Black holes in two spacetime dimensions

J. D. Brown and Marc Henneaux*
Center for Theoretical Physics, University of Texas at Austin, Austin, Texas 78712

Claudio Teitelboim
Center for Theoretical Physics, University of Texas at Austin, Austin, Texas 78712
and Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile
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It is shown that the analog of the black hole exists in two-dimensional gravity. It is given by a metric which solves the vacuum field equation (constant curvature) everywhere except on a singular line. This geometry possesses an event horizon. There is as well an analog of Hawking radiation with temperature proportional to the strength (mass) of the singularity.

The natural analog of the vacuum Einstein equations with a cosmological constant in two spacetime dimensions is given by the requirement of constant scalar curvature,

$$R-\Lambda=0 . \tag{1}$$

This has of course been evident for a long time, but only recently has it been realized that one can derive (1) from a local action principle.[1,2] This development has put two-dimensional gravity on a firm footing as a dynamical field theory.

The two-dimensional theory is extremely simple but still rich in content and its formal structure is remarkably similar to the one found in higher dimensions. One then naturally asks whether there is an analog of the black hole as well. It appears that this is indeed so as we show in this paper.

The analysis will be first carried out for $\Lambda=0$ to grasp more easily the central features and will then be extended to $\Lambda\neq 0$.

The most general stationary metric in two spacetime dimensions is also static and explicitly reads

$$ds^2=\exp[2\sigma(x)](-dt^2+dx^2) . \tag{2}$$

The field equation with a point source at the origin is

$$R(x)=4a\exp(-\sigma)\delta(x) , \tag{3}$$

where a is the strength of the source and has the dimensions of an inverse length. The factor $\exp(-\sigma)$ guarantees that the right-hand side of (3) is a spacetime scalar.

The general solution of (3) is given by

$$\sigma(x)=-a\,|\,x\,|\exp(\beta)+\beta+\gamma x , \tag{4}$$

where β and γ are two integration constants. We will first treat the particular case obtained by choosing these constants in such a way that (i) t is the proper time along the source world line (this yields $\beta=0$), and (ii) the gravitational field is "spherically symmetric," i.e., invariant under the inversion $x\to -x$ (this yields $\gamma=0$). Nonspherically symmetric solutions are also of some interest and will be briefly considered later.

If the absolute value of x were replaced by x itself in σ, the metric (2) would be flat everywhere and could be transformed to the standard Minkowskian form by means of the coordinate transformation

$$aT=e^{-ax}\sinh(at) , \tag{5a}$$

$$aX=e^{-ax}\cosh(at) \tag{5b}$$

($a\neq 0$). The coordinates t,x which both run over the real line, would then be Rindler-type coordinates[3] associated with an accelerated observer in hyperbolic motion (constant acceleration a) in flat spacetime. They only cover the quadrant of Minskowsi space bounded by the lightlike line whose equations are $X^2-T^2=0$, $aX\geq 0$. This line is at $aX=+\infty$ in the original coordinate system and is the event horizon of the accelerated observer. (More precisely, it is only "half" of the horizon but we will loosely refer to it as the "horizon" for simplicity. The entire straight line $aX=|\,a\,|\,T$ is in fact the future event horizon of the observer, whereas the entire straight line $aX=-|\,a\,|\,T$ is the past event horizon.[4])

However, because σ involves $|\,x\,|$ instead of x, new interesting features emerge and the solution (2) turns out to possess nontrivial global properties very different from those of flat, empty space. This happens only when a is positive.

What happens in the case at hand is that the regions $x>0$ and $x<0$ located on both side of the source's (timelike) world line $x=0$ are still flat (in two dimensions, zero scalar curvature implies vanishing Riemann tensor), but must be patched together in an unorthodox fashion along $x=0$. Accordingly, the gravitational field in two dimensions does not show itself through local geodesic deviations, but only through nontrivial global effects.

This can be understood by verifying that the δ-function behavior of the curvature at $x=0$ means, in intrinsic terms, that the acceleration vector $D_u u$ of the source, measured in a frame which varies continuously as one goes from one side of the source world line to the other, has a jump with a magnitude equal to $2a$. Here, u is the unit, future-pointing vector tangent to the world line and $D_u u$ is its covariant derivative along itself. The jump in the acceleration vector ("extrinsic curvature") is of course absent in standard Minkowski space and entirely accounts for the new properties of the solutions.

To describe explicitly how the regions $x>0$ and $x<0$ are patched along $x=0$, it is convenient to perform the coordinate transformation (5) for $x>0$, and (5) with x replaced by $-x$ for $x<0$. This coordinate transformation has the following effects: (i) it maps the source world line $x=0$ on the hyperbolic timelike line S defined by $a^2(X^2-T^2)=1$, $aX>0$; and (ii) it maps both regions $x>0$ and $x<0$ on the *same* section of the Minkowski plane bounded by that hyperbolic world line S. In that section the coordinates X,T are Lorentzian.

One way of visualizing the resulting geometry is to cut a flat sheet of paper along a hyperbolic line and to keep the part either to the left or to the right of the line [depending on the sign of a (see below)]. The solution (2,4) with $\beta=\gamma=0$ is just represented by *both faces* of what remains of the sheet of paper. One can go from one face to the other continuously, by crossing the source world line $x=0$.

The source looks exactly the same observed from both faces, as either accelerating inwards or outwards with acceleration $|a|$, depending on the sign of a. This symmetry is just an expression of reflection invariance $x\to -x$ and stands in sharp contrast with the Minkowski-space situation, where an accelerated object is viewed differently by an observer located to its right or to its left (the object accelerates inwards from one side, outwards from the other).

It is important to realize that the (timelike) line $x=0$ lies in the manifold, although the curvature is singular there. Indeed, the curvature singularity is merely due to our assumption of a point source, and disappears upon smearing it, without changing the global structure of the spacetime. An appropriate analog in four spacetime dimensions would be a thin shell of matter which is a physical object in spacetime, but on which the Einstein tensor has a δ-function singularity.

The global features of the solution (2) are easily read off from its description in terms of patches of Minkowski-space sections. One finds the properties that follow.

When a is negative, the conformal factor $\exp(-2a|x|)$ vanishes nowhere. This means that one keeps (and duplicates) that side of the hyperbolic trajectory of the source which does not contain the lightlike horizon $X^2-T^2=0$ (see Fig. 1). The global causal structure of this spacetime is not very interesting since it is just that of Minkowski space. For this reason, the case of negative strength a will be discarded from now on.

When a is positive, the conformal factor $\exp(-2a|x|)$ vanishes at both $x=\pm\infty$. This indicates that one now keeps (and duplicates) that part of Minkowski space which contains the source horizon (see Fig. 2). [The coordinate system (x,t) does not extend beyond the horizon, but because the metric is regular there, it can be analytically continued across the horizon to cover the entire portion of Minkowski space to the left of the source world line.]

It is straightforward to see that the lightlike line $x=+\infty$ of Fig. 2 is an event horizon not only for the source at $x=0$, but also for all inertial observers lying on the $x<0$ side of the solution. Indeed, every signal these observers would send to (or would receive from) the positive half $x>0$, would have to cross first the source world line $x=0$, for which one knows (and one easily checks on Fig. 2) that the line $x=+\infty$ acts as a horizon. Similarly, the lightlike line $x=-\infty$ is an event horizon for both the

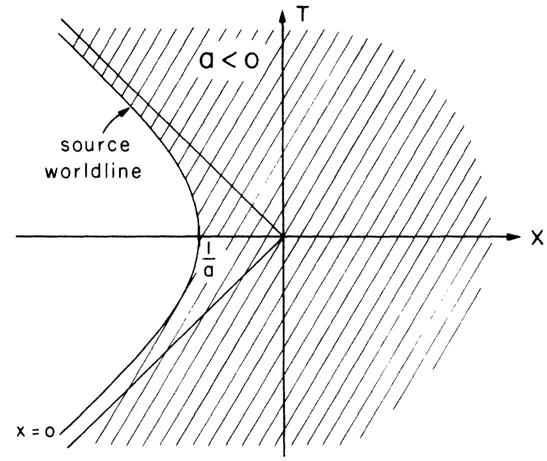

FIG. 1. The solution consists of the unshaded region and a copy of itself. The point source is located at $x=0$ and is in hyperbolic motion. The original coordinate system (x,t) with $x>0$ covers the section of Minkowski space to the left of the source world line (unshaded region). The other half of the solution $(x<0)$ is identical with this first half. Both halves are glued together along the source trajectory.

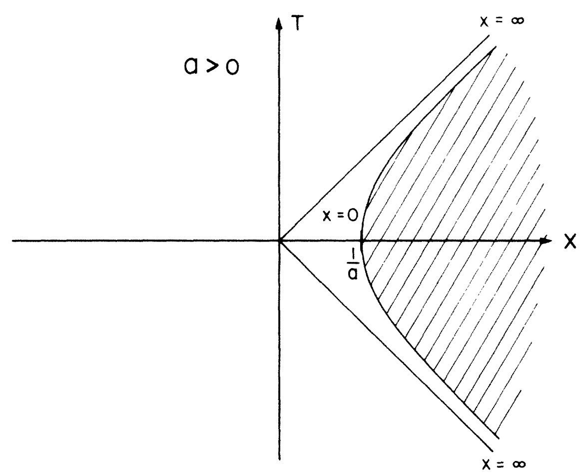

FIG. 2. The solution consists of the unshaded region and a copy of itself. The point source is located at $x=0$. The original coordinate system (x,t) with $x>0$ only covers the portion of Minkowski space limited by the world line of the source and its event horizon at $x=+\infty$. The metric is, however, regular at $x=+\infty$ and can be analytically continued across this lightlike line. The half of the solution to the left of the source is accordingly the entire unshaded region of the figure. Because of the symmetry $x\to -x$, the other half of the solution $(x<0)$ is identical with the first half. It must be patched to the region $x<0$ along the source world line.

source and the inertial observers lying on the positive side ($x>0$) of the solution. Given that there is a horizon, it does not seem inappropriate to call our solution a black hole, even though the source is naked since it stands in front of the horizon.

The global structure of this black-hole spacetime can be displayed by stretching the (X,T) coordinates in the vicinity of the source so as to put both sides of the solution in the same plane (see Fig. 3). This cannot be done, however, by a conformal transformation since the lightlike lines $x=+\infty$ and $x=-\infty$ do not cross each other and, hence, must be bent. This means that light rays are not represented by straight lines at 45°, and the interpretation of the diagram is less direct than that of the standard Kruskal coordinate diagram. By adding points at infinity in the usual way, one can, however, get a Penrose diagram with light rays at 45° to the vertical (see Fig. 4).

Incidentally we mention in passing another analogy with the four-dimensional case, namely, the fact that the strength a can be represented by a "surface integral" at $x=\pm\infty$,

$$2a\exp(\beta)=-\sigma'(+\infty)+\sigma'(-\infty) \tag{6}$$

and in that sense may be considered as the analog of the Schwarzschild mass. Note that if $|x|$ were replaced by x in $\sigma(x)$, the total "mass" defined by the right-hand side of (6) would vanish, which would seem natural since the solution reduces then to flat empty space.

There is, however, a difference in this respect with the Schwarzschild case; namely, there appears to be no compelling argument to exclude either positive or negative a's [since the source of the gravitational field in two dimensions is not the (0-0) component of the energy-momentum tensor and has no definite sign].

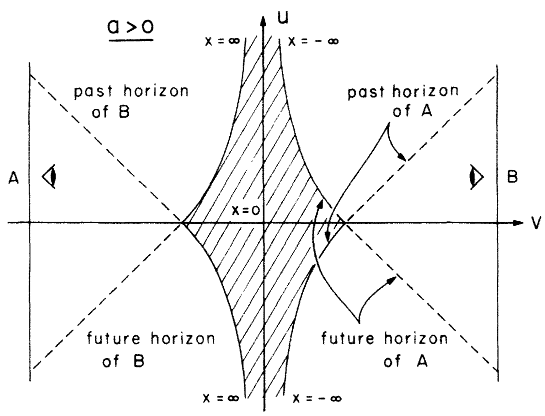

FIG. 3. Maximal analytic extension of the $a>0$ solution. The lines $x=+\infty$ and $x=-\infty$ are lightlike; $x=+\infty$ is an event horizon for the inertial observer B, whereas $x=-\infty$ is an event horizon for the inertial observer A. More precisely, the line $x=-\infty$, $t>0$ together with its prolongation below the $t=0$ axis is the future horizon of A (A can have no knowledge of events occurring to the right of that line) whereas the line $x=-\infty$, $t<0$ together with its prolongation above the $t=0$ axis is A's past horizon (A cannot influence events occurring to the right of that line). The original coordinate system (x,t) only covers the shaded region.

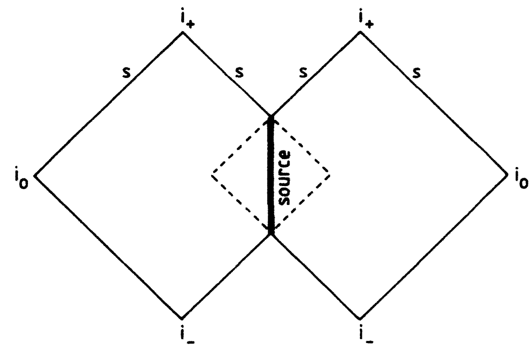

FIG. 4. Penrose diagram for the $a>0$ solution. The horizons are the dashed lines.

Next, we observe that there is an analog of Hawking radiation in the spacetime (2,4) with $\beta=\gamma=0$ and $a>0$. This is verified[5] by going to the Euclidean section of that spacetime by means of the change $t=-i\tau$, which yields

$$ds_E{}^2=\exp(-2a\,|x|)(d\tau^2+dx^2)\ . \tag{7}$$

The metric (7) has a conical singularity at $|x|=\infty$ unless one demands that τ should be a periodic coordinate with period

$$\frac{2\pi}{a}\ . \tag{8}$$

The need for a periodic behavior in the imaginary time is most easily seen by making the change of variables

$$r=a^{-1}\exp(-ax)\quad(x>0) \tag{9}$$

which brings the metric to the familiar polar form

$$ds^2=dr^2+a^2r^2d\tau^2\ . \tag{10}$$

[The transformation (9) is appropriate for the $x>0$ section of the spacetime. This section is glued on the circle $x=0$ ($r=a^{-1}$) to a copy of itself corresponding to $x<0$. In the $x<0$ region, one must replace (9) by $r=a^{-1}\exp(ax)$.]

It should be noticed that there is no conical singularity to avoid when $a<0$ because the origin of the polar coordinate system is not included in the Euclidean section in that case.

From the period (8) one can read off the associated temperature T as

$$T=\beta^{-1}=\frac{\hbar a}{2\pi}\ . \tag{11}$$

Notice that contrary to the situation in the Schwarzschild case, T decreases as the strength of the singularity decreases.

The relation (11) coincides with the formula obtained by Unruh[6] in his analysis of an accelerating detector with proper acceleration a in empty flat spacetime. This is not surprising since the Minkowskian metric in Rindler-type coordinates coincides with (2,4) for either $x>0$ or $x<0$ (but not for both). However, the key point in our analysis is the presence of a source with given strength (having the dimensions of an acceleration). This acceleration is an intrinsic characteristic of spacetime and not an arbitrary number related to particular coordinates or choice of observers.

Furthermore, the presence of the source reflects itself in the $|x|$ in the exponent of (2), which implies in turn that the global structure of the solution is not trivial since there is now an event horizon for the "inertial," i.e., unaccelerated observers at infinity. (Note that the horizon of the inertial observers is just inherited from the horizons of the accelerated ones.) Thus it appears to us that in the present case the radiation with temperature (11) is an analog of that found in the Schwarzschild spacetime.[5] Even though the formula (11) has been derived in the static coordinate system adapted to the source, inertial observers also detect a radiation bath (whose precise temperature and spectrum depend on the vacuum state just as for accelerated observers and whose detailed properties will not be discussed here).

Similar phenomena (occurrence of horizons) also characterize the "nonspherically symmetric" solutions of Eq. (3). For instance, with

$$\sigma=-a(|x|+x) \quad a>0\ , \tag{12}$$

one finds that the source $x=0$ follows a geodesic when viewed from $x<0$, but is in hyperbolic motion with proper acceleration $2a$ when viewed from the positive side $x>0$. As a result, the lightlike line $x=+\infty$, where the conformal factor $\exp(2\sigma)$ vanishes, appears as an event horizon for the inertial observers located at negative x. We will not include here a detailed study of the metric (12) and other possible solutions.

Last, we briefly comment on how our results extend to the theory with a cosmological constant Λ. Again, the elementary solution with a point source at the origin [right-hand side of (1) replaced by $4a\delta(x)\exp(-\sigma)$] is obtained by patching sections of the de Sitter space ($\Lambda>0$) or anti-de Sitter space ($\Lambda<0$) along curves of constant proper acceleration a, in such a way that there is a jump in the acceleration vector of the source world line as one crosses it.

In the de Sitter case, one finds

$$ds^2=-\frac{\cos^2[(|x|-x_0)/\rho]}{\cos^2(x_0/\rho)}dt^2+dx^2 \tag{13a}$$

with

$$a=\rho^{-1}\tan(x_0/\rho)\ . \tag{13b}$$

Here, ρ is the radius of curvature related to Λ through $\Lambda\rho^2=2$.

In the anti-de Sitter case, the metric reads explicitly

$$ds^2=-\frac{\cosh^2[(|x|-x_0)/\rho]}{\cosh^2(x_0/\rho)}dt^2+dx^2 \tag{14a}$$

with

$$a=\rho^{-1}\tanh(x_0/\rho) \tag{14b}$$

when the strength $|a|$ of the source is smaller than ρ^{-1}, and

$$ds^2=-\frac{\sinh^2[(|x|-x_0)/\rho]}{\sinh^2(x_0/\rho)}dt^2+dx^2 \tag{15a}$$

with

$$a=\rho^{-1}\coth(x_0/\rho) \tag{15b}$$

when $|a|>\rho^{-1}$. The radius ρ is now given by $-\Lambda\rho^2=2$ (see Fig. 5). The Hawking temperature turns out to be equal to

$$T=\frac{\hbar}{2\pi}(a^2-\rho^{-2})^{1/2} \quad \text{for } \Lambda<0 \tag{16}$$

and

$$T=\frac{\hbar}{2\pi}(a^2+\rho^{-2})^{1/2} \quad \text{for } \Lambda>0\ . \tag{17}$$

Equation (16) is valid only when $a>\rho^{-1}$. There is no Hawking radiation when $|a|<\rho^{-1}$ because in that case there is no event horizon since the accelerated observers move along integral curves of everywhere timelike Killing vector fields. Moreover, when $a<0$, the Euclidean section of (15a) is everywhere regular as in the $\Lambda=0$ case. No restriction at all on a appears for the de Sitter case ($\Lambda>0$). All absolute values of a are allowed because accelerated observers have event horizons even in the limiting case of vanishing acceleration, and both signs of a are permitted because there is always a conical singularity to avoid in the Euclidean section of (13a), even when $a<0$.

It should be pointed out that the metric with negative Λ also describes a charged black hole immersed in a $\Lambda=0$ background since the electromagnetic field $F_{\lambda\mu}$ of a point charge e reduces to

$$F_{01}=\tfrac{1}{2}e\,\mathrm{sgn}(x)\exp[2\sigma(x)] \tag{18}$$

in two dimensions. Hence, the electromagnetic field effectively induces a cosmological constant equal to $(-e^2/8)$ in the Einstein equation (1). According to the

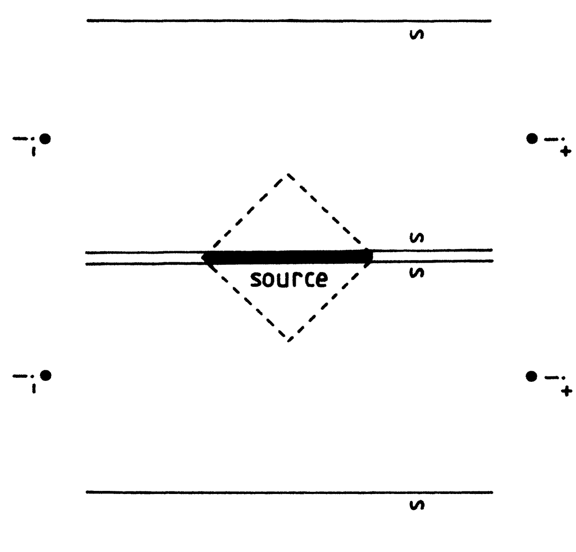

FIG. 5. Penrose diagram for the $a>\rho^{-1}$, $\Lambda<0$ solution. The Penrose diagram for the de Sitter case ($\Lambda>0$) is the same as when $a=0$ and is not represented.

above analysis, the two-dimensional charged black hole is similar to its four-dimensional analog, since a horizon exists only when $|e| < 4a$.

(Here, and before, we have set the "gravitational constant" k appearing in Ref. 1 equal to unity and we have written the Maxwell Lagrangian as $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Also, the relative sign between the gravitational and Maxwell actions is taken here in such a way that the kinetic term of the ϕ field of Ref. 1, which describes gravity, and the kinetic term of the electromagnetic field are both positive. However, because this ϕ field is the conformal degree of freedom of the metric, and because this degree of freedom possesses negative kinetic energy in higher dimensions, one might argue that one should couple gravity and matter with the opposite relative sign.[2] In that case the charged black hole would be equivalent to an uncharged black hole in a $\Lambda > 0$ background, and no bound on $|e|$ would exist.)

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*Permanent address: Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine, C.P. 231, B-1050 Bruxelles, Belgium.

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