Technical aspect in the light-cone gauge

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The validity of the operations $q_\mu \to 0$ and $q_\mu = 0$ is examined in the light-cone gauge in the context of the quark-quark-gluon vertex $\Lambda_\mu(p+q,p)$.

I. INTRODUCTION

The purpose of this Brief Report is to discuss in more detail a technical aspect in the light-cone gauge concerning the equality of the operations $\lim_{q\to 0} g(q)$ and $g(q)|_{q=0}$, where $g(q)$ is a function of the external momentum q. The light-cone gauge has recently been applied[1] to the one-loop quark self-energy $\Sigma(p)$ and the quark-quark-gluon vertex function $\Lambda_\mu(q+p,p)$ which obey the Ward identity

$$q_\mu \Lambda_\mu(q+p,p) = -\Sigma(q+p) + \Sigma(p) , \quad (1)$$

as well as to the three-gluon vertex function[2] $\Gamma^{abc}_{\mu\nu\rho}(p,q,-(q+p))$. An important test of the light-cone formalism is whether or not $\Lambda_\mu(q+p,p)$ and $\Gamma^{abc}_{\mu\nu\rho}(p,q,-(q+p))$ reduce correctly to $\Lambda_\mu(p,p)$ and $\Gamma^{abc}_{\mu\nu\rho}(p,0,-p)$ in the limit as $q\to 0$. To answer this question, one must examine the Feynman integral

$$I(p,q) = \int d^{2\omega}k \, [(k-p)^2 k\cdot n \, (k-q)\cdot n]^{-1}, \quad n^2 = 0 , \quad (2)$$

which occurs in the computations of both $\Lambda_\mu(q+p,p)$ and $\Gamma^{abc}_{\mu\nu\rho}(p,q,-(q+p))$. Our aim is to show with the aid of distributions that $\lim_{q\to 0} I(p,q) = I(p,q=0)$.

II. THE INTEGRAL $I(p,q)$

We begin by evaluating the integral $I(p,q)$ in Eq. (2). Use of the light-cone prescription[3]

$$\frac{1}{k\cdot n} \to \lim_{\epsilon\to 0} \frac{k\cdot n^*}{k\cdot n k\cdot n^* + i\epsilon}, \quad \epsilon > 0 ,$$
$$n_\mu = (n_0, \mathbf{n}), \quad n^*_\mu = (n_0, -\mathbf{n}) \quad (3)$$

gives

$$I(p,q) = \left[\frac{2i\pi^2 \Gamma(\omega-1)\Gamma(2-\omega)}{n\cdot n^* \Gamma(\omega)} \right] J(p,q) , \quad (4)$$

where

$$J(p,q) = N(p,q)/q\cdot n ,$$
$$N(p,q) = (p-q)\cdot n^* [A(p,q)]^{\omega-2} - p\cdot n^* [B(p)]^{\omega-2} ,$$
$$A(p,q) = -2(p-q)\cdot n \, (p-q)\cdot n^* / n\cdot n^* ,$$
$$B(p) = -2p\cdot n p\cdot n^* / n\cdot n^* . \quad (5)$$

We must check if

$$\int d^{2\omega}k \lim_{q\to 0} [(k-p)^2 k\cdot n \, (k-q)\cdot n]^{-1} \stackrel{?}{=} \lim_{q\to 0} [\text{RHS of Eq. (4)}] . \quad (6)$$

Consider first

$$\int d^{2\omega}k \lim_{q\to 0} [(k-p)^2 k\cdot n \, (k-q)\cdot n]^{-1} = \int d^{2\omega}k \, [(k-p)^2 (k\cdot n)^2]^{-1} \equiv I(p,q=0) . \quad (7)$$

The integral on the right-hand side of (7) gives

$$\int d^{2\omega}k \, [(k-p)^2 (k\cdot n)^2]^{-1} \underset{\omega\to 2}{\sim} \frac{(n^*)^2 \Gamma(2-\omega)}{(n\cdot n^*)^2} \times \text{factor} + \frac{2i\pi^2 p\cdot n^*}{n\cdot n^* p\cdot n}$$
$$= \frac{2i\pi^2 p\cdot n^*}{n\cdot n^* p\cdot n}, \quad (n^*)^2 = 0 , \quad (8)$$

and is seen to be finite.

Next consider the right-hand side of (4) where substitution of $q=0$ into the numerator and denominator gives $J \sim 0/0$. Since this naive approach leads to a wrong result, we shall make use of the theory of distributions.[4] To simplify the analysis, we work with $n_\mu = (1,0,0,1)/\sqrt{2}$, $n^*_\mu = (1,0,0,-1)/\sqrt{2}$, and $n\cdot n^* = 1$, so that $q\cdot n \equiv q_- = (q_0 - q_3)/\sqrt{2}$ and $q\cdot n^* \equiv q_+ = (q_0+q_3)/\sqrt{2}$, with $q_\mu = (q_0, \mathbf{q})$. Accordingly, prescription (3) says that

$$\frac{1}{[q\cdot n]} = \left[\frac{q_+}{q_+ q_- + i\epsilon} \right]_{\epsilon\to 0} \equiv P\left[\frac{1}{q_-}\right] - i\pi \, \mathrm{sgn}(q_+) \delta(q_-) , \quad (9)$$

and similarly,

$$\frac{q\cdot n^*}{q\cdot n} = \frac{q_+}{[q_-]} = \left[\frac{(q_+)^2}{q_+ q_- + i\epsilon} \right]_{\epsilon\to 0}$$
$$\equiv q_+ P\left[\frac{1}{q_-}\right] - i\pi |q_+| \delta(q_-) , \quad (10)$$

where P stands for the Cauchy principal value. The expression $1/[q_-]$ is a continuous linear functional over a suitable space of test functions[4] $\phi(q_-)$. Hence $1/[q_-] \equiv f$ is a distribution which assigns to $\phi(q_-)$ the complex number

$$\langle f,\phi\rangle = \int \frac{dq_-\phi(q_-)}{[q_-]} = \int dq_-\phi(q_-)P\left(\frac{1}{q_-}\right) - i\pi\,\mathrm{sgn}(q_+)\phi(0) . \quad (11)$$

Similarly,

$$\langle q_+ f,\phi\rangle = q_+\int dq_-\phi(q_-)P\left(\frac{1}{q_-}\right) - i\pi|q_+|\phi(0) . \quad (12)$$

We see from (11) that $1/[q_-]$ is in general different from zero and well defined in the context of distributions, while the right-hand side of (12) implies that, for any test function ϕ,

$$\lim_{q_+\to 0}\frac{q_+}{[q_-]} = \lim_{q_+\to 0}\langle q_+ f,\phi\rangle = 0 ; \quad (13)$$

in this sense

$$\left.\frac{q_+}{[q_-]}\right|_{q=0} = 0 . \quad (14)$$

Let us apply the above results to the expression $J(p,q) = N(p,q)/[q\cdot n]$ in Eq. (4). Employing the plus-minus notation we deduce from (5) that

$$J(p,q) = (p_+ - q_+)[-2(p_- - q_-)(p_+ - q_+)]^{\omega-2}/[q_-] - p_+(-2p_+p_-)^{\omega-2}/[q_-] . \quad (15)$$

Expansion of the numerator about $(q_+,q_-)=(0,0)$ yields ($\omega\neq 2$ and $p_\pm\neq 0$)

$$J(p,q) = \frac{(-2)^{\omega-2}}{[q_-]}[(p_+)^{\omega-1}(p_-)^{\omega-2} + (2-\omega)(p_+)^{\omega-1}(p_-)^{\omega-3}q_- - (\omega-1)(p_+p_-)^{\omega-2}q_+ - (\omega-1)(2-\omega)(p_+)^{\omega-2}(p_-)^{\omega-3}q_+q_- + O(q_-^2) + O(q_+^2)] - \frac{p_+(-2p_+p_-)^{\omega-2}}{[q_-]} .$$

Since $q_-P(1/q_-)=1$ and $q_-\delta(q_-)=0$, we have $q_-/[q_-]=1$. Hence

$$J(p,q) = (-2)^{\omega-2}[(2-\omega)(p_+)^{\omega-1}(p_-)^{\omega-3} - (\omega-1)(2-\omega)(p_+)^{\omega-2}(p_-)^{\omega-3}q_+ - (\omega-1)(p_+p_-)^{\omega-2}q_+/[q_-] + O(q_-) + O(q_+^2)/[q_-]] , \quad (16)$$

and

$$\lim_{q\to 0} J(p,q) = (2-\omega)\left(\frac{p_+}{p_-}\right)(-2p_+p_-)^{\omega-2} .$$

Finally,

$$\lim_{q\to 0}[\text{RHS of (4)}] = \frac{2i\pi^2\Gamma(\omega-1)\Gamma(2-\omega)(2-\omega)}{n\cdot n^*\Gamma(\omega)}\left(\frac{p_+}{p_-}\right) \times(-2p_+p_-)^{\omega-2} \underset{\omega\to 2}{\sim} \frac{2i\pi^2 p\cdot n^*}{n\cdot n^* p\cdot n} , \quad (17)$$

which agrees with (8). The limiting procedure $q\to 0$ gives, therefore, *consistent* results for the integral (2), just as in the axial gauge or planar gauge.

The crucial question is what happens when we set $q=0$ on both sides of Eq. (4). The left-hand side of (4) yields the value [cf. Eq. (7)]

$$I(p,q=0) = \int d^{2\omega}k[(k-p)^2(k\cdot n)^2]^{-1} = \frac{2i\pi^2 p\cdot n^*}{n\cdot n^* p\cdot n},\quad \omega\to 2 , \quad (18)$$

while

$$[\text{RHS of (4)}]\Big|_{q=0} = \frac{2i\pi^2\Gamma(\omega-1)\Gamma(2-\omega)}{n\cdot n^*\Gamma(\omega)}J(p,q=0) .$$

Using (14) and (16) we find $J(p,q=0)=(2-\omega)p\cdot n^*/p\cdot n$, so that

$$[\text{RHS of (4)}]\Big|_{q=0} = \frac{2i\pi^2 p\cdot n^*}{n\cdot n^* p\cdot n},\quad \omega\to 2 , \quad (19)$$

in agreement with (18). We have, therefore, demonstrated that in the light-cone gauge

$$\lim_{q\to 0} I(p,q) = I(p,q=0) . \quad (20)$$

III. THE VERTEX FUNCTION $\Lambda_\mu(q+p,p)$

To complete the discussion, let us return to the quark-quark-gluon vertex[1] $\Lambda_\mu(q+p,p)$, expressing it in the form

$$\Lambda_\mu(q+p,p) = \frac{-g^2}{12\pi^2}\Gamma(2-\omega)(\lambda_\mu^1 + \lambda_\mu^2) , \quad (21)$$

$$\lambda_\mu^1 = \gamma_\mu - \frac{2\not{n}^*}{n\cdot n^*}n_\mu - \frac{5\not{n}}{2n\cdot n^*}n_\mu^*,\quad \omega\to 2 , \quad (21a)$$

$$\lambda_\mu^2 = \frac{9\not{n}n_\mu}{2n\cdot n^*}\left(\frac{(q+p)\cdot n^*[A(p,q,m)]^{\omega-2} - p\cdot n^*[B(p,m)]^{\omega-2}}{q\cdot n}\right) , \quad (21b)$$

where

$$A(p,q,m)=B(p,m)-\frac{2p\cdot np\cdot n^*}{n\cdot n^*}\left[\frac{q\cdot n}{p\cdot n}+\frac{q\cdot n^*}{p\cdot n^*}+\frac{q\cdot nq\cdot n^*}{p\cdot np\cdot n^*}\right],$$

$$B(p,m)=m^2-2p\cdot np\cdot n^*/n\cdot n^*, \quad (22)$$

m being the quark mass and g the QCD coupling constant. To be "safe" we have kept the ω dependence in the contentious, nonlocal component λ_μ^2.

The λ_μ^2 term arises precisely from integrals of type (2) [see also Eq. (4)], so that the arguments given between Eqs. (2) and (20) apply here as well. We see, in particular, that $\lim_{q\to 0}\lambda_\mu^2(p,q)=\lambda_\mu^2(p,q=0)=0$, so that $\lim_{q\to 0}\Lambda_\mu(q+p,p)=\Lambda_\mu(p,p)$.

Our final remark concerns the relation

$$\Lambda_\mu(p,p)\neq-\partial\Sigma(p)/\partial p_\mu \ . \quad (23)$$

By differentiating (1) with respect to q_ν, it is easy to convince oneself that

$$\Lambda_\nu(q+p,p)+q_\mu\frac{\partial}{\partial q_\nu}\Lambda_\mu(q+p,p)=-\frac{\partial}{\partial q_\nu}\Sigma(q+p)\ , \quad (24)$$

$$\Sigma(p)=\frac{-g^2}{12\pi^2}\Gamma(2-\omega)[\not p-2m-(\not p\not n^*\not n+\not n\not n^*\not p)/n\cdot n^*]\ , \quad (25)$$

which leads to the inequality (23). This problem has also been investigated by Bassetto and Soldati.[5] We note in this context that the equality $\Lambda_\mu(p,p)=-\partial\Sigma(p)/\partial p_\mu$ holds in QCD in the axial gauge and in QED in a linear gauge.

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