior to the hypersurface $X=1$ ($r=2m$) is apparently free of singularities. This agrees with our interpretation since the surface of a realistic astronomical body is expected to lie outside $r=2m$. Furthermore, these conclusions are expected to hold even when the body rotates (and the NUT parameter vanishes).

An important characteristic of the gravitational field of the rotating deformed mass is the set of relativistic multipole moments. These would have to be determined according to an invariant prescription such as that of Geroch and Hansen.[14,15] Preliminary calculations indicate that these moments consist of the Newtonian multipole moments plus relativistic corrections. A more detailed determination of the physical properties of the general solution, especially the explicit calculation of the multipole moments of the rotating source in terms of the parameters presented here, remains a task for the future.

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