Similarity solutions for the self-dual SU(2) fields

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We present new solutions to Yang's self-dual SU(2) equations. These solutions have the property that they are self-similar, together with some of their elliptical and transcendental extensions.

I. INTRODUCTION

Since the publication of Yang's original work[1] on the self-dual SU(2) gauge theory, much effort has been put forward to integrate these equations.[2] Although this work was carried out in the four-dimensional Euclidean space—in connection with instantons—there is always freedom to suppress two of the coordinates and study the theory in a plane. Such a reduced formalism turns out to have much in common with Einstein's equations that admit two Killing vectors, a topic to which much access has been attained in the relativity community.[3] By the same token, we integrate Yang's equations once more in analogy with the similarity integral that we had obtained previously in the Einstein-Maxwell theory.[4]

Static, axially symmetric self-dual Yang's equations in the R gauge can be derived by the variation of the action

$$E[\phi,\Psi,\overline{\Psi}]=\int\frac{|\nabla\Psi|^2+(\nabla\phi)^2}{\phi^2}\rho\,d\rho\,dz \quad , \tag{1}$$

where the real function ϕ and the complex function Ψ depend only on ρ and z. This action is in the form of an energy functional of the harmonic maps, $f:M\rightarrow M'$, where the respective manifolds are

$$M:ds^2=d\rho^2+dz^2+\rho^2 d\varphi^2 \quad , \tag{2}$$

$$M':ds'^2=\frac{|d\Psi|^2+d\phi^2}{\phi^2} \quad . \tag{3}$$

Our purpose is to consider composite maps, where a third manifold M'' is introduced between the manifolds M and M'. By a corollary,[5] the composition of a geodesic map and a harmonic map itself is harmonic; henceforth we consider the harmonic maps from M'' into M', where

$$M'':ds''^2=dv^2 \tag{4}$$

is a one-dimensional manifold. The requirement that the maps from M into M'' are harmonic restricts v to be the arbitrary harmonic function in the ρ, z coordinates, i.e.,

$$v_{\rho\rho}+\frac{1}{\rho}v_\rho+v_{zz}=0 \quad .$$

The energy functional of the maps from M'' into M' reads

$$E[\phi,\Psi,\overline{\Psi}]=\int dv\frac{|\Psi'|^2+\phi'^2}{\phi^2} \quad , \tag{5}$$

where prime denotes d/dv, so that Yang's equations reduce to

$$\phi\phi''-\phi'^2+|\Psi'|^2=0 \quad , \tag{6}$$

$$\phi\Psi''-2\phi'\Psi'=0 \quad , \tag{7}$$

$$\phi\overline{\Psi}''-2\phi'\overline{\Psi}'=0 \quad . \tag{8}$$

The integration of these equations is in the sequel.

II. COMPLETE SIMILARITY INTEGRAL

One notices first that Eqs. (6)–(8) are equivalent to

$$\Psi'=m_0\phi^2 \tag{9}$$

and

$$\phi\phi''-\phi'^2+|m_0|^2\phi^4=0 \quad , \tag{10}$$

where m_0 is a complex integration constant. Equation (10) is the typical Liouville equation for the function $\ln\phi$, which admits the solution

$$\phi=\frac{1}{\cosh(|m_0|v)} \quad , \tag{11}$$

and, therefore,

$$\Psi=\frac{m_0}{|m_0|}\tanh(|m_0|v)+n_0 \quad , \tag{12}$$

with n_0 another constant. We note that since a constant is harmonic, we have the freedom to omit an additive constant to the variable v.

The foregoing solution can be obtained alternatively by making use of the Hamilton-Jacobi (HJ) theory. The HJ functional is to be parametrized by v and the HJ equation reads

$$\frac{\partial S}{\partial v}+H\left(\Psi,\phi,\frac{\partial S}{\partial\Psi},\frac{\partial S}{\partial\phi}\right)=0 \quad , \tag{13}$$

where the Hamiltonian is defined by

$$H=|g|^{-1/2}g'^{AB}\frac{\partial S}{\partial f^A}\frac{\partial S}{\partial f^B} \quad , \tag{14}$$

and where f^A are the coordinates of the M' metric. Choosing $\Psi=\chi e^{i\lambda}$, the HJ equation becomes

$$\phi^{-2}\frac{\partial S}{\partial v}+\left(\frac{\partial S}{\partial\chi}\right)^2+\left(\frac{\partial S}{\partial\phi}\right)^2+\frac{1}{\chi^2}\left(\frac{\partial S}{\partial\lambda}\right)^2=0 \quad , \tag{15}$$

whose separable solution can be expressed by

$$S=-a_1v+a_3\lambda+\int^{\chi}\left(a_2-\frac{a_3^2}{\chi^2}\right)^{1/2}d\chi+\int\left(\frac{a_1^2}{\phi^2}-a_2\right)^{1/2}d\phi\ . \tag{16}$$

Here a_1, a_2, and a_3 are nontrivial constants and the self-dual similarity solution sought reduces then to the equations

$$\frac{\partial S}{\partial a_1}=0,\quad \frac{\partial S}{\partial a_2}=0,\quad \frac{\partial S}{\partial a_3}=0\ . \tag{17}$$

Although this solution is a rather simple one, it has the feature that its independent variable occurs as an arbitrary harmonic function.

III. AN ELLIPTICAL SOLUTION

We reparametrize the foregoing functions by

$$\phi=y\cos\Omega\ ,$$

$$\Omega=y\sin\Omega\, e^{i\lambda+ib\tilde{v}}\quad (b \text{ is a real constant})\ , \tag{18}$$

and consider the harmonic map between

$$M'':ds''^2=dv^2+d\tilde{v}^2 \tag{19}$$

and

$$M':ds'^2=\frac{|\Psi|^2+d\phi^2}{\phi^2}\ . \tag{20}$$

As is observed, we make M'' a two-dimensional manifold, where $\tilde{v}$ is a new function whose Jacobian with v must not vanish everywhere. The functions y, Ψ, and λ are still only functions of v. The energy functional constructed from M'' into M' will yield the Lagrangian

$$L=\left(\frac{y'}{y\cos\Omega}\right)^2+\left(\frac{\Omega'}{\cos\Omega}\right)^2+\tan^2\Omega\,(\lambda'^2+b^2)\ . \tag{21}$$

The Yang equations resulting from the variational principles admit the first integrals

$$\tan^2\Omega\,\lambda'=c_0\ ,$$

$$y'=a_0y\cos^2\Omega\ , \tag{22}$$

where c_0 and a_0 are both real integration constants. The equation for Ω turns out to be nontrivial:

$$\Omega''-\sin\Omega\cos^3\Omega\, a_0^2+\tan\Omega\,(\Omega'^2-c_0^2\cot^2\Omega-b^2)=0\ . \tag{23}$$

Defining a new function by $M=\text{arctanh}\,(\sin\Omega)$, this equation is transformed into

$$M''-a_0^2\frac{\sinh M}{\cosh^3 M}-c_0^2\frac{\cosh M}{\sinh^3 M}-b^2\sinh M\cosh M=0\ ,$$

which is equivalent to the expression

$$\int\frac{dR}{[b^2R^3+(b^2+l)R^2+(l-a_0^2-c_0^2)R-c_0^2]^{1/2}}=2v\ . \tag{24}$$

Note that we have redefined $R=\sinh^2 M$ and l is a new constant of integration. It is known that for $b\neq 0$ this can be transformed into the standard elliptical forms by the proper choice of the constants A and B in the transformation[6] $R=(A+Bx)/(1+x)$; however, we shall not pursue it further here. Assuming this has been carried out, the final solution is

$$\lambda=\text{const}+c_0\int\frac{dv}{R} \tag{25}$$

and

$$y=\text{const}\,\exp\left(a_0\int\frac{dv}{1+R}\right)\ .$$

IV. A TRANSCENDENT SOLUTION

As a final class of solutions we show that the self-dual Yang-Mills equations admit solutions expressible in terms of Painleve's fifth transcendents. Although this class was discovered before,[7] we shall rederive it by an alternative method.

We choose the M'' manifold to be in one of the following forms:

(i) $ds''^2=e^{2v}dv^2+d\tilde{v}^2+e^{2v}d\varphi^2\ ,$ (26)

(ii) $ds''^2=dv^2+d\tilde{v}^2e^{-v}+e^{v}d\varphi^2\ ,$ (27)

and Ψ is chosen as in Sec. III, $\Psi=y\sin\Omega\, e^{i(\lambda+\beta\tilde{v})}$, where β is a real constant. The Lagrangian of the new map takes the form

$$L=\left(\frac{y'}{y\cos\Omega}\right)^2+\left(\frac{\Omega'}{\cos\Omega}\right)^2+\tan^2\Omega\,(\lambda'^2+\beta^2e^{2v})\ , \tag{28}$$

and the Ω equation is modified to

$$\frac{d^2w}{dv^2}-\left(\frac{dw}{dv}\right)^2\left(\frac{1}{2w}+\frac{1}{w-1}\right)-2(1-w)^2\left(a_0^2w+\frac{c_0^2}{w}\right)-2\beta^2e^{2v}w=0\ , \tag{29}$$

where $w=\sin^2\Omega$. Changing the independent variable by $x=e^{2v}$, this equation becomes

$$w_{xx}+\frac{1}{x}w_x-w_x^2\left(\frac{1}{2w}+\frac{1}{w-1}\right)-\frac{(1-w)^2}{2x^2}\left(a_0^2w+\frac{c_0^2}{w}\right)-\frac{\beta^2}{2x}w=0\ ,$$

which is a particular Painleve's fifth transcendent, whose general form is

$$w_{xx}+\frac{1}{x}w_x-w_x^2\left(\frac{1}{2w}+\frac{1}{w-1}\right)+\frac{(1-w)^2}{x^2}\left(\alpha w+\frac{\epsilon}{w}\right)+\frac{\gamma}{x}w+\delta w\frac{w+1}{w-1}=0\ . \tag{30}$$

It is crucial that either $\delta\neq 0$ (as in Einstein-Maxwell theory[8]) or $\gamma\neq 0$ (in this case) in order that the transcendental nature remains. Let us note however that we have yet to meet the case where both $\gamma\neq 0$ and $\delta\neq 0$. The final solution is

$$\lambda=\text{const}+\int (w^{-1}-1)dv\ ,\quad \phi=\text{const}(1-w)^{1/2}\exp\left[a_0\int (1-w)dv\right]\ ,\quad e^{-i\lambda}\Psi=\frac{w^{1/2}}{(1-w)^{1/2}}\phi e^{i\beta\tilde{v}}\ , \tag{31}$$

where v is harmonic and $\tilde{v}$ is an arbitrary function. We must add, however, that once we want to recover axial symmetry in the problem, we are bound to make the choices $v=\ln\rho$ and $\tilde{v}=z$ for the base manifold (26), which will result in the particular solution already given in Ref. 7.

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